Chapter 1 Random Processes

“Probabilities” is considered an important background to communication study.
1.1 Mathematical Models

To model a system mathematically is the basis of its analysis, either analytically or empirically.

Two models are usually considered:

- **Deterministic model**
  - No uncertainty about its time-dependent behavior at any instance of time.

- **Random or stochastic model**
  - Uncertain about its time-dependent behavior at any instance of time,
  - but *certain* on the *statistical* behavior at any instance of time.
1.1 Examples of Stochastic Models

- Channel noise and interference
- Source of information, such as voice
A1.1: Relative Frequency

☐ How to determine the probability of “head appearance” for a coin?

☐ Answer: Relative frequency.

Specifically, by carrying out $n$ coin-tossing experiments, the relative frequency of head appearance is equal to $N_n(A)/n$, where $N_n(A)$ is the number of head appearance in these $n$ random experiments.
A1.1: Relative Frequency

- Is relative frequency close to the true probability (of head appearance)?
  - It could occur that **4-out-of-10** tossing results are “head” for a **fair** coin!

- Can one guarantee that the true “head appearance probability” remains **unchanged** (i.e., **time-invariant**) in each experiment (performed at different time instance)?
A1.1: Relative Frequency

- Similarly, the previous question can be extended to “In a communication system, can we estimate the noise by repetitive measurements at consecutive but different time instance?”

- Some assumptions on the statistical models are necessary!
A1.1: Axioms of Probability

Definition of a *Probability System* \((S, F, P)\) (also named *Probability Space*)

1. Sample space \(S\)
   - All possible outcomes (sample points) of the experiment

2. Event space \(F\)
   - Subset of sample space, which can be probabilistically measured.
   - \(\emptyset \in F\) and \(S \in F\)
   - \(A \in F\) implies \(A^c \in F\).
   - \(A \in F\) and \(B \in F\) imply \(A \cup B \in F\).

3. Probability measure \(P\)

\[ A \in F\] and \(B \in F\) imply \(A \cap B \in F\).
A1.1: Axioms of Probability

3. Probability measure $P$

- A probability measure satisfies:
  - $P(S) = 1$ and $P(\text{EmptySet}) = 0$
  - For any $A$ in $F$, $0 \leq P(A) \leq 1$.
  - For any two mutually exclusive events $A$ and $B$, $P(A \cup B) = P(A) + P(B)$

These Axioms coincide with the relative-frequency expectation.

1. $N_n(S) = n$ and $N_n(\text{emptySet}) = 0$.
2. $0 \leq \frac{N_n(A)}{n} \leq 1$.
3. $N_n(A \cup B) = N_n(A) + N_n(B)$
A1.1: Axioms of Probability

- Example. Rolling a dice.

- \( S = \{ \begin{array}{c}
\text{\ding{192}} \\
\text{\ding{193}} \\
\text{\ding{194}} \\
\text{\ding{195}} \\
\text{\ding{196}} \\
\text{\ding{197}} \\
\text{\ding{198}} \\
\text{\ding{199}} \\
\end{array} \} \)

- \( F = \{ \text{EmptySet}, \{\begin{array}{c}
\text{\ding{192}} \\
\text{\ding{193}} \\
\text{\ding{194}} \\
\end{array}\}, \{\begin{array}{c}
\text{\ding{193}} \\
\text{\ding{194}} \\
\text{\ding{195}} \\
\end{array}\}, S \} \)

- \( P \) satisfies
  - \( P(\text{EmptySet}) = 0 \)
  - \( P(\{\begin{array}{c}
\text{\ding{193}} \\
\text{\ding{194}} \\
\text{\ding{195}} \\
\end{array}\}) = 0.4 \)
  - \( P(\{\begin{array}{c}
\text{\ding{193}} \\
\text{\ding{194}} \\
\text{\ding{195}} \\
\text{\ding{196}} \\
\end{array}\}) = 0.6 \)
  - \( P(S) = 1. \)
A1.1: Properties from Axioms

All the properties are induced from Axioms

Example 1. \(P(A^c) = 1 - P(A)\).

*Proof.* Since \(A\) and \(A^c\) are mutually exclusive events, 
\[P(A) + P(A^c) = P(A \cup A^c) = P(S) = 1.\]

Example 2. \(P(A \cup B) = P(A) + P(B) - P(A \cap B)\).

*Proof.* 
\[
P(A \cup B) = P(A/B) + P(B/A) + P(A \cap B) = [P(A/B) + P(A \cap B)] + [P(B/A) + P(A \cap B)] - P(A \cap B) = P(A) + P(B) - P(A \cap B).
\]
A1.1: Conditional Probability

- Definition of conditional probability

$$P(B \mid A) \approx \frac{N_n(A \cap B)}{N_n(A)} = \frac{P(A \cap B)}{P(A)}$$

- Independence of events $P(B \mid A) = P(B)$
  - A knowledge of occurrence of event $A$ tells us no more about the probability of occurrence of event $B$ than we knew without this knowledge.
  - Hence, they are statistically independent.
A1.2: Random Variable

- A probability system \((S, F, P)\) can be “visualized” (or observed, or recorded) through a **real-valued** random variable \(X\).

  - After mapping the *sample point* in the sample space to a *real number*, the *cumulative distribution function* (cdf) can be defined as:

\[
F_X(x) = \Pr[X \leq x] = P\{s \in S : X(s) \leq x\}
\]
A1.2: Random Variable

Since the event \([X \leq x]\) has to be \textit{probabilistically measurable} for any real number \(x\), the event space \(F\) must consist of all the elements of the form \([X \leq x]\).

In previous example, the event space \(F\) must contain the intersections and unions of the following 6 sets.

\[
\begin{align*}
\{ & \} = \{ s \in S : X(s) \leq 1 \} \\
\{ & \ \} = \{ s \in S : X(s) \leq 2 \} \\
\{ & \ \} = \{ s \in S : X(s) \leq 3 \} \\
\{ & \ \} = \{ s \in S : X(s) \leq 4 \} \\
\{ & \ \} = \{ s \in S : X(s) \leq 5 \} \\
\{ & \ \} = \{ s \in S : X(s) \leq 6 \}
\end{align*}
\]

Otherwise, the cdf is not well-defined.
A1.2: Random Variable

It can be proved that we can construct a well-defined probability system \((S, F, P)\) for any random variable \(X\) and its cdf \(F_X\).

- So to define a real-valued random variable by its cdf is *good* enough from engineering standpoint.
- In other words, it is not necessary to mention the probability system \((S, F, P)\) before defining a random variable.
A1.2: Random Variable

- It can be proved that any function satisfying:
  1. $\lim_{x \to -\infty} G(x) = 0$ and $\lim_{x \to \infty} G(x) = 1$.
  2. Right-continuous.
  3. Non-decreasing.

is a legitimate cdf for some random variable.

- It suffices to check the above three properties for $F_X(x)$ to well-define a random variable.
A1.2: Random Variable

- A non-negative function \( f_X(x) \) satisfies
  \[
  \Pr(X \leq x) = \int_{-\infty}^{x} f_X(t) dt
  \]
  is called the \textit{probability density function} (pdf) of random variable \( X \).

- If the pdf of \( X \) exists, then
  \[
  f_X(x) = \frac{\partial F_X(x)}{\partial x}
  \]
A1.2: Random Variable

- It is not necessarily true that
  - If \[ f_X(x) = \frac{\partial F_X(x)}{\partial x}, \]
    then the pdf of \( X \) exists and equals \( f_X(x) \).
A1.2: Random Vector

We can extend a random variable to a (multi-dimensional) random vector.

