1.

The autocorrelation function of X(t) is

$$R_X(\tau) = E[X(t+\tau)X(t)]$$

$$= E[A^2\cos(2\pi f_c t + 2\pi f_c \tau + \Theta)\cos(2\pi f_c t + \Theta)]$$

$$= \frac{A^2}{2}E[\cos(4\pi f_c t + 2\pi f_c \tau + 2\Theta)] + \frac{A^2}{2}E[\cos(2\pi f_c \tau)]$$

$$= \frac{A^2}{2}\int_{-\pi}^{\pi}\frac{1}{2\pi}\cos(4\pi f_c t + 2\pi f_c \tau + 2\theta)\,d\theta + \frac{A^2}{2}\cos(2\pi f_c \tau)$$

The first term integrates to zero, and so we get

$$R_X(\tau) = \frac{A^2}{2}\cos(2\pi f_c \tau)$$

2.

(a) $R_{XY}(\tau) = E[X(t+\tau)\,Y(t)]$

Replacing τ with $-\tau$:

$$R_{XY}(-\tau) = E[X(t-\tau)\,Y(t)]$$

Next, replacing $t-\tau$ with t, we get

$$R_{XY}(-\tau) = E[Y(t+\tau)\,X(t)]$$

$$= R_{YX}(\tau)$$

(b) Form the non-negative quantity

$$E[\{X(t+\tau) \pm Y(t)\}^2] = E[X^2(t+\tau) \pm 2X(t+\tau)\,Y(t) + Y^2(t)]$$

$$= E[X^2(t+\tau)] \pm 2E[X(t+\tau)\,Y(t)] + E[Y^2(t)]$$

$$= R_X(0) \pm 2R_{XY}(\tau) + R_Y(0)$$

Hence,

$$R_X(0) \pm 2R_{XY}(\tau) + R_Y(0) \geq 0$$

or

$$|R_{XY}(\tau)| \leq \frac{1}{2}[R_X(0) + R_Y(0)]$$

3.

Strictly Stationary：

Mathematically, this can be formulated as that for any t_1, t_2, …, t_k and τ:

$$F_{X(t_1+\tau),X(t_2+\tau),\ldots,X(t_k+\tau)}(x_1,x_2,\ldots,x_k)$$
$$=F_{X(t_1),X(t_2),\ldots,X(t_k)}(x_1,x_2,\ldots,x_k)$$

Wide Sense Stationary：

Definition (**Wide-Sense Stationarity**) A random process $X(t)$ is WSS if

$$\begin{cases}\mu_X(t)=\text{constant};\\ C_X(t_1,t_2)=C_X(t_1-t_2)\end{cases} \quad \text{or} \begin{cases}\mu_X(t)=\text{constant};\\ R_X(t_1,t_2)=R_X(t_1-t_2).\end{cases}$$

4.

(a) The integrator output at time t is

$$Y(t)=\int_0^t X(\tau)\,d\tau$$

$$=A\int_0^t \cos(2\pi f_c\tau)\,d\tau$$

$$=\frac{A}{2\pi f_c}\sin(2\pi f_c t)$$

Therefore,

$$E[Y(t)]=\frac{\sin(2\pi f_c t)}{2\pi f_c}E[A]=0$$

$$\mathrm{Var}[Y(t)]=\frac{\sin^2(2\pi f_c t)}{(2\pi f_c)^2}\mathrm{Var}[A]$$

$$=\frac{\sin^2(2\pi f_c t)}{(2\pi f_c)^2}\sigma_A^2 \qquad (1)$$

From Eq. (1) we note that the variance of Y(t) depends on time t, and so Y(t) is nonstationary.

(b)

For a random process to be ergodic it has to be stationary. Since $Y(t)$ is nonstationary, it follows that it is not ergodic.