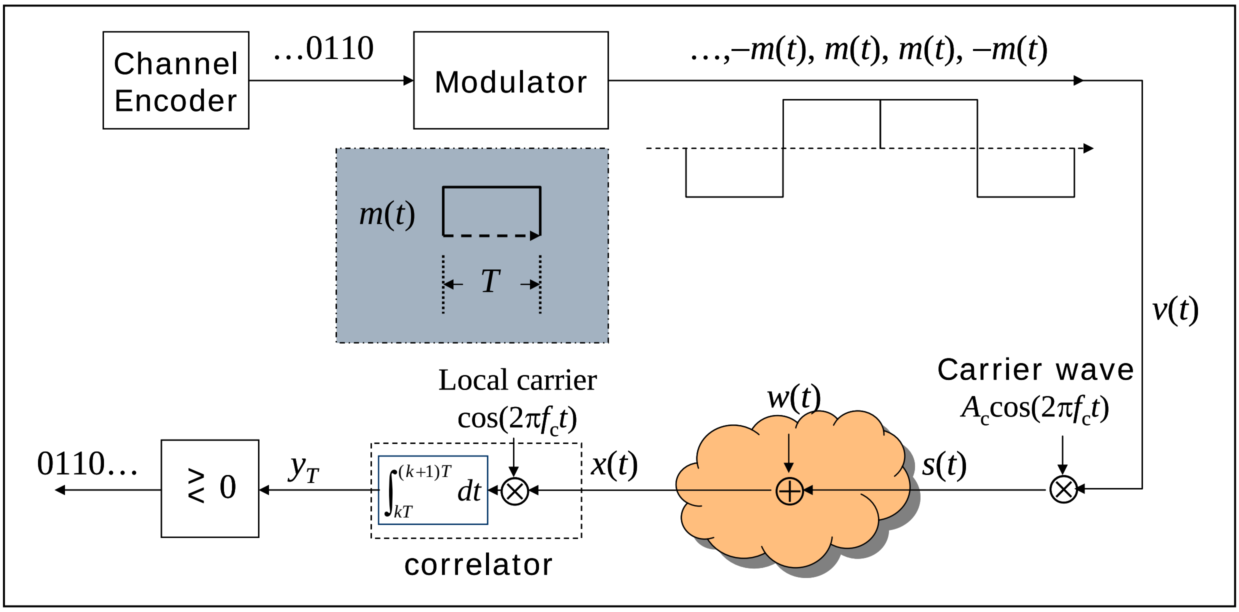
# 通訊原理九十五學年度第二學期第一次期中考題解答

1.




(a) (12%) Suppose that the integration in the above correlator becomes $\int_{kT+\alpha T}^{(k+1)T+\alpha T} dt$, where $0 \le \alpha < 1$ is the "non-synchronization ratio". Show that if $v(t) = \sum_{n=-\infty}^{\infty} I_n \cdot g(t-nT)$, then

$$y_T = \frac{A_c T}{2} I_k - \frac{A_c T}{2}\left(\alpha + \frac{\sin(4\pi f_c T\alpha)}{4\pi f_c T}\right)(I_k - I_{k+1}) + \int_{(k+\alpha)T}^{(k+1+\alpha)T} w(t)\cos(2\pi f_c t)dt,$$

where $g(t) = \begin{cases} 1, & 0 \le t < T \\ 0, & \text{otherwise} \end{cases}$ and $f_c$ is a multiple of $1/T$. (Hint:

$$\int_{kT+\alpha T}^{(k+1)T+\alpha T} g(t-nT)dt = \begin{cases} (1-\alpha)T, & n = k \\ \alpha T, & n = k+1 \\ 0, & \text{otherwise} \end{cases}.)$$

(b) (4%) Prove that if $4\pi f_c T\alpha$ is small enough such that $\sin(4\pi f_c T\alpha) \approx 4\pi f_c T\alpha$, then the second term in the $y_T$ expression in (a) becomes $-A_c T\alpha(I_k - I_{k+1})$.

(c) (4%) Suppose $\sin(x)$ is well-approximated by $x$ for $|x| < 0.1\pi$. Let $f_c = 100MHz$ and $T = 1\mu s$. Determine the range of $\alpha$ such that the second term mentioned in (b) can be well-approximated by $-A_c T\alpha(I_k - I_{k+1})$.