For two random variables $X$ and $Y$, the joint cdf is defined as:

$$F_{X,Y}(x, y) = \Pr(X \leq x \text{ and } Y \leq y)$$

$$= P(\{s \in S : X(s) \leq x\} \cap \{s \in S : Y(s) \leq y\})$$

Again, the event $[X \leq x \text{ and } Y \leq y]$ must be \textit{probabilistically measurable} in some probability system $(S, F, P)$ for any real numbers $x$ and $y$. 
A1.2: Random Vector

As continuing from previous example, the event space $F$ must contain the intersections and unions of the following 4 sets.

$\{s \in S : X(s) \leq 1 \text{ and } Y(s) \leq 1\}$

$\{s \in S : X(s) \leq 2 \text{ and } Y(s) \leq 1\}$

$\{s \in S : X(s) \leq 1 \text{ and } Y(s) \leq 2\}$

$\{s \in S : X(s) \leq 2 \text{ and } Y(s) \leq 2\}$

$(X,Y): S \rightarrow \mathbb{R} \times \mathbb{R}$

\[\begin{align*}
& (1,1) \\
& (2,1) \\
& (1,1) \\
& (2,2) \\
& (1,2) \\
& (2,2)
\end{align*}\]
A1.2: Random Vector

- If its joint density \( f_{X,Y}(x,y) \) exists, then

\[
f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}
\]

- The conditional density function of \( Y \) given that \([X = x]\) is

\[
f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}
\]

provided that \( f_X(x) \neq 0 \).
1.2 Random Process

- Random process: An extension of *multi-dimensional* random vectors
  - Representation of two-dimensional random vector
    - \((X,Y) = (X(1), X(2)) = \{X(j), j \in I\}\), where the index set \(I\) equals \(\{1, 2\}\).
  - Representation of \(m\)-dimensional random vector
    - \(\{X(j), j \in I\}\), where the index set \(I\) equals \(\{1, 2, \ldots, m\}\).
1.2 Random Process

- How about \( \{X(t), \ t \in \mathbb{R}\} \)?
  - It is no longer a random vector since the index set is continuous!
  - This is a suitable model for, e.g., a noise because a noise often exists continuously in time.
  - Its cdf is well-defined through the mapping:

\[
X(t) : S \rightarrow \mathbb{R}
\]

for every \( t \in \mathbb{R} \).
Define, or view, a random process via its inherited probability system

Sample Space $S$

- $X(t,s_1)$
- $X(t,s_2)$
- $X(t,s_3)$
- $X(t,s_4)$
- $X(t,s_5)$
- $X(t,s_6)$

outcome corresponding to the first sample
1.2 Random Process

- $X(t, s_j)$ is called a *sample function* (or a *realization*) of the random process for *sample point* $s_j$.
- $X(t, s_j)$ is *deterministic*.
1.2 Random Process

Notably, with a probability system \((S, F, P)\) over which the random process is defined, *any finite-dimensional joint cdf* is well-defined.

For example,

\[
\Pr[X(t_1) \leq x_1 \text{ and } X(t_2) \leq x_2 \text{ and } X(t_3) \leq x_3]
\]

\[
= P\left(\{s \in S : X(t_1, s) \leq x_1\} \cap \{s \in S : X(t_2, s) \leq x_2\} \cap \{s \in S : X(t_3, s) \leq x_3\}\right)
\]
1.2 Random Process

- **Summary**
  - A random variable
    - maps \( s \) to a **real number**.
  - A random vector
    - maps \( s \) to a **multi-dimensional real vector**.
  - A random process
    - maps \( s \) to a **real-valued deterministic function**.

- \( s \) is where the probability is truly defined; yet the image of the mapping is what we can observe, manipulate and experiment over.
1.2 Random Process

- Question: Can we map $s$ to two or more real-valued deterministic functions?
- Answer: Yes, such as $(X(t), Y(t))$.

Then, we can discuss any finite-dimensional joint distribution of $X(t)$ and $Y(t)$, such as, the joint distribution of

$$(X(t_1), X(t_2), X(t_3), Y(t_1), Y(t_4))$$
1.3 (Strictly) Stationary

- For strict stationarity, the statistical property of a random process encountered in real world is often *independent* of the time at which the observation (or experiment) is initiated.

- Mathematically, this can be formulated as that for any $t_1, t_2, \ldots, t_k$ and $\tau$:

$$F_{X(t_1+\tau), X(t_2+\tau), \ldots, X(t_k+\tau)}(x_1, x_2, \ldots, x_k)$$

$$= F_{X(t_1), X(t_2), \ldots, X(t_k)}(x_1, x_2, \ldots, x_k)$$
1.3 (Strictly) Stationary

Example 1.1. Suppose that any finite-dimensional cdf of a random process $X(t)$ is defined. Find the probability of the joint event.

$$A = [a_1 < X(t_1) \leq b_1 \text{ and } a_2 < X(t_2) \leq b_2]$$

Answer:

$$P(A) = F_{X(t_1),X(t_2)}(b_1, b_2) - F_{X(t_1),X(t_2)}(b_1, a_2) - F_{X(t_1),X(t_2)}(a_1, b_2) + F_{X(t_1),X(t_2)}(a_1, a_2)$$
1.3 (Strictly) Stationary

possible sample functions
1.3 (Strictly) Stationary

Example 1.1. Further assume that $X(t)$ is strictly stationary.

Then, $P(A)$ is also equal to:

$$P(A) = F_{X(t_1+\tau),X(t_2+\tau)}(b_1,b_2) - F_{X(t_1+\tau),X(t_2+\tau)}(b_1,a_2) - F_{X(t_1+\tau),X(t_2+\tau)}(a_1,b_2) + F_{X(t_1+\tau),X(t_2+\tau)}(a_1,a_2)$$
1.3 (Strictly) Stationary

Why introducing “stationarity?”

- With stationarity, we can be certain that the observations made at different time instances have the same distributions!
- For example, $X(0), X(T), X(2T), X(3T), \ldots$.

Suppose that $\Pr[X(0) = 0] = \Pr[X(0)=1] = \frac{1}{2}$. Can we guarantee that the relative frequency (i.e., their average) of “1’s appearance” for experiments performed at several different times approach $\frac{1}{2}$ by stationarity? No, we need an additional assumption!
1.4 Mean

- The mean of a random process $X(t)$ at time $t$ is equal to:

$$\mu_X(t) = E[X(t)] = \int_{-\infty}^{\infty} x \cdot f_{X(t)}(x)dx$$

where $f_{X(t)}(\cdot)$ is the pdf of $X(t)$ at time $t$.

- If $X(t)$ is stationary, the mean $\mu_X(t)$ is independent of $t$, and is a constant for all $t$. 
1.4 Autocorrelation

- The autocorrelation function of a real random process $X(t)$ is given by:

\[ R_X(t_1, t_2) = E[X(t_1)X(t_2)] \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f_{X(t_1), X(t_2)}(x_1, x_2) dx_1 dx_2 \]

If $X(t)$ is complex, $R_X(t_1, t_2) = E[X(t_1)X^*(t_2)]$.

