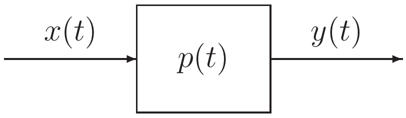
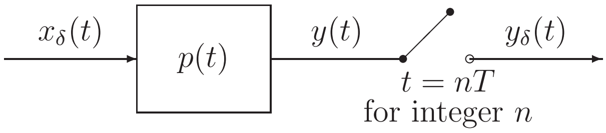
Sample Problems for the Tenth Quiz

1. (Alternative proof for Nyquist Criterion)

   (a) Find the LTI filter design with impulse response $p(t)$ that guarantees $y(t) = x(t)$.



   (b) Find the LTI filter design with impulse response $p(t)$ that guarantees $y_\delta(t) = x_\delta(t)$, where $x_\delta(t) = \sum_{k=-\infty}^{\infty} x(nT)\delta(t - nT)$.



Solution.

(a) Since $Y(f) = X(f)P(f)$, we must have $P(f) = 1$ in order to have $Y(f) = X(f)$.

(b) It is clear from each block of the system that

$$\begin{cases} Y(f) = X_\delta(f)P(f) \\ Y_\delta(f) = \frac{1}{T}\sum_{n=-\infty}^{\infty} Y\left(f - \frac{n}{T}\right) \end{cases}$$

Hence,

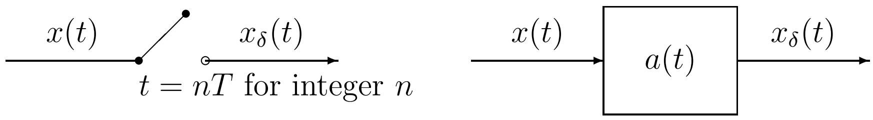
$$\begin{aligned} Y_\delta(f) &= \frac{1}{T}\sum_{n=-\infty}^{\infty} X_\delta\left(f - \frac{n}{T}\right) P\left(f - \frac{n}{T}\right) \\ &= \frac{1}{T}\sum_{n=-\infty}^{\infty} X_\delta(f)\, P\left(f - \frac{n}{T}\right) \\ &\quad \text{since } X_\delta(f) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(f - \frac{k}{T}\right) \text{ does not change by } \frac{n}{T} \text{ right shift.} \\ &= X_\delta(f)\left(\frac{1}{T}\sum_{n=-\infty}^{\infty} P\left(f - \frac{n}{T}\right)\right), \end{aligned}$$

which implies that we must have

$$\frac{1}{T}\sum_{n=-\infty}^{\infty} P\left(f-\frac{n}{T}\right)=1$$

in order to have $Y_\delta(f)=X_\delta(f)$.

2. Is a sampler linear? Can we find an equivalent linear time-invariant filter with impulse response $a(t)$ such that $x_\delta(t)=x(t)\star a(t)$?



Hint: $x_\delta(t)=\sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)$.

Solution. Given that $x_\delta(t)$ and $y_\delta(t)$ are sampler outputs respectively for inputs $x(t)$ and $y(t)$, the output due to input $x(t)+y(t)$ must be $x_\delta(t)+y_\delta(t)$; so, a sampler is a linear system.

Now suppose such $a(t)$ exists. Then, its Fourier transform $A(f)$ must satisfy

$$X(f)A(f)=X_\delta(f)=\frac{1}{T}\sum_{n=-\infty}^{\infty} X\left(f-\frac{n}{T}\right)$$

and $A(f)$ must be "nothing to do" with the input (i.e., its spectrum should not be "input-dependent"). However, for general $X(f)$, we have that

$$A(f)=\frac{1}{T}\sum_{n=-\infty}^{\infty}\frac{X\left(f-\frac{n}{T}\right)}{X(f)}=\frac{1}{T}+\frac{1}{T}\sum_{n=-\infty,n\neq 0}^{\infty}\frac{X\left(f-\frac{n}{T}\right)}{X(f)}$$

We can easily construct two examples of $X(f)$ such that the resulting $A(f)$ are different!

Note: So, a linear system may not be represented as a LTI filter since it may be input-dependent.

3. (a) Show that the time-averaged PSD of

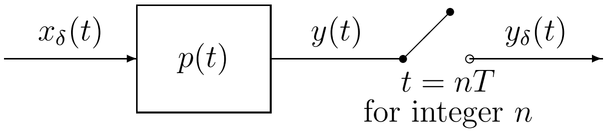
$$s(t)=\sum_{n=-\infty}^{\infty} a_n\, g(t-nT)$$

for WSS $\{a_n\}_{n=-\infty}^{\infty}$ is

$$\overline{\text{PSD}}_s(f) = |G(f)|^2 \left(\frac{1}{T} \sum_{n=-\infty}^{\infty} R_a(n) e^{-\imath 2\pi f n T} \right)$$

where $G(f)$ is the Fourier transform of $g(t)$ and $R_a(n) = E[a_{n+m} a_n^*]$ is the autocorrelation function of WSS $\{a_n\}_{n=-\infty}^{\infty}$.

(b) Use (a) to find the time-averaged PSD of $x_\delta(t)$ and $y(t)$ below, where $x_\delta(t) = \sum_{k=-\infty}^{\infty} a_k\, \delta(t - kT)$ and $\{a_n\}_{n=-\infty}^{\infty}$ WSS. Note that here $p(t)$ is a general LTI filter and may not satisfy the Nyquist Criterion (and is possibly a design of correlative level coding).



(c) Continue from (b). We note that $y_\delta(t) = \sum_{n=-\infty}^{\infty} c_n\, \delta(t - nT)$, where $c_n = y(nT)$. If $c_n = a_n + a_{n-1}$ as the duo-binary correlative level coding, is $\{c_n\}_{n=-\infty}^{\infty}$ also WSS? Represent the time-averaged PSD of $y_\delta(t)$ in terms of that of $x_\delta(t)$.