Answers:

(a)

$$
\begin{aligned}
y_T &= \int_{kT+\alpha T}^{(k+1)T+\alpha T} x(t)\cos(2\pi f_c t)dt = \int_{kT+\alpha T}^{(k+1)T+\alpha T} [s(t)+w(t)]\cos(2\pi f_c t)dt \\
&= A_c \int_{kT+\alpha T}^{(k+1)T+\alpha T} [v(t)\cos(2\pi f_c t) + w(t)]\cos(2\pi f_c t)dt \\
&= A_c \int_{kT+\alpha T}^{(k+1)T+\alpha T} v(t)\cos^2(2\pi f_c t)dt + \int_{kT+\alpha T}^{(k+1)T+\alpha T} w(t)\cos(2\pi f_c t)dt \\
&= A_c \int_{kT+\alpha T}^{(k+1)T+\alpha T} v(t)\frac{1+\cos(4\pi f_c t)}{2}dt + \int_{kT+\alpha T}^{(k+1)T+\alpha T} w(t)\cos(2\pi f_c t)dt \\
&= \frac{A_c}{2}\int_{kT+\alpha T}^{(k+1)T+\alpha T} \sum_{n-\infty}^{\infty} I_n g(t-nT)dt + \frac{A_c}{2}\int_{kT+\alpha T}^{(k+1)T+\alpha T} \sum_{n-\infty}^{\infty} I_n g(t-nT)\cos(4\pi f_c t)dt \\
&\quad + \int_{kT+\alpha T}^{(k+1)T+\alpha T} w(t)\cos(2\pi f_c t)dt \\
&= \frac{A_c}{2}\sum_{n=-\infty}^{\infty} I_n \int_{kT+\alpha T}^{(k+1)T+\alpha T} g(t-nT)dt + \frac{A_c}{2}\sum_{n-\infty}^{\infty} I_n \int_{kT+\alpha T}^{(k+1)T+\alpha T} g(t-nT)\cos(4\pi f_c t)dt \\
&\quad + \int_{kT+\alpha T}^{(k+1)T+\alpha T} w(t)\cos(2\pi f_c t)dt \\
&= \frac{A_c}{2}[I_k(1-\alpha)T + I_{k+1}\alpha T] + \frac{A_c}{2} I_k \int_{(k+\alpha)T}^{(k+1)T} \cos(4\pi f_c t)dt + \frac{A_c}{2} I_{k+1}\int_{(k+1)T}^{(k+1+\alpha)T} \cos(4\pi f_c t)dt \\
&\quad + \int_{(k+\alpha)T}^{(k+1+\alpha)T} w(t)\cos(2\pi f_c t)dt \\
&= \frac{A_c}{2}\left[I_k(1-\alpha)T + I_{k+1}\alpha T\right] + \frac{A_c}{8\pi f_c}\left[-I_k \sin(4\pi f_c T(k+\alpha)) + I_{k+1}\sin(4\pi f_c T(k+1+\alpha)\right] \\
&\quad + \int_{(k+\alpha)T}^{(k+1+\alpha)T} w(t)\cos(2\pi f_c t)dt \\
&= \frac{A_c}{2}\left[I_k(1-\alpha)T + I_{k+1}\alpha T\right] + \frac{A_c}{8\pi f_c}\left[-I_k \sin(4\pi f_c T\alpha) + I_{k+1}\sin(4\pi f_c T\alpha)\right] \\
&\quad + \int_{(k+\alpha)T}^{(k+1+\alpha)T} w(t)\cos(2\pi f_c t)dt \\
&= \frac{A_c T}{2} I_k - \frac{A_c T}{2}\left(\alpha + \frac{\sin(4\pi f_c T\alpha)}{4\pi f_c T}\right)(I_k - I_{k+1}) + \int_{(k+\alpha)T}^{(k+1+\alpha)T} w(t)\cos(2\pi f_c t)dt
\end{aligned}
$$

(b)

$$
\begin{aligned}
y_T &= \frac{A_c T}{2} I_k - \frac{A_c T}{2}\left(\alpha + \frac{\sin(4\pi f_c T\alpha)}{4\pi f_c T}\right)(I_k - I_{k+1}) + \int_{(k+\alpha)T}^{(k+1+\alpha)T} w(t)\cos(2\pi f_c t)dt \\
&\approx \frac{A_c T}{2} I_k - \frac{A_c T}{2}\left(\alpha + \frac{4\pi f_c T\alpha}{4\pi f_c T}\right)(I_k - I_{k+1}) + \int_{(k+\alpha)T}^{(k+1+\alpha)T} w(t)\cos(2\pi f_c t)dt \\
&= \frac{A_c T}{2} I_k - A_c T\alpha(I_k - I_{k+1}) + \int_{(k+\alpha)T}^{(k+1+\alpha)T} w(t)\cos(2\pi f_c t)dt
\end{aligned}
$$

(c) $|4\pi f_c T\alpha| = |4\pi(100M)(1\mu s)\alpha| \le 0.1\pi$ implies $0 \le \alpha < 0.1/400 = 0.00025$.

2. Which of the below processes is wise-sense stationary? Which of the below processes is not wide-sense stationary but cyclostationary? (Note: If any of the below processes is not WSS but cyclostationary, please also determine its minimum period.)
   (a) (6%) $X_1(t) = \sin(2\pi f_c t + \Theta)$, where $\Theta$ is uniformly distributed over $[-p, p)$ for some $0 < p < \pi$.
   (b) (6%) $X_2(t) = A\cos(2\pi f_c t)$, where $A$ is uniformly distributed over $[-1,+1)$.

(c) (6%) $X_3(t) = U\cos(2\pi f_c t) + V\sin(2\pi f_c t)$, where $U$ and $V$ are independently distributed with zero-mean and unit variance.