- If $X(t)$ is stationary, the autocorrelation function $R_X(t_1, t_2)$ is equal to $R_X(t_1 - t_2, 0)$. 
1.4 Autocorrelation

\[ R_X(t_1, t_2) = E[X(t_1)X(t_2)] \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f_{X(t_1), X(t_2)}(x_1, x_2) dx_1 dx_2 \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f_{X(t_1-t_2), X(0)}(x_1, x_2) dx_1 dx_2 \]

\[ = E[X(t_1 - t_2)X(0)] \]

\[ = R_X(t_1 - t_2, 0) \]

\[ = R_X(t_1 - t_2) \]

A short-hand for autocorrelation function of a stationary process
Chapter 1

1.4 Autocorrelation

- Conceptually, autocorrelation function = “power correlation” between two time instances $t_1$ and $t_2$. 
1.4 Autocovariance

“Variance” is the degree of variation to the standard value (i.e., mean).
Autocovariance function $C_X(t_1, t_2)$ is given by:

$$C_X(t_1, t_2) = E[(X(t_1) - \mu_X(t_1))(X(t_2) - \mu_X(t_2))]$$

$$= E[X(t_1)X(t_2) - X(t_1)\mu_X(t_2) - \mu_X(t_1)X(t_2) + \mu_X(t_1)\mu_X(t_2)]$$

$$= E[X(t_1)X(t_2)] - E[X(t_1)]\mu_X(t_2) - \mu_X(t_1)E[X(t_2)] + \mu_X(t_1)\mu_X(t_2)$$

$$= R_X(t_1, t_2) - \mu_X(t_1)\mu_X(t_2) - \mu_X(t_1)\mu_X(t_2) + \mu_X(t_1)\mu_X(t_2)$$

$$= R_X(t_1, t_2) - \mu_X(t_1)\mu_X(t_2)$$

If $X(t)$ is complex, $C_X(t_1, t_2) = R_X(t_1, t_2) - \mu_X(t_1)\mu^*_X(t_2)$. 

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1.4 Autocovariance

If $X(t)$ is stationary, the autocovariance function $C_X(t_1, t_2)$ becomes

$$C_X(t_1, t_2) = R_X(t_1, t_2) - \mu_X(t_1)\mu_X(t_2)$$
$$= R_X(t_1 - t_2, 0) - \mu_X^2$$
$$= C_X(t_1 - t_2, 0)$$
$$= C_X(t_1 - t_2)$$

If $X(t)$ complex, $C_X(t_1, t_2) = R_X(t_1 - t_2) - |\mu_X|^2 = C_X(t_1 - t_2)$. 
1.4 Wide-Sense Stationary (WSS)

Since in most cases of practical interest, only the first two moments (i.e., $\mu_X(t)$ and $C_X(t_1, t_2)$) are concerned, an alternative definition of stationarity is introduced.

Definition (Wide-Sense Stationarity) A random process $X(t)$ is WSS if

$$
\begin{align*}
\mu_X(t) &= \text{constant}; \\
C_X(t_1, t_2) &= C_X(t_1 - t_2)
\end{align*}
$$

or

$$
\begin{align*}
\mu_X(t) &= \text{constant}; \\
R_X(t_1, t_2) &= R_X(t_1 - t_2).
\end{align*}
$$
1.4 Wide-Sense Stationary (WSS)

- Alternative names for WSS include
  - weakly stationary
  - stationary in the weak sense
  - second-order stationary

- If the first two moments of a random process exist (i.e., are finite), then strictly stationary implies weakly stationary (but not vice versa).
1.4 Cyclostationarity

- Definition (Cyclostationarity) A random process $X(t)$ is cyclostationary if there exists a constant $T$ such that

$$
\begin{align*}
\mu_X(t + T) &= \mu_X(t); \\
C_X(t_1 + T, t_2 + T) &= C_X(t_1, t_2).
\end{align*}
$$
1.4 Properties of Autocorrelation Function for WSS Random Process

1. **Second Moment:** \( R_X(0) = E[X^2(t)] > 0 \).

2. **Symmetry:** \( R_X(\tau) = R_X(-\tau) \).

   - In concept, autocorrelation function = “power correlation” between two time instances \( t_1 \) and \( t_2 \).
   - For a WSS process, this “power correlation” only depends on time difference.

If \( X(t) \) complex, \( R_X(0) = E[|X(t)|^2] \) and \( R_X(\tau) = R_X^*(-\tau) \).
1.4 Properties of Autocorrelation Function for WSS Random Process

3. Peak at zero: $|R_X(\tau)| \leq R_X(0)$

*Proof:* \[0 \leq E[(X(t + \tau) \pm X(t))^2] = E[X^2(t + \tau)] + E[X^2(t)] \pm 2E[X(t + \tau)X(t)] = R_X(0) + R_X(0) \pm 2R_X(\tau) = 2R_X(0) \pm 2R_X(\tau)\]

Hence, $-R_X(0) \leq R_X(\tau) \leq R_X(0)$ with equality holding when

\[\Pr[X(t + \tau) = X(t)] = \Pr[X(\tau) = X(0)] = 1.\]

If $X(t)$ complex, $E[|X(t + \tau) \pm X(t)|^2] \geq 0$ implies $|\Re\{R_X(\tau)\}| \leq R_X(0)$. 

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1.4 Properties of Autocorrelation Function for WSS Random Process

- Operational meaning of autocorrelation function:
  - The “power” correlation of a random process at $\tau$ seconds apart.
  - The smaller $R_X(\tau)$ is, the less correlation between $X(t)$ and $X(t+\tau)$ is.
1.4 Properties of Autocorrelation Function for WSS Random Process

- If $R_X(\tau)$ decreases faster, the correlation decreases faster.
1.4 Decorrelation Time

Some researchers define the decorrelation time as:

$$\min \left\{ \tau \geq 0 : |R_X(\tau)| \leq \frac{1}{100} R_X(0) \right\}$$
Example 1.2 Signal with Random Phase

- Let $X(t) = A \cos(2\pi f_c t + \Theta)$, where $\Theta$ is uniformly distributed over $[-\pi, \pi)$.

- *Application*: A local carrier at the receiver side may have a random “phase difference” with respect to the phase of the carrier at the transmitter side.
Example 1.2 Signal with Random Phase

Channel Encoder

...0110

Modulator

...\,-m(t), m(t), m(t), -m(t)

\[ y(t) = A \cos(2\pi f_c t) \]

Local carrier \( \cos(2\pi f_c t + \Theta) \)

correlator

\[ s(t) = w(t) + x(t) \]

Carrier wave \( A_c \cos(2\pi f_c t) \)

\[ X(t) = A \cos(2\pi f_c t + \Theta) \]
Example 1.2 Signal with Random Phase

Then \[ \mu_X(t) = E[A \cos(2\pi f_c t + \Theta)] \]

\[ = \int_{-\pi}^{\pi} A \cos(2\pi f_c t + \Theta) \frac{1}{2\pi} d\theta \]

\[ = \frac{A}{2\pi} \int_{-\pi}^{\pi} \cos(\theta + 2\pi f_c t) d\theta \]

\[ = \frac{A}{2\pi} \left( \sin(\theta + 2\pi f_c t) \right)_{-\pi}^{\pi} \]

\[ = \frac{A}{2\pi} \left( \sin(\pi + 2\pi f_c t) - \sin(-\pi + 2\pi f_c t) \right) \]

\[ = 0. \]
Example 1.2 Signal with Random Phase

\[ R_X(t_1, t_2) = E[A \cos(2\pi f_c t_1 + \Theta) \cdot A \cos(2\pi f_c t_2 + \Theta)] \]

\[
= A^2 \int_{-\pi}^{\pi} \cos(\theta + 2\pi f_c t_1) \cos(\theta + 2\pi f_c t_2) \frac{1}{2\pi} d\theta 
\]

