Sample Problems for the 8th Quiz

1. (a) Prove that the Fourier transform of the sampled signal of $g(t)$, i.e.,

$$g_\delta(t) \triangleq \sum_{n=-\infty}^{\infty} g(nT_s)\,\delta(t-nT)$$

is

$$G_\delta(f) = f_s \sum_{n=-\infty}^{\infty} G(f-mf_s),$$

where $f_s = 1/T_s$ is the sampling rate, and T_s is the sampling period.

(b) Prove that $g(t)$ can be recovered from its samples using interpolation technique below:

$$g(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\cdot 2UT_s\text{sinc}(2U(t-nT_s)) \tag{1}$$

if

$$\int_{-\infty}^{\infty} G(f)e^{\imath 2\pi f t}dt = \int_{-U}^{U} G(f)e^{\imath 2\pi f t}dt. \tag{2}$$

(c) Show that the interpolation in (1) can be implemented via linear filtering technique with input $g_\delta(t)$ and output $g(t)$.

(d) Now, generalize (1) to

$$s(t) = \sum_{n=-\infty}^{\infty} m(nT_s)\cdot h(t-nT_s),$$

by replacing $g(nT_s)$ with $m(nT_s)$, and also by letting $h(t) = 2UT_s$ sinc$(2Ut)$. A PAM signal $s(t)$ is formed. Do we need an equalizer to recover $m(t)$ if $U = W$, where W is the bandwidth of $m(t)$ and $f_s > 2W$?

Solution.

(a) See Slides 3-5 $\sim$ 3-6.

(b) See Slide 3-11.

Note that U is not necessarily to be equal to the bandwidth of $g(t)$! What we need is the condition of (2).

(c) Just show that (1) can be written as convolution operation. See Slide 3-13 $\sim$ 3-14.

(d) Since $H(f) = T_s \cdot \text{rect}(T_s f)$, which is a constant within the bandwidth of $m(t)$, no equalizer is necessary.

Note that in practice, $h(t) = 2WT_s\text{sinc}(2Wt)$ cannot be realized; hence, an equalizer is usually necessary except that $T \ll T_s$, in which case $H(f)$ is approximately flat within the bandwidth of $m(t)$.

2. Let $g(t)$ be a band-limited signal with bandwidth W. Give a sampling rate f_s that will cause *aliasing*.

Solution. See Slide 3-9.

3. Give an example of impulse response that is causal. Give an example of impulse response that is not causal.

Solution. See Slide 3-23 and create examples yourself.

4. Answer the following questions.

(a) Are PDM and PPM constant amplitude?

(b) What is the difference between midrise and midtread symmetric quantizations?

Solution.

(a) See Slide 3-29 for an answer.

(b) A symmetric quantizer with output as zero at a vicinity of input zero is of midtread type, while one which shows transition in level at input zero is a midriser.

5. (a) Prove that for a uniformly distributed zero-mean input M and a full-load quantizer $g(\cdot)$, the output SNR of the quantization system is equal to

$$10\log_{10}(3P) - 20\log_{10}(m_{\max}) + 20\log_{10}(L) \text{ dB}.$$

where L is the total number of representation levels for quantizer $g(\cdot)$, $P = E[M^2]$ is the power of the input and

$$m_{\max} = \sup\{m : \Pr[M > m] > 0\}$$

is the largest possible value of $|M|$.

(b) How many dB gains can be achieved if we double L?

Solution.

(a) See Slides 3-36 $\sim$ 3-38.

(b) $20\log_{10}(2) \approx 6$ dB.

6. Find the best representation levels v_1, v_2, v_3, v_4 for a given set of partition intervals $[m_1, m_2)$, $[m_2, m_3)$, $[m_3, m_4)$ and $[m_4, m_5)$ with $m_1 < m_2 < m_3 < m_4 < m_5$, if $d(m, v_i) = (m - v_i)^2$ and m is uniformly distributed over $[m_1, m_5)$. Answer the same question if $d(m, v_i) = |m - v_i|$.

Solution. From Slide 3-45,

$$v_{j,\text{optimal}} = E[M | m_j \le m < m_{j+1}], \qquad \text{if } d(m, v_i) = (m - v_i)^2.$$

Since m is still uniformly distributed given $m \in [m_j, m_{j+1}]$,

$$v_{j,\text{optimal}} = \frac{m_j + m_{j+1}}{2}.$$

When $d(m, v_i) = |m - v_i|$, the best representation levels should be given by

$$\begin{aligned}
(v_1^*, v_2^*, v_3^*, v_4^*) &= \arg \min_{v_1,v_2,v_3,v_4} \sum_{k=1}^{4} \int_{m_k}^{m_{k+1}} d(m, v_k) f_M(m) dm \\
&= \arg \min_{v_1,v_2,v_3,v_4} \sum_{k=1}^{4} \int_{m_k}^{m_{k+1}} |m - v_k| \cdot \frac{1}{(m_5 - m_1)} dm \\
&= \arg \min_{v_1,v_2,v_3,v_4} \sum_{k=1}^{4} \int_{m_k}^{m_{k+1}} |m - v_k| dm \\
&= \Bigg(\arg\min_{v_1} \int_{m_1}^{m_2} |m - v_1| dm, \arg\min_{v_2} \int_{m_2}^{m_3} |m - v_2| dm, \\
&\qquad \arg\min_{v_3} \int_{m_3}^{m_4} |m - v_3| dm, \arg\min_{v_4} \int_{m_4}^{m_5} |m - v_4| dm \Bigg)
\end{aligned}$$

Since

$$\int_{m_1}^{m_2} |m - v_1| dm = \begin{cases} \frac{(m_2 - m_1)(m_2 + m_1 - 2v_1)}{2}, & v_1 < m_1 \\ \frac{(v_1 - m_1)^2}{2} + \frac{(m_2 - v_1)^2}{2}, & m_1 \le v_1 < m_2 \\ \frac{(m_2 - m_1)(2v_1 - m_2 - m_1)}{2}, & v_1 \ge m_2 \end{cases},$$

we obtain still that

$$\arg\min_{v_1} \int_{m_1}^{m_2} |m - v_1| dm = \frac{m_1 + m_2}{2}.$$

Analogously,

$$\arg\min_{v_j} \int_{m_j}^{m_{j+1}} |m - v_j| dm = \frac{m_j + m_{j+1}}{2}.$$

7. Given the tables in Slides 3-52, 3-53, 3-57 and 3-58, find the A/μ-law quantizer output for a given input.

   Solution. Self-practice.