Answers:

(a)

$$\mu_{X_1}(t) = E[X_1(t)] = \int_{-p}^{p} \sin(2\pi f_c t + \theta)\frac{1}{2p}d\theta$$

$$= \frac{1}{2p}[\cos(2\pi f_c t - p) - \cos(2\pi f_c t + p)] = \frac{1}{p}\sin(2\pi f_c t)\sin(p)$$

and

$$R_{X_1}(t,s) = E[X_1(t)X_1(s)] = \frac{1}{2p}\int_{-p}^{p}\sin(2\pi f_c t + \theta)\sin(2\pi f_c s + \theta)d\theta$$

$$= \frac{1}{4p}\int_{-p}^{p}[\cos(2\pi f_c(t-s)) - \cos(2\pi f_c(t+s) + 2\theta)]d\theta$$

$$= \frac{1}{2}\cos(2\pi f_c(t-s)) - \frac{1}{4p}\cos(2\pi f_c(t+s))\sin(2p)$$

Hence, $X_1(t)$ is not weakly stationary because the mean function is not a constant, i.e., the mean function is not independent of $t$. However, $X_1(t)$ is cyclostationary with period $1/f_c$ since $\mu_{X_1}(t+1/f_c) = \mu_{X_1}(t)$ and $R_{X_1}(t+1/f_c, s+1/f_c) = R_{X_1}(t,s)$.

(b)

$$\mu_{X_2}(t) = E[A\cos(2\pi f_c t)] = E[A]\cos(2\pi f_c t) = 0.$$

and

$$R_{X_2}(t,s) = E[X_2(t)X_2(s)] = \frac{1}{2}\int_{-1}^{1} a^2\cos(2\pi f_c t)\cos(2\pi f_c s)da$$

$$= \frac{1}{3}\cos(2\pi f_c t)\cos(2\pi f_c s) = \frac{1}{6}[\cos(2\pi f_c(t-s) + \cos(2\pi f_c(t+s))]$$

Again, $X_2(t)$ is not weakly stationary because the autocorrelation function depends on the time. However, $X_2(t)$ is cyclostationary with period $1/(2f_c)$ since $\mu_{X_2}(t+1/(2f_c)) = \mu_{X_2}(t) = 0$ and $R_{X_2}(t+1/(2f_c), s+1/(2f_c)) = R_{X_2}(t,s)$.

(c)

$$\mu_{X_3}(t) = E[U\cos(2\pi f_c t) + V\sin(2\pi f_c t)] = E[U]\cos(2\pi f_c t) + E[V]\sin(2\pi f_c t) = 0.$$

and

$$R_{X_3}(t,s) = E[(U\cos(2\pi f_c t) + V\sin(2\pi f_c t))(U\cos(2\pi f_c s) + V\sin(2\pi f_c s))]$$

$$= E[U^2]\cos(2\pi f_c t)\cos(2\pi f_c s) + E[V^2]\sin(2\pi f_c t)\sin(2\pi f_c s)$$

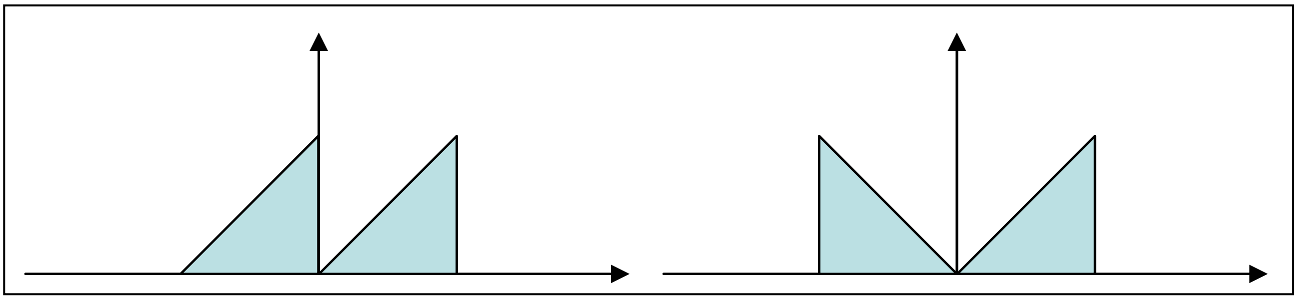
$$+ E[U]E[V]\cos(2\pi f_c t)\sin(2\pi f_c s) + E[V]E[U]\sin(2\pi f_c t)\cos(2\pi f_c s)$$