\[
= \frac{A^2}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2} \left( \cos[(\theta + 2\pi f_c t_1) + (\theta + 2\pi f_c t_2)] + \cos[(\theta + 2\pi f_c t_1) - (\theta + 2\pi f_c t_2)] \right) d\theta 
\]

\[
= \frac{A^2}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2} \left( \cos(2\theta + 2\pi f_c (t_1 + t_2)) + \cos(2\pi f_c (t_1 - t_2)) \right) d\theta 
\]

\[
= \frac{A^2}{2} \cos(2\pi f_c (t_1 - t_2)). \quad \text{Hence, } X(t) \text{ is WSS.}
\]
Example 1.2 Signal with Random Phase

\[ R_X(\tau) \]

\[ \frac{1}{f_c} \]
Example 1.3 Random Binary Wave

Let

$$X(t) = \sum_{n=-\infty}^{\infty} A \cdot I_n \cdot p(t - nT - t_d)$$

where ..., $I_{-2}$, $I_{-1}$, $I_0$, $I_1$, $I_2$, ... are independent, and each $I_j$ is either $-1$ or $+1$ with equal probability, and

$$p(t) = \begin{cases} 
1, & 0 \leq t < T \\
0, & \text{otherwise}
\end{cases}$$
Example 1.3 Random Binary Wave
Example 1.3 Random Binary Wave

Channel Encoder \( \ldots0110 \) \rightarrow \text{Modulator} \rightarrow \ldots, -m(t), m(t), m(t), -m(t)

\( m(t) = p(t) \)

\( T \)

\( w(t) = 0 \)

\( x(t) = A \)

correlator

\( \int_{-t_d}^{T-t_d} dt \)

No (or ignore) carrier wave

\( \int_0^T dt \)

\( X(t) = A \ p(t-t_d) \)
Example 1.3 Random Binary Wave

Then by assuming that \( t_d \) is uniformly distributed over \([0, T)\), we obtain:

\[
\mu_X(t) = E\left[ \sum_{n=-\infty}^{\infty} A \cdot I_n \cdot p(t - nT - t_d) \right] \\
= \sum_{n=-\infty}^{\infty} A \cdot E[I_n] \cdot E[p(t - nT - t_d)] \\
= \sum_{n=-\infty}^{\infty} A \cdot 0 \cdot E[p(t - nT - t_d)] \\
= 0
\]
Example 1.3 Random Binary Wave

A useful probabilistic rule: \( E[X] = E[E[X|Y]] \)

So we have:

\[
E[X(t_1)X(t_2)] = E\left[ E\left[ X(t_1)X(t_2) | t_d \right] \right]
\]

Note:

\[
\begin{align*}
E[X|Y] &= \int x f_{X|Y}(x|y) dx = g(y) \\
E[E[X|Y]] &= \int g(y) f_Y(y) dy
\end{align*}
\]
\[
E[X(t_1)X(t_2)|t_d]
\]
\[
= E\left[\left( \sum_{n=-\infty}^{\infty} A \cdot I_n \cdot p(t_1 - nT - t_d) \right) \left( \sum_{m=-\infty}^{\infty} A \cdot I_m \cdot p(t_2 - mT - t_d) \right) \right]_{t_d}
\]
\[
= A^2 \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} E[I_n I_m | t_d] E[p(t_1 - nT - t_d)p(t_2 - mT - t_d) | t_d]
\]
\[
= A^2 \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} E[I_n I_m] p(t_1 - nT - t_d)p(t_2 - mT - t_d)
\]
\[
= A^2 \sum_{n=-\infty}^{\infty} E[I_n^2] p(t_1 - nT - t_d)p(t_2 - nT - t_d)
\]
\[
= A^2 \sum_{n=-\infty}^{\infty} p(t_1 - nT - t_d)p(t_2 - nT - t_d)
\]

Since \( E[I_n I_m] = E[I_n]E[I_m] = 0 \) for \( n \neq m \).
Among $-\infty < n < \infty$, there is at most one $n$ that can make

$$p(t_1 - nT - t_d)p(t_2 - nT - t_d) = 1.$$ 

Without loss of generality, we let $t_1 = mT + \xi$ and $t_2 = t_1 - \tau$, where $m$ is an integer and $0 \leq \xi < T$. 

$$p(t_1 - nT - t_d) = p(t_2 - nT - t_d) = 1$$

$$\Leftrightarrow 0 \leq t_1 - nT - t_d < T \text{ and } 0 \leq t_2 - nT - t_d < T$$

$$\Leftrightarrow \frac{t_1 - t_d}{T} - 1 < n \leq \frac{t_1 - t_d}{T} \text{ and } \frac{t_2 - t_d}{T} - 1 < n \leq \frac{t_2 - t_d}{T}$$

$$\Leftrightarrow \left[ \frac{t_1 - t_d}{T} \right] = \left[ \frac{t_2 - t_d}{T} \right]$$

$$\Leftrightarrow \left[ \frac{mT + \xi - t_d}{T} \right] = \left[ \frac{mT + \xi - \tau - t_d}{T} \right]$$
\[ \Leftrightarrow \quad m + \left\lfloor \frac{\xi - t_d}{T} \right\rfloor = m + \left\lfloor \frac{\xi - \tau - t_d}{T} \right\rfloor \]

\[ \Leftrightarrow \quad \left\lfloor \frac{\xi - t_d}{T} \right\rfloor = \left\lfloor \frac{\xi - \tau - t_d}{T} \right\rfloor \]

\[ \Leftrightarrow \quad \left\lfloor \frac{\xi - t_d}{T} \right\rfloor = \left\lfloor \frac{\xi - \tau - t_d}{T} \right\rfloor = -1 \text{ or } 0 \]

(Note that \( \left\lfloor \frac{\xi - t_d}{T} \right\rfloor \) can only be either \(-1\) or \(0\) since \(0 \leq \xi, t_d < T\).)

\[ \Leftrightarrow \quad (\xi < t_d \leq \xi + T \text{ and } \xi - \tau < t_d \leq \xi - \tau + T) \]

or \((\xi - T < t_d \leq \xi \text{ and } \xi - \tau - T < t_d \leq \xi - \tau)\)

\[ \Leftrightarrow \quad (\xi < t_d \leq \xi - \tau + T) \quad \text{or} \quad (\xi - T < t_d \leq \xi - \tau) \quad \text{for} \quad 0 \leq \tau < T \]

\[ \Leftrightarrow \begin{cases} 
(\xi < t_d < T) \quad \text{or} \quad (0 \leq t_d \leq \xi - \tau), \quad 0 \leq \tau \leq \xi; \\
\xi < t_d \leq \xi - \tau + T, \quad \xi < \tau < T, 
\end{cases} \]

where in the last step, we use again \(0 \leq t_d < T\).
As a result,

\[
E[X(t_1)X(t_2)] = E\left[ E[X(t_1)X(t_2) | t_d] \right]
\]

\[
= \begin{cases} 
\int_0^T A^2 \frac{1}{T} dt_d + \int_0^{\xi-\tau} A^2 \frac{1}{T} dt_d, & 0 \leq \tau \leq \xi; \\
\int_{\xi-\tau+T}^{\xi} A^2 \frac{1}{T} dt_d, & \xi < \tau < T, \\
\int_{\xi}^{T} A^2 \frac{1}{T} dt_d, & \xi \leq \tau \leq T,
\end{cases}
\]

\[
= A^2 \left( \frac{T - \tau}{T} \right), & 0 \leq \tau < T
\]

\[
\begin{cases} 
A^2 \left( 1 - \frac{|	au|}{T} \right), & |\tau| < T; \\
0, & \text{otherwise}
\end{cases}
\]
Example 1.3 Random Binary Wave

\[ R_X(\tau) \]

\[ A^2 \]

\[ -T \quad T \]
1.4 Cross-Correlation

- The cross-correlation between two processes $X(t)$ and $Y(t)$ is:

$$R_{X,Y}(t,u) = E[X(t)Y(u)]$$

If $X(t)$ and $Y(t)$ complex, $R_{X,Y}(t,u) = E[X(t)Y^*(u)]$.