Solution.

(a) See Slides 3-64 $\sim$ 3-65.

(b)

$$\overline{\text{PSD}}_{x_\delta}(f) = |\Delta(f)|^2 \left(\frac{1}{T} \sum_{n=-\infty}^{\infty} R_a(n) e^{-\imath 2\pi f n T} \right) = \frac{1}{T} \sum_{n=-\infty}^{\infty} R_a(n) e^{-\imath 2\pi f n T}$$

$$\overline{\text{PSD}}_y(f) = |P(f)|^2 \overline{\text{PSD}}_{x_\delta}(f) = |P(f)|^2 \left(\frac{1}{T} \sum_{n=-\infty}^{\infty} R_a(n) e^{-\imath 2\pi f n T} \right)$$

where the time-averaged PSD of $y(t)$ can also be obtained directly from $y(t) = \sum_{n=-\infty}^{\infty} a_n\, p(t - nT)$ and the formula in (a).

(c) $\{c_n\}$ is the output due to WSS $\{a_n\}$ and a discrete LTI filter; so, it is surely WSS.

$$\begin{aligned} R_c(n) &= E[c_{m+n} c_m^*] \\ &= E[(a_{m+n} + a_{m+n-1})(a_m^* + a_{m-1}^*)] \\ &= 2R_a(n) + R_a(n+1) + R_a(n-1). \end{aligned}$$

$$
\begin{aligned}
\overline{\text{PSD}}_{y_\delta}(f) &= \frac{1}{T}\sum_{n=-\infty}^{\infty} R_c(n)e^{-\imath 2\pi f n T} \\
&= \frac{1}{T}\sum_{n=-\infty}^{\infty} (2R_a(n)+R_a(n+1)+R_a(n-1))e^{-\imath 2\pi f n T} \\
&= 2\overline{\text{PSD}}_{x_\delta}(f)+\overline{\text{PSD}}_{x_\delta}(f)e^{\imath 2\pi f t}+\overline{\text{PSD}}_{x_\delta}(f)e^{-\imath 2\pi f t} \\
&= \overline{\text{PSD}}_{x_\delta}(f)\left(2+e^{\imath 2\pi f t}+e^{-\imath 2\pi f t}\right) \\
&= \overline{\text{PSD}}_{x_\delta}(f)\left(4\cos^2(\pi f t)\right) \\
\Big(&= \overline{\text{PSD}}_{x_\delta}(f)\,|H_{\text{I}}(f)|^2\,\Big)
\end{aligned}
$$

4. From the slides, we know that the error propagation of correlative level coding can be eliminated by pre-coding. Two examples are given in our lectures, which are:

$$
\text{Duo-binary}\begin{cases} b_k = \tilde{b}_k \oplus \tilde{b}_{k-1} \\ a_k = 2\tilde{b}_k - 1 \\ c_k = a_k + a_{k-1} \end{cases}
$$

$\Rightarrow b_k$ can be fully determined by c_k.

$$
\text{Modified Duo-binary}\begin{cases} b_k = \tilde{b}_k \oplus \tilde{b}_{k-2} \\ a_k = 2\tilde{b}_k - 1 \\ c_k = a_k - a_{k-2} \end{cases}
$$

$\Rightarrow b_k$ can be fully determined by c_k.

Give one pre-coding scheme for

$$
\begin{cases} b_k = ?? \\ a_k = 2\tilde{b}_k - 1 \\ c_k = a_k + 2a_{k-1} + a_{k-2} \end{cases} \qquad \Rightarrow b_k \text{ can be fully determined by } c_k.
$$

Hint: $b_k = (\alpha\tilde{b}_k)\oplus(\beta\tilde{b}_{k-1})\oplus(\gamma\tilde{b}_{k-2})$ for some $(\alpha,\beta,\gamma)\in\{0,1\}^3$. There are eight possible candidates for precoding design.

Solution.

$\tilde{b}_{k-2}$	$\tilde{b}_{k-1}$	$\tilde{b}_k$	a_{k-2}	a_{k-1}	a_k	c_k	b_k
0	0	0	-1	-1	-1	-4	0
0	0	1	-1	-1	1	-2	α
0	1	0	-1	1	-1	0	β
0	1	1	-1	1	1	2	$\beta \oplus \alpha$
1	0	0	1	-1	-1	-2	γ
1	0	1	1	-1	1	0	$\gamma \oplus \alpha$
1	1	0	1	1	-1	2	$\gamma \oplus \beta$
1	1	1	1	1	1	4	$\gamma \oplus \beta \oplus \alpha$

Since $c_k = 0$ should be mapped to the same b_k, we have $\beta = \alpha \oplus \gamma$. By $c_k = -2$, we obtain $\alpha = \gamma$. By $c_k = 2$, we know $\gamma \oplus \beta = \beta \oplus \alpha$.

This results in $(\alpha, \beta, \gamma) = (0, 0, 0)$ or $(1, 0, 1)$, where $(0, 0, 0)$ obviously is not a legitimate choice.

c_k	b_k	$(\alpha, \beta, \gamma) = (1, 0, 1)$
-4	0	0
-2	α	1
0	β	0
2	$\alpha \oplus \beta$	1
-2	γ	1
0	$\gamma \oplus \alpha$	0
2	$\gamma \oplus \beta$	1
4	$\gamma \oplus \beta \oplus \alpha$	0

Accordingly, the precoding is $b_k = \tilde{b}_k \oplus \tilde{b}_{k-2}$ (and the decision rule is $b_k = 1$ if $1 \leq |c_k| < 3$, and $b_k = 0$, otherwise).