$$= \cos(2\pi f_c t)\cos(2\pi f_c s) + \sin(2\pi f_c t)\sin(2\pi f_c s)$$

$$= \cos(2\pi f_c(t-s))$$

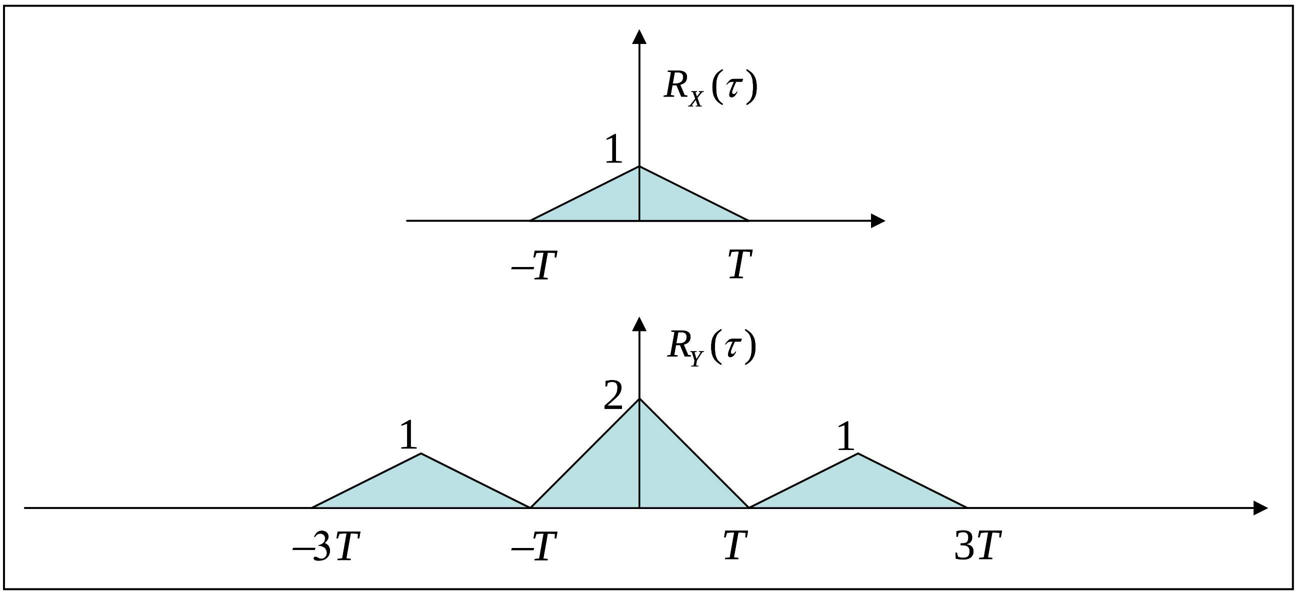
Hence, $X_3(t)$ is WSS (and hence, is cyclostationary with any positive period).

3. (a) (4%) Please explain why the functions shown below cannot be the legitimate autocorrelation functions of some WSS processes.



(b) (4%) Can the functions in (a) be the power density spectrums of some real WSS processes? If the answer is negative, please state the reason why the function cannot be the PSD. (Hint: Consider the function as the PSD of the output process due to a white WSS input through a real LTI filter.)

(c) (10%) Suppose that the autocorrelation functions of real-valued WSS processes $X(t)$ and $Y(t)$ are respectively of the forms shown below. Determine the relation between $X(t)$ and $Y(t)$. (Hint: $S_Y(f) = S_X(f)|H(f)|^2$ for some real $h(\tau)$.)




Answers:

(a) The first one is not symmetric, and the second one does not peak at zero.

(b) The spectrum of a real LTI filter should be symmetric. Hence, the first one **cannot** be a PSD of a real WSS process, however the second one **can** be the PSD of some WSS process.

(c)

$$R_Y(\tau) = R_X(\tau + 2T) + 2R_X(\tau) + R_X(\tau - 2T)$$

implies that

$$\begin{aligned} S_Y(f) &= S_X(f)e^{j4\pi fT} + 2S_X(f) + S_X(f)e^{-j4\pi fT} \\ &= S_X(f)[2 + 2\cos(4\pi fT)] \\ &= S_X(f)(2\cos(2\pi fT))^2 \\ &= S_X(f)|H(f)|^2 \end{aligned}$$

Hence, since that both $X(t)$ and $Y(t)$ are real implies that it suffices to have a real $h(t)$ (which in turn implies $H^*(-f) = H(f)$), we can assign $H(f) = 2\cos(2\pi fT) = e^{j2\pi fT} + e^{-j2\pi fT}$ and yield $h(\tau) = \delta(\tau + T) + \delta(\tau - T)$. Accordingly,

$$\begin{aligned} Y(t) &= X(t) * h(t) \\ &= \int_{-\infty}^{\infty} h(\tau) X(t-\tau) d\tau \\ &= \int_{-\infty}^{\infty} \left(\delta(\tau+T) + \delta(\tau-T)\right) X(t-\tau) d\tau \\ &= X(t+T) + X(t-T). \end{aligned}$$

4. (a) (6%) Prove that real cyclostationary input process induces real cyclostationary output process if the filter is real LTI.

(b) (6%) Should the minimum cyclostationary period of the output process be the same as the minimum cyclostationary period of the input process? Is it possible that a real cyclostationary but non-WSS input process induces a real WSS output process for some real LTI filter? Justify your answer by constructing an example.