- Sometimes, their correlation matrix is given instead for convenience:

$$R_{X,Y}(t,u) = \begin{bmatrix} R_X(t,u) & R_{X,Y}(t,u) \\ R_{Y,X}(t,u) & R_Y(t,u) \end{bmatrix}$$
1.4 Cross-Correlation

If $X(t)$ and $Y(t)$ are jointly wide-sense stationary, then

$$R_{X,Y}(t,u) = R_{X,Y}(t-u)$$

$$= \begin{bmatrix} R_X(t-u) & R_{X,Y}(t-u) \\ R_{Y,X}(t-u) & R_Y(t-u) \end{bmatrix}$$
Consider a pair of quadrature decomposition of $X(t)$ as:

$$
\begin{align*}
X_I(t) &= X(t) \cos(2\pi f_c t + \Theta) \\
X_Q(t) &= X(t) \sin(2\pi f_c t + \Theta)
\end{align*}
$$

where $\Theta$ is independent of $X(t)$ and is uniformly distributed over $[0, 2\pi)$.

For example, $X(t) = A \cdot p(t - t_d)$. 

Example 1.4 Quadrature-Modulated Processes

\[
R_{X_I, X_Q}(t, u) = E[X_I(t)X_Q(u)]
\]

\[
= E[X(t)\cos(2\pi f_c t + \Theta) \cdot X(u)\sin(2\pi f_c u + \Theta)]
\]

\[
= E[X(t)X(u)]E[\sin(2\pi f_c u + \Theta)\cos(2\pi f_c t + \Theta)]
\]

\[
= R_X(t, u)E\left[\frac{\sin(2\pi f_c (t + u) + 2\Theta) + \sin(2\pi f_c (u - t))}{2}\right]_{=0}
\]

\[
= -\frac{1}{2} \sin(2\pi f_c (t - u))R_X(t, u)
\]
Example 1.4 Quadrature-Modulated Processes

\[ R_X(t, u) = A^2 \left(1 - \frac{|t-u|}{T}\right) 1\{|t-u| < T\} \]
Set \( A = T = 1 \) and \( f_c = 4 \).
Example 1.4 Quadrature-Modulated Processes

- Notably, if \( t = u \), i.e., synchronize in two quadrature components, then

\[
R_{X_i,X_q}(t,t) = 0
\]

which indicates that simultaneous observations of the quadrature-modulated processes are “orthogonal” to each other!

(See Slide 1-79 for a formal definition of orthogonality.)
1.5 Ergodic Process

☐ For a random-process-modeled noise (or random-process-modeled source) $X(t)$, how can we know its mean and variance?

- Answer: Relative frequency.
- Specifically, by measuring $X(t_1), X(t_2), \ldots, X(t_n)$, and calculating their average, it is generally expected that this time average will be close to its mean.

☐ Question is “Will this time average be close to its mean, if $X(t)$ is stationary?”

- For a stationary process, the mean function $\mu_X(t)$ is independent of time $t$. 
1.5 Ergodic Process

- The answer is negative!
- An additional *ergodicity* assumption is necessary for *time average* converging to the *ensemble average* $\mu_X$. 
1.5 Time Average versus Ensemble Average

Example.

- $X(t)$ is stationary.
- For any $t$, $X(t)$ is uniformly distributed over \{1, 2, 3, 4, 5, 6\}.
- Then ensemble average is equal to:

$$1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = 3.5$$
1.5 Time Average versus Ensemble Average

- We make a series of observations at time 0, $T$, $2T$, …, $10T$ to obtain 1, 2, 3, 4, 3, 2, 5, 6, 4, 1. (They are deterministic!)

- Then, the *time average* is equal to:

$$
\frac{1 + 2 + 3 + 4 + 3 + 2 + 5 + 6 + 4 + 1}{10} = 3.1
$$
1.5 Ergodicity

**Definition.** A stationary process $X(t)$ is *ergodic in the mean* if

1. $\Pr\left[ \lim_{T \to \infty} \mu_X(T) = \mu_X \right] = 1$, and
2. $\lim_{T \to \infty} \text{Var}[\mu_X(T)] = 0$

where

$$\mu_X(T) = \frac{1}{2T} \int_{-T}^{T} X(t) dt$$
1.5 Ergodicity

**Definition.** A stationary process $X(t)$ is *ergodic in the autocorrelation function* if

1. $\Pr \left[ \lim_{T \to \infty} R_X(\tau;T) = R_X(\tau) \right] = 1$, and
2. $\lim_{T \to \infty} \text{Var}[R_X(\tau;T)] = 0$

where

$$R_X(\tau;T) = \frac{1}{2T} \int_{-T}^{T} X(t+\tau)X(t)dt$$
1.5 Ergodic Process

- Experiments (or observations) on the same process can only be performed at different time.
- “Stationarity” only guarantees that the observations made at different time come from the same distribution.
A1.3 Statistical Average

- Alternative names of *ensemble average*
  - *Mean*
  - *Expected value or expectation value*
  - *Sample average* (Recall that sample space consists of all possible outcomes!)

- How about the sample average of a function $g(\cdot)$ of a random variable $X$?

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$
A1.3 Statistical Average

- The *nth moment* of random variable $X$
  \[ E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x) \, dx \]
  - The 2nd moment is also named *mean-square value*.

- The *nth central moment* of random variable $X$
  \[ E[(X - \mu_X)^n] = \int_{-\infty}^{\infty} (x - \mu_X)^n f_X(x) \, dx \]
  - The 2nd central moment is also named *variance*.
  - *Square root of the 2nd central moment* is also named *standard deviation*.
A1.3 Joint Moments

The joint moment of $X$ and $Y$ is given by:

$$E[X^i Y^k] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^i y^j f_{X,Y}(x, y) dx dy$$

- When $i = j = 1$, the joint moment is specifically named correlation.
- The correlation of centered random variables is specifically named covariance.

$$\operatorname{Cov}[X, Y] = E[(X - \mu_X)(Y - \mu_Y)] = E[XY] - \mu_X \mu_Y$$
A1.3 Joint Moments

- Two random variables, $X$ and $Y$, are uncorrelated if, and only if, $\text{Cov}[X, Y] = 0$.
- Two random variables, $X$ and $Y$, are orthogonal if, and only if, $E[XY] = 0$.
- The covariance, normalized by two standard deviations, is named correlation coefficient of $X$ and $Y$.

\[
\rho = \frac{\text{Cov}[X, Y]}{\sigma_X \sigma_Y}
\]
A1.3 Characteristic Function

- Characteristic function is indeed the inverse Fourier transform of the pdf.

\[
\phi_X(\nu) = E[\exp(j\nu X)] = \int_{-\infty}^{\infty} f_X(x) \exp(j\nu x) \, dx
\]

\[
f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_X(\nu) \exp(-j\nu x) \, d\nu
\]
1.6 Transmission of a Random Process Through a Stable Linear Time-Invariant Filter

- **Linear**
  - $Y(t)$ is a linear function of $X(t)$.
  - Specifically, $Y(t)$ is a weighted sum of $X(t)$.

- **Time-invariant**
  - The weights are time-independent.

- **Stable**
  - *Dirichlet’s condition* (defined later) and $\int_{-\infty}^{\infty} |h(\tau)|^2 d\tau < \infty$
  - “Stability” implies that the output is an energy function, which has finite power (second moment), if the input is an energy function.
Example of LTI Filter: Mobile Radio Channel

\[ Y(t) = \alpha_1 s(t - \tau_1) + \alpha_2 s(t - \tau_2) + \alpha_3 s(t - \tau_3) \]

where \( h(\tau_i) = \alpha_i \).
Example of LTI Filter: Mobile Radio Channel

\[ Y(t) = \int_{-\infty}^{\infty} h(\tau) X(t - \tau) d\tau \]
1.6 Transmission of a Random Process Through a Linear Time-Invariant Filter

- What are the mean and autocorrelation functions of the LTI filter output \( Y(t) \)?
  - Suppose \( X(t) \) is stationary and has finite mean.
  - Suppose \( \int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty \)
  - Then

\[
\mu_Y(t) = E[Y(t)] = E\left[ \int_{-\infty}^{\infty} h(\tau)X(t-\tau)d\tau \right] = \int_{-\infty}^{\infty} h(\tau)E[X(t-\tau)]d\tau = \mu_X \int_{-\infty}^{\infty} h(\tau)d\tau
\]
1.6 Zero-Frequency (DC) Response

The mean of the LTI filter output process is equal to the mean of the stationary filter input multiplied by the DC response of the system.

\[ \mu_Y = \mu_X \int_{-\infty}^{\infty} h(\tau) d\tau \]
1.6 Autocorrelation

If complex, \( R_Y(t, u) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1)h^*(\tau_2)R_X(t - \tau_1, u - \tau_2)d\tau_2 d\tau_1. \)

\[ R_Y(t, u) = E[Y(t)Y(u)] \]

\[ = E\left[ \int_{-\infty}^{\infty} h(\tau_1)X(t - \tau_1)d\tau_1 \cdot \int_{-\infty}^{\infty} h(\tau_2)X(u - \tau_2)d\tau_2 \right] \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)E[X(t - \tau_1)X(u - \tau_2)]d\tau_2 d\tau_1 \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(t - \tau_1, u - \tau_2)d\tau_2 d\tau_1 \]

\( \Rightarrow \) If \( X(t) \) WSS,

then \( R_Y(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(\tau - \tau_1 + \tau_2)d\tau_2 d\tau_1 \)
1.6 WSS Input Induces WSS Output

- From the above derivations, we conclude:
  - For a stable LTI filter, a WSS input induces a WSS output.
  - In general (not necessarily WSS),
    \[ \mu_Y(t) = \int_{-\infty}^{\infty} h(\tau)\mu_X(t - \tau)d\tau \]
    \[ R_Y(t, u) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1)h^*(\tau_2)R_X(t - \tau_1, u - \tau_2)d\tau_2d\tau_1 \]
  - As the above two quantities also relate in the “convolution” form, a spectrum analysis is perhaps better in characterizing their relationship.
A2.1 Fourier Analysis

- Fourier Transform Pair

Fourier Transform of $g(t)$:
$$G(f) = \int_{-\infty}^{\infty} g(t) \exp(-j2\pi ft) dt$$

Inverse Fourier Transform of $g(t)$:
$$g(t) = \int_{-\infty}^{\infty} G(f) \exp(j2\pi ft) df$$

- $G(f)$ is the frequency spectrum content of a signal $g(t)$.
  - $|G(f)|$ magnitude spectrum
  - arg\{G(f)\} phase spectrum
A2.1 Dirichlet’s Condition

- **Dirichlet’s condition**
  - In every finite interval, $g(t)$ has a finite number of local maxima and minima, and a finite number of discontinuity points.

- **Sufficient conditions for the existence of Fourier transform**
  - $g(t)$ satisfies Dirichlet’s condition
  - Absolute integrability: $\int_{-\infty}^{\infty} |g(t)| \, dt < \infty$
A2.1 Dirichlet’s Condition

“Existence” means that the Fourier transform pair is valid only for **continuity** points.

![Box]

\[ g(t) = \begin{cases} 
1, & -1 < t < 1; \\
0, & |t| \geq 1.
\end{cases} \quad \text{and} \quad \bar{g}(t) = \begin{cases} 
1, & -1 \leq t \leq 1; \\
0, & |t| > 1.
\end{cases} \]

has the **same** spectrum (Fourier transform) \( G(f) \).

However, the above two functions are not equal at \( t = 1 \) and \( t = -1 \) !
A2.1 Dirac Delta Function

- It is a function that exists only in principle.
- **Define** the Dirac delta function as a function \( \delta(t) \) satisfies:

\[
\delta(t) = \begin{cases} 
\infty, & t = 0; \\
0, & t \neq 0.
\end{cases}
\]

and

\[
\int_{-\infty}^{\infty} \delta(t)dt = 1.
\]

- \( \delta(t) \) can be thought of as a limit of a unit-area pulse function.

\[
\lim_{n \to \infty} s_n(t) = \delta(t), \quad \text{where} \quad s_n(t) = \begin{cases} 
n, & -\frac{1}{2n} < t < \frac{1}{2n}; \\
0, & \text{otherwise.}
\end{cases}
\]
A2.1 Properties of Dirac Delta Function

- Sifting property
  - If \( g(t) \) is continuous at \( t_0 \), then
    \[
    \int_{-\infty}^{\infty} g(t) \delta(t - t_0) \, dt = g(t_0)
    \]
    \[
    \left( \int_{-\infty}^{\infty} g(t) s_n(t - t_0) \, dt = \int_{t_0 - 1/(2n)}^{t_0 + 1/(2n)} g(t) \cdot n \cdot dt \to g(t_0) \right)
    \]
  - The sifting property is not necessarily true if \( g(t) \) is discontinuous at \( t_0 \).
A2.1 Properties of Dirac Delta Function

- Replication property
  
  For every continuous point of \( g(t) \),

  \[
  g(t) = \int_{-\infty}^{\infty} g(\tau) \delta(t - \tau) d\tau
  \]

- Constant spectrum

  \[
  \int_{-\infty}^{\infty} \delta(t) \exp(-j2\pi ft) dt = \int_{-\infty}^{\infty} \delta(t - 0) \exp(-j2\pi ft) dt = 1.
  \]
A2.1 Properties of Dirac Delta Function

- Scaling after integration

Although

\[ \delta(t) = 2 \cdot \delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases} \]

their integrations are different

\[ \int_{-\infty}^{\infty} \delta(t) dt = 1 \quad \text{while} \quad \int_{-\infty}^{\infty} 2\delta(t) dt = 2. \]

Hence, the “multiplicative constant” to the Dirac delta function is significant, and shall never be ignored!
A2.1 Properties of Dirac Delta Function

- More properties
  - Multiplication convention

\[
\delta(t - a)\delta(t - b) = \begin{cases} 
\delta(t - a), & \text{if } a = b \\
0, & \text{if } a \neq b
\end{cases}
\]

\[
\int_{-\infty}^{\infty} \delta(t - a)\delta(t - b)dt = \int_{-\infty}^{\infty} \delta(t - a)\delta(a - b)dt \\
= \delta(a - b)\int_{-\infty}^{\infty} \delta(t - a)dt \\
= \delta(a - b)
\]
A2.1 Fourier Series