Proof:

(a) Denote by $X(t)$ and $Y(t)$ the real input and real output processes, respectively. Let $h(\tau)$ be the real LTI filter. Then,

$$\mu_Y(t) = E\left[\int_{-\infty}^{\infty} h(\tau) X(t-\tau) d\tau\right] = \int_{-\infty}^{\infty} h(\tau) E[X(t-\tau)] d\tau = \int_{-\infty}^{\infty} h(\tau) \mu_X(t-\tau) d\tau$$

and

$$\begin{aligned} R_Y(t_1, t_2) &= E\left[\int_{-\infty}^{\infty} h(\tau_1) X(t_1-\tau_1) d\tau_1 \cdot \int_{-\infty}^{\infty} h(\tau_2) X(t_2-\tau_2) d\tau_2\right] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1) h(\tau_2) E[X(t_1-\tau_1) X(t_2-\tau_2)] d\tau_2 d\tau_1 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1) h(\tau_2) R_X(t_1-\tau_1, t_2-\tau_2) d\tau_2 d\tau_1 \end{aligned}$$

Given that the cyclostationary period of $X(t)$ is $T$, we have:

$$\mu_Y(t+T) = \int_{-\infty}^{\infty} h(\tau) \mu_X(t+T-\tau) d\tau = \int_{-\infty}^{\infty} h(\tau) \mu_X(t-\tau) d\tau = \mu_Y(t)$$

and

$$\begin{aligned} R_Y(t_1+T, t_2+T) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1) h(\tau_2) R_X(t_1+T-\tau_1, t_2+T-\tau_2) d\tau_2 d\tau_1 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1) h(\tau_2) R_X(t_1-\tau_1, t_2-\tau_2) d\tau_2 d\tau_1 \\ &= R_Y(t_1, t_2) \end{aligned}$$

Hence, $Y(t)$ is cyclostationary with minimum period at most $T$.

(b) It is not necessary that the minimum cyclostationary period of the output process is the same as the minimum cyclostationary period of the input process. In other words, the minimum cyclostationary period of the output process may be smaller than the minimum cyclostationary period of the input process. Take $h(\tau) = 1$ for $0 \le \tau < T$ as an example, where $T$ is the minimum cyclostationary period of the input process. Then,

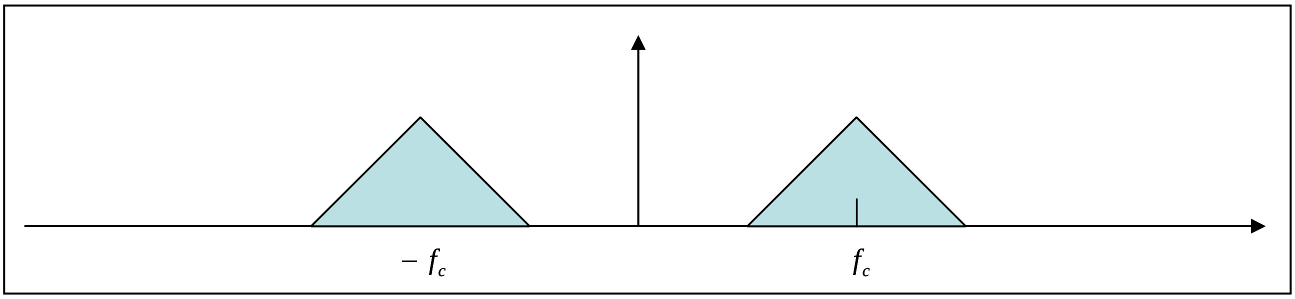
$$\mu_Y(t) = \int_{-\infty}^{\infty} h(\tau) \mu_X(t-\tau) d\tau = \int_0^T \mu_X(t-\tau) d\tau = \text{constant},$$

and

$$
\begin{aligned}
R_Y(t_1,t_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(t_1-\tau_1,t_2-\tau_2)d\tau_2 d\tau_1 \\
&= \int_0^T\int_0^T R_X(t_1-\tau_1,t_2-\tau_2)d\tau_2 d\tau_1 \\
&= \int_{t_1-T}^{t_1}\int_{t_2-T}^{t_2} R_X(s_1,s_2)ds_2 ds_1 \\
&= \int_0^T\int_0^T R_X(s_1,s_2)ds_2 ds_1 \\
&= \text{constant.}
\end{aligned}
$$

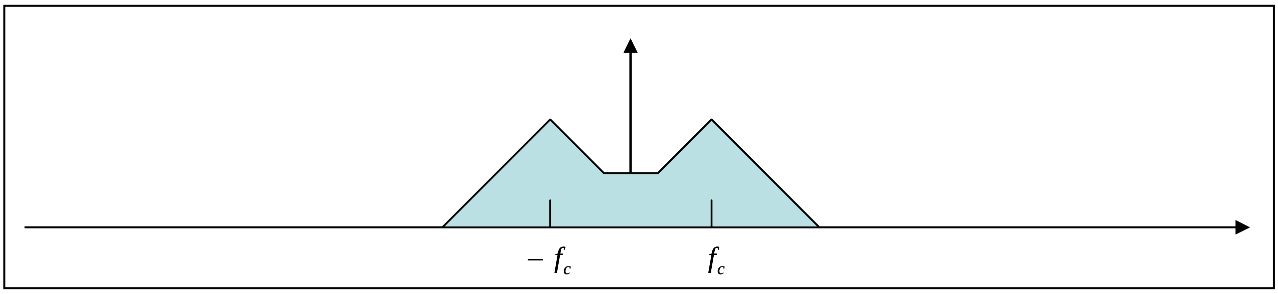
Hence, $Y(t)$ is a real WSS process (or cyclostationary whose period can be any positive number.)