- The Fourier transform of a periodic function does not exist!
  - E.g., for integer $k$,
    \[
    g(t) = \begin{cases} 
    1, & 2k \leq t < 2k + 1; \\
    0, & \text{otherwise}.
    \end{cases}
    \]
A2.1 Fourier Series

**Theorem:** If $g_T(t)$ is a bounded periodic function with period $T$ that satisfies *Dirichlet condition*, then

$$g_T(t) = \sum_{n=-\infty}^{\infty} c_n \exp\left( j \frac{2\pi n}{T} t \right)$$

at every *continuity* points of $g_T(t)$, where

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} g_T(t) \exp\left( - j \frac{2\pi n}{T} t \right) dt$$
A2.1 Relation between a Periodic Function and its Generating Function

Define the generating function of a periodic function $g_T(t)$ with period $T$ as:

$$g(t) = \begin{cases} 
g_T(t), & -T/2 \leq t < T/2; \\
0, & \text{otherwise}. 
\end{cases}$$

Then

$$g_T(t) = \sum_{m=-\infty}^{\infty} g(t - mT)$$
A2.1 Relation between a Periodic Function and its Generating Function

Let $G(f)$ be the spectrum of $g(t)$ (which is assumed to exist).

Then from Theorem in Slide 1-97,

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} g_T(t) \exp\left(-j \frac{2\pi n}{T} t\right) dt$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} \left( \sum_{m=\infty}^{\infty} g(t - mT) \right) \exp\left(-j \frac{2\pi n}{T} t\right) dt$$

$$= \frac{1}{T} \sum_{m=-\infty}^{\infty} \int_{-T/2}^{T/2} g(t - mT) \exp\left(-j \frac{2\pi n}{T} t\right) dt, \quad s = t - mT$$
(Continue from the previous slide.)

\[ c_n = \frac{1}{T} \sum_{m=-\infty}^{\infty} \int_{-T/2-mT}^{T/2-mT} g(s) \exp \left( -j \frac{2\pi n}{T} (s + mT) \right) ds \]

\[ = \frac{1}{T} \sum_{m=-\infty}^{\infty} \int_{-T/2-mT}^{T/2-mT} g(s) \exp \left( -j \frac{2\pi n}{T} s \right) ds \]

\[ = \frac{1}{T} \int_{-\infty}^{\infty} g(s) \exp \left( -j2\pi \frac{n}{T} s \right) ds \]

\[ = \frac{1}{T} G \left( \frac{n}{T} \right) \]
A2.1 Relation between a Periodic Function and its Generating Function

This concludes to the Poisson’s sum formula.

\[
g_T(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} G\left(\frac{n}{T}\right) \exp\left(j2\pi \frac{n}{T} t\right)
\]
A linear filter satisfies the principle of superposition, i.e.,

\[ x_1(t) + x_2(t) \rightarrow h(\tau) \rightarrow y_1(t) + y_2(t) \]
A2.1 Linearity and Convolution

- A linear time-invariant filter can be described by convolution integral $y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$.

- Example of a non-linear system

$$y(t) = x(t) - 0.1 \cdot x^3(t)$$
A2.1 Linearity and Convolution

Time-Convolution = Spectrum-Multiplication

\[ y(t) = \int_{-\infty}^{\infty} h(\tau) x(t - \tau) d\tau \]

and

\[ x(t) = \int_{-\infty}^{\infty} x(f) \exp(j2\pi ft) df \]

\[ h(\tau) = \int_{-\infty}^{\infty} H(f) \exp(j2\pi f \tau) df \]

\[ y(f) = \int_{-\infty}^{\infty} y(t) \exp(-j2\pi ft) dt \]

\[ = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} h(\tau) x(t - \tau) d\tau \right] \exp(-j2\pi ft) dt \]

\[ = \int_{-\infty}^{\infty} h(\tau) \left[ \int_{-\infty}^{\infty} x(t - \tau) \exp(-j2\pi ft) dt \right] d\tau \]
\[ y(f') = \int_{-\infty}^{\infty} h(\tau) \left[ \int_{-\infty}^{\infty} x(s) \exp(-j2\pi f (s + \tau)) dt \right] d\tau, \quad s = t - \tau \]

\[ = \int_{-\infty}^{\infty} h(\tau) \exp(-j2\pi f \tau) \left[ \int_{-\infty}^{\infty} x(s) \exp(-j2\pi f s) ds \right] d\tau \]

\[ = x(f') \int_{-\infty}^{\infty} h(\tau) \exp(-j2\pi f \tau) d\tau \]

\[ = x(f')H(f') \]
A2.1 Impulse Response of LTI Filter

- Impulse response = Filter response to Dirac delta function (application of replication property)

\[ y(t) = \int_{-\infty}^{\infty} h(\tau) x(t - \tau) d\tau = \int_{-\infty}^{\infty} h(\tau) \delta(t - \tau) d\tau = h(t). \]
A2.1 Frequency Response of LTI Filter

- Frequency response = Filter response to a complex exponential input of unit amplitude and frequency $f$

\[
y(t) = \int_{-\infty}^{\infty} h(\tau) x(t - \tau) d\tau
\]

\[
= \int_{-\infty}^{\infty} h(\tau) \exp(j2\pi f(t - \tau)) d\tau
\]

\[
= \exp(j2\pi ft) \int_{-\infty}^{\infty} h(\tau) \exp(-j2\pi ft) d\tau
\]

\[
= \exp(j2\pi ft) H(f)
\]
A2.1 Measures for Frequency Response

Expression 1

\[ H(f) = |H(f)| \cdot \exp[j\beta(f)], \quad \text{where} \quad \begin{cases} |H(f)| & \text{amplitude response} \\ \beta(f) & \text{phase response} \end{cases} \]

Expression 2

\[
\log H(f) = \log |H(f)| + j\beta(f) \\
= \alpha(f) + j\beta(f) \quad \text{where} \quad \begin{cases} \alpha(f) & \text{gain} \\ \beta(f) & \text{phase response} \end{cases}
\]

\[
\alpha(f) = \ln |H(f)| \quad \text{nepers} \\
= 20\log_{10} |H(f)| \text{dB}
\]
A2.1 Fourier Analysis

- Remember to self-study Tables A6.2 and A6.3.

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<th>TABLE A6.3 Fourier-transform pairs</th>
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<td><strong>Time Function</strong></td>
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<td>$rect\left(\frac{t}{T}\right)$</td>
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<td>$\text{sinc}(2Wt)$</td>
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<td>$\exp(-at) u(t)$, $a &gt; 0$</td>
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<tr>
<td>$\exp(-a</td>
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<td>$\Delta(t) := \begin{cases} 1 - \frac{</td>
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<td>$\delta(t)$</td>
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<tr>
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</tr>
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<td>$\sin(2\pi f_0 t)$</td>
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<td>$\text{sgn}(t)$</td>
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<td>$\frac{1}{j\pi t}$</td>
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<tr>
<td>$u(t)$</td>
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<tr>
<td>$\sum_{n=-\infty}^{\infty} \delta(t-nT_0)$</td>
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Notes: $u(t)$ = unit step function  
$\delta(t)$ = delta function or unit impulse  
$\text{rect}(t)$ = rectangular function of unit amplitude and unit duration centered on the origin  
$\text{sgn}(t)$ = signum function  
$\text{sinc}(t) = \text{sinc}$ function

Note: $e^{-at} u(t)$, $\text{Re}\{a\} > 0$  
$\frac{1}{a+j2\pi f}$
1.7 Power Spectral Density

**Deterministic** $x(t)$

$LTI$ $h(\tau)$

$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau$

**WSS $X(t)$**

$LTI$ $h(\tau)$

$Y(t) = \int_{-\infty}^{\infty} h(\tau)X(t - \tau)d\tau$

$\mu_Y(t) = \int_{-\infty}^{\infty} h(\tau)\mu_X(t - \tau)d\tau$

$R_Y(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1)h^*(\tau_2)R_X(\tau - \tau_1 + \tau_2)d\tau_2d\tau_1$
1.7 Power Spectral Density

How about the spectrum relation between filter input and filter output?