5. (a) (4%) Write down the formula of Hilbert transform pair.
(b) (4%) Why Hilbert transform is also named 90 degree phase shifter?
(c) (6%) Suppose that the spectrum of function $g(t)$ is as follows.




Draw the spectrums of pre-envelope $g_+(t)$ and complex envelope $\tilde{g}(t)$ of $g(t)$. What is the mathematical relation between $\tilde{g}(t)$ and $g(t)$?

(d) (6%) Suppose that the spectrum of function $f(t)$ is as follows.

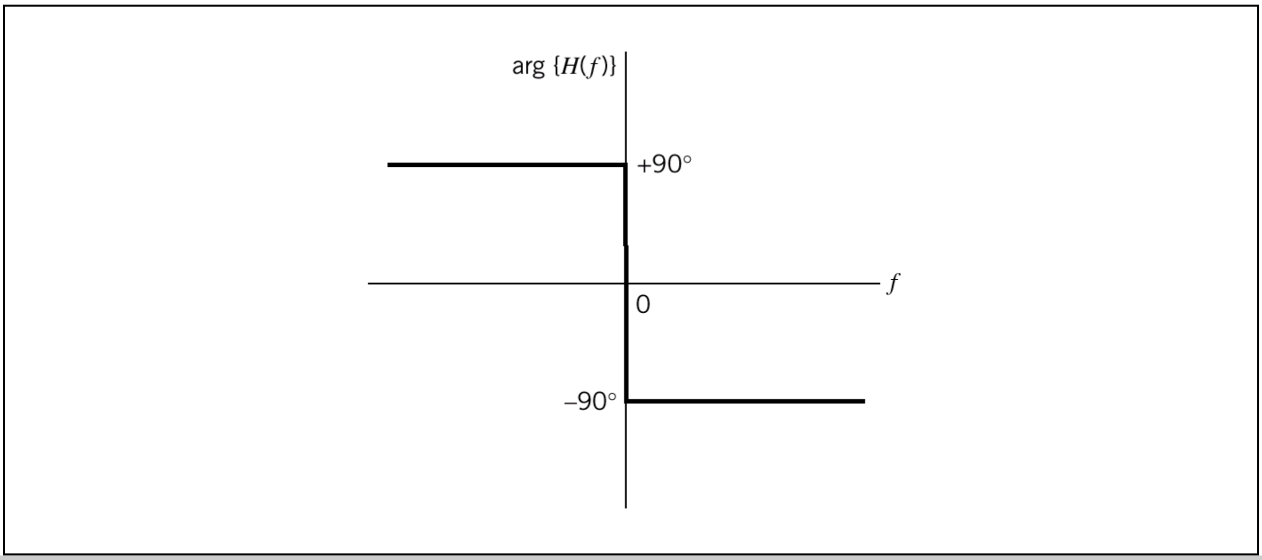



Draw the spectrums of pre-envelope $f_+(t)$ and complex envelope $\tilde{f}(t)$ of $f(t)$. What is the mathematical relation between $\tilde{f}(t)$ and $f(t)$?

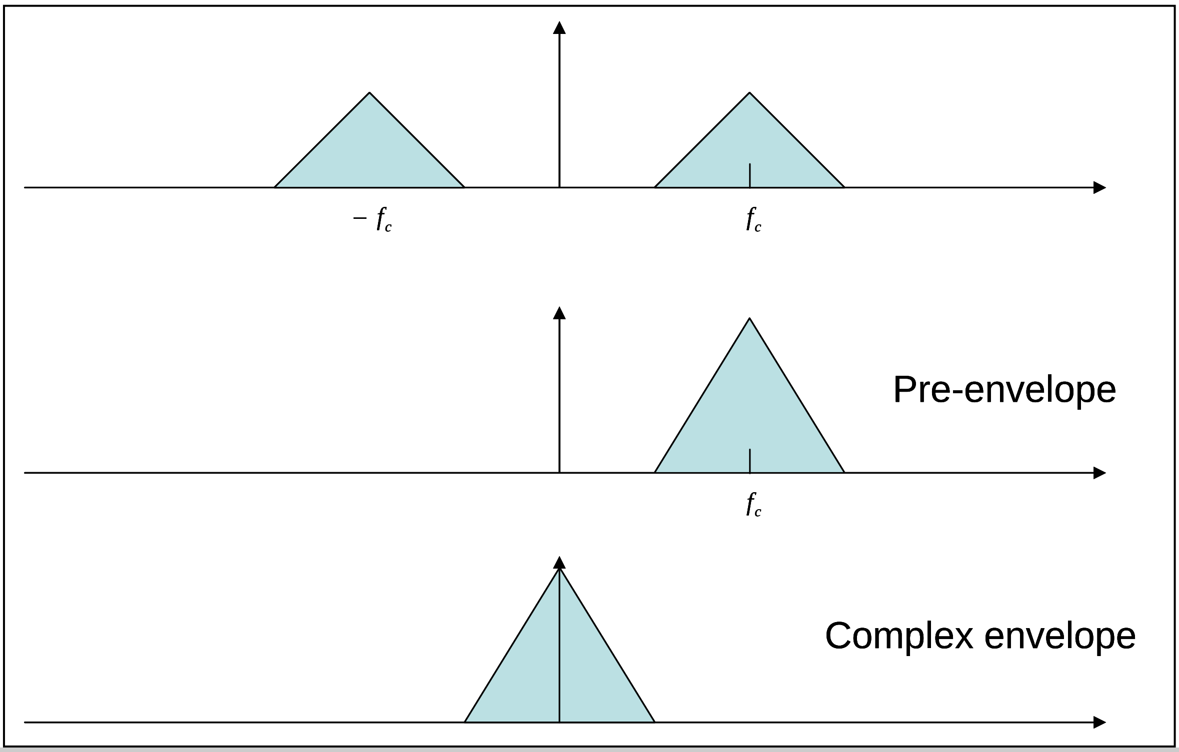
Answers:

(a) Hilbert Transform Pair $\begin{cases} \hat{g}(t) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty}\dfrac{g(\tau)}{t-\tau}d\tau \\ g(t) = -\dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty}\dfrac{\hat{g}(\tau)}{t-\tau}d\tau \end{cases}$

(b) It is simply because the phase of the transfer function of Hilbert transform is either 90 degree or -90 degree as shown below.

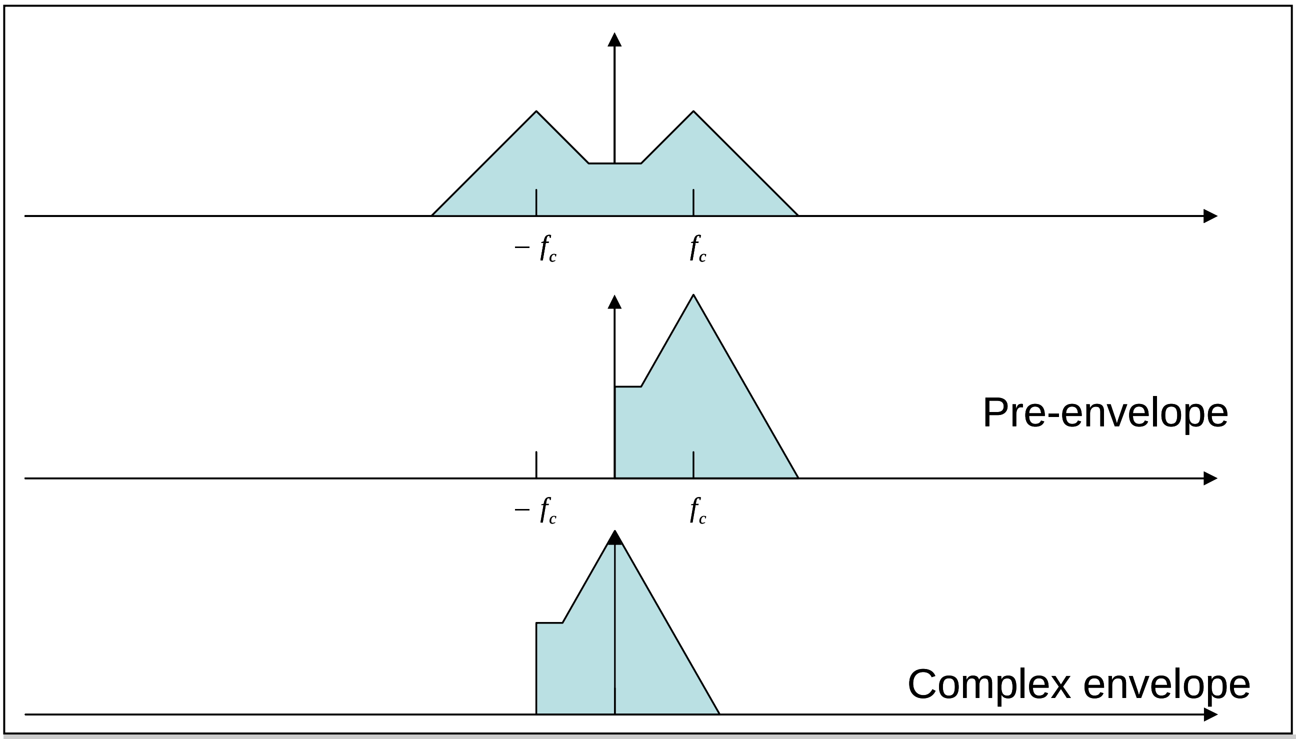



(c)




$$g(t) = \text{Re}\{\tilde{g}(t)e^{j2\pi f_c t}\}$$

(d)




$$f(t) = \mathrm{Re}\left\{\tilde{f}(t)e^{j2\pi f_c t}\right\}$$

6. (12%) Prove that $N(t) = N_I(t)\cos(2\pi f_c t) - N_Q(t)\sin(2\pi f_c t)$ is cyclostationary if $N_I(t)$ and $N_Q(t)$ are zero-mean and jointly cyclostationary with period $T$, and $f_c$ is a multiple of $1/T$. (Hint: Represent $R_N(t_1, t_2)$ in terms of $R_{N_I}(t_1, t_2)$, $R_{N_Q}(t_1, t_2)$, $R_{N_I, N_Q}(t_1, t_2)$ and $R_{N_Q, N_I}(t_1, t_2)$. Then, show that $R_N(t_1 + T, t_2 + T) = R(t_1, t_2)$.)

Proof:

$$
\begin{aligned}
& R_N(t_1, t_2) \\
&= E[N(t_1)N(t_2)] \\
&= E[(N_I(t_1)\cos(2\pi f_c t_1) - N_Q(t_1)\sin(2\pi f_c t_1))(N_I(t_2)\cos(2\pi f_c t_2) - N_Q(t)\sin(2\pi f_c t_2)] \\
&= R_{N_I}(t_1, t_2)\cos(2\pi f_c t_1)\cos(2\pi f_c t_2) - R_{N_I, N_Q}(t_1, t_2)\cos(2\pi f_c t_1)\sin(2\pi f_c t_2) \\
&\quad - R_{N_Q N_I}(t_1, t_2)\sin(2\pi f_c t_1)\cos(2\pi f_c t_2) + R_{N_Q}(t_1, t_2)\sin(2\pi f_c t_1)\sin(2\pi f_c t_2) \\
&= R_{N_I}(t_1, t_2)\frac{\cos(2\pi f_c(t_1 - t_2)) + \cos(2\pi f_c(t_1 + t_2))}{2} - R_{N_I, N_Q}(t_1, t_2)\frac{\sin(2\pi f_c(t_1 + t_2)) - \sin(2\pi f_c(t_1 - t_2))}{2} \\
&\quad - R_{N_Q N_I}(t_1, t_2)\frac{\sin(2\pi f_c(t_1 + t_2)) + \sin(2\pi f_c(t_1 - t_2))}{2} + R_{N_Q}(t_1, t_2)\frac{\cos(2\pi f_c(t_1 - t_2)) - \cos(2\pi f_c(t_1 + t_2))}{2} \\
&= \frac{1}{2}[R_{N_I}(t_1, t_2) + R_{N_Q}(t_1, t_2)]\cos(2\pi f_c(t_1 - t_2)) + \frac{1}{2}[R_{N_I, N_Q}(t_1, t_2) - R_{N_Q N_I}(t_1, t_2)]\sin(2\pi f_c(t_1 - t_2)) \\
&\quad + \frac{1}{2}[R_{N_I}(t_1, t_2) - R_{N_Q}(t_1, t_2)]\cos(2\pi f_c(t_1 + t_2)) - \frac{1}{2}[R_{N_I, N_Q}(t_1, t_2) + R_{N_Q N_I}(t_1, t_2)]\sin(2\pi f_c(t_1 + t_2))
\end{aligned}
$$

Then,

$$
\begin{aligned}
&R_N(t_1+T,t_2+T)\\
&=\frac{1}{2}[R_{N_I}(t_1+T,t_2+T)+R_{N_Q}(t_1+T,t_2+T)]\cos(2\pi f_c(t_1-t_2))\\
&+\frac{1}{2}[R_{N_I,N_Q}(t_1+T,t_2+T)-R_{N_Q N_I}(t_1+T,t_2+T)]\sin(2\pi f_c(t_1-t_2))\\
&+\frac{1}{2}[R_{N_I}(t_1+T,t_2+T)-R_{N_Q}(t_1+T,t_2+T)]\cos(2\pi f_c(t_1+t_2+2T))\\
&-\frac{1}{2}[R_{N_I,N_Q}(t_1+T,t_2+T)+R_{N_Q N_I}(t_1+T,t_2+T)]\sin(2\pi f_c(t_1+t_2+2T))\\
&=\frac{1}{2}[R_{N_I}(t_1,t_2)+R_{N_Q}(t_1,t_2)]\cos(2\pi f_c(t_1-t_2))\\
&+\frac{1}{2}[R_{N_I,N_Q}(t_1,t_2)-R_{N_Q N_I}(t_1,t_2)]\sin(2\pi f_c(t_1-t_2))\\
&+\frac{1}{2}[R_{N_I}(t_1,t_2)-R_{N_Q}(t_1,t_2)]\cos(2\pi f_c(t_1+t_2))\\
&-\frac{1}{2}[R_{N_I,N_Q}(t_1,t_2)+R_{N_Q N_I}(t_1,t_2)]\sin(2\pi f_c(t_1+t_2))\\
&=R_N(t_1,t_2)
\end{aligned},
$$

It can be easily shown that the mean function of $N(t)$ is zero, which is of course period! Hence, $N(t)$ is cyclostationary.