An apparent relation is:

\[ Y(f) = H(f)X(f) \]

\[ y(f) = H(f)x(f) \]
1.7 Power Spectral Density

- This is however not adequate for a random process.
  - For a random process, what concerns us is the relation between the input statistic and output statistic.
1.7 Power Spectral Density

How about the relation of the first two moments between filter input and output?

Spectrum relation of mean processes

\[ \mu_Y(t) = E[Y(t)] = E \left[ \int_{-\infty}^{\infty} h(\tau) X(t-\tau) d\tau \right] \]

\[ = \int_{-\infty}^{\infty} h(\tau) \mu_X(t-\tau) d\tau \]

\[ \Rightarrow \mu_Y(f) = \mu_X(f) H(f) \]
補充: Time-Average Autocorrelation Function

- For a **non-stationary** process, we can use the *time-average autocorrelation function* to define the *average power correlation* for a given time difference.

\[
\overline{R}_X(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} E[X(t+\tau)X(t)]dt
\]

- It is implicitly assumed that \(\overline{R}_X(\tau)\) is independent of the location of the integration window. Hence,

\[
\overline{R}_X(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T/2}^{3T/2} E[X(t+\tau)X(t)]dt
\]
補充: Time-Average Autocorrelation Function

- E.g., for a WSS process,

\[
R_X(\tau) = E[X(t + \tau)X(t)]
\]

- E.g., for a deterministic function,

\[
R_X(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} E[x(t + \tau)x(t)] dt
\]

\[
= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t + \tau)x(t) dt
\]
補充: Time-Average Autocorrelation Function

- E.g., for a cyclostationary process,

\[ \bar{R}_X(\tau) = \frac{1}{2T} \int_{-T}^{T} E[X(t + \tau)X(t)] dt, \]

where \( T \) is the cyclostationary period of \( X(t) \).
The *time-average power spectral density* is the Fourier transform of the *time-average autocorrelation function*.

\[
\overline{S}_X(f) = \int_{-\infty}^{\infty} \left( \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} E[X(t + \tau)X(t)] dt \right) e^{-j2\pi f \tau} d\tau \\
= \lim_{T \to \infty} \frac{1}{2T} E\left[\int_{-\infty}^{\infty} X(t + \tau)X_{2T}(t) dt \right] e^{-j2\pi f \tau} d\tau \\
= \lim_{T \to \infty} \frac{1}{2T} E[X(f)X_{2T}^*(f)] \text{ where } X_{2T}(t) = X(t) \cdot 1_{\{t | \leq T\}}.
\]
補充: Time-Average Autocorrelation Function

- For a WSS process, $\bar{S}_X(f) = S_X(f)$.
- For a deterministic process,

$$\bar{S}_X(f) = \lim_{T \to \infty} \frac{1}{2T} x(f)x^*_{2T}(f).$$
1.7 Power Spectral Density

Relation of time-average PSDs

\[ R_Y(t + \tau, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1) h^*(\tau_2) R_X(t + \tau - \tau_1, t - \tau_2) d\tau_2 d\tau_1 \]

\[ \bar{R}_Y(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1) h^*(\tau_2) R_X(t + \tau - \tau_1, t - \tau_2) d\tau_2 d\tau_1 dt \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1) h^*(\tau_2) \left( \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} R_X(t + \tau - \tau_1, t - \tau_2) dt \right) d\tau_2 d\tau_1 \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1) h^*(\tau_2) \bar{R}_X(\tau - \tau_1 + \tau_2) d\tau_2 d\tau_1 \]
\[ S_Y(f) = \int_{-\infty}^{\infty} R_Y(\tau) e^{-j2\pi f \tau} d\tau \]

\[ = \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1) h^*(\tau_2) R_X(\tau - \tau_1 + \tau_2) d\tau_2 d\tau_1 \right) e^{-j2\pi f \tau} d\tau \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1) h^*(\tau_2) \left( \int_{-\infty}^{\infty} R_X(\tau - \tau_1 + \tau_2) e^{-j2\pi f \tau} d\tau \right) d\tau_2 d\tau_1 \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1) h^*(\tau_2) \left( \int_{-\infty}^{\infty} R_X(u) e^{-j2\pi f(u+\tau_1-\tau_2)} du \right) d\tau_2 d\tau_1 \]

(We let \( u = \tau - \tau_1 + \tau_2 \).)

\[ = \left( \int_{-\infty}^{\infty} h(\tau_1) e^{-j2\pi f \tau_1} d\tau_1 \right) \left( \int_{-\infty}^{\infty} h^*(\tau_2) e^{j2\pi f \tau_2} d\tau_2 \right) \left( \int_{-\infty}^{\infty} R_X(u) e^{-j2\pi f u} du \right) \]

\[ = \left( \int_{-\infty}^{\infty} h(\tau_1) e^{-j2\pi f \tau_1} d\tau_1 \right) \left( \int_{-\infty}^{\infty} h(\tau_2) e^{-j2\pi f \tau_2} d\tau_2 \right)^* \left( \int_{-\infty}^{\infty} R_X(u) e^{-j2\pi f u} du \right) \]

\[ = H(f) H^*(f) S_X(f) \]

\[ = |H(f)|^2 \tilde{S}_X(f) \]
1.7 Power Spectral Density under WSS Input

For a WSS filter input,

\[ \mu_X(t) = \text{constant} = \mu_X \]
\[ \Rightarrow \mu_X(f) = \int_{-\infty}^{\infty} \mu_X \exp(-j2\pi ft) dt = \mu_X \delta(f) \]

\[ \bar{R}_X(\tau) = R_X(\tau) \]
\[ \Rightarrow S_Y(f) = \bar{S}_Y(f) = |H(f)|^2 \bar{S}_X(f) = |H(f)|^2 S_X(f) \]
1.7 Power Spectral Density under WSS Input

- **Observation**

\[
E[Y^2(t)] = R_Y(0) = \int_{-\infty}^{\infty} S_Y(f)df = \int_{-\infty}^{\infty} |H(f)|^2 S_X(f)df
\]

- \(E[Y^2(t)]\) is generally viewed as the *average power* of the WSS filter output process \(Y(t)\).

- This *average power* distributes over each spectrum frequency \(f\) through \(S_Y(f)\). (Hence, the *total average power* is equal to the integration of \(S_Y(f)\).)

- Thus, \(S_Y(f)\) is named the *power spectral density (PSD)* of \(Y(t)\).
1.7 Power Spectral Density under WSS Input

- The unit of $E[Y^2(t)]$ is, e.g., Watt.
- So the unit of $S_Y(f)$ is therefore Watt per Hz.
1.7 Operational Meaning of PSD

Example. Assume $h(\tau)$ is real, and $|H(f)|$ is given by:

![Diagram showing PSD with $|H(f)| = 1.0$ at $f = \pm f_c$.](image)
1.7 Operational Meaning of PSD

Then \( E[Y^2(t)] = R_Y(0) = \int_{-\infty}^{\infty} S_Y(f)df \)

\[
= \int_{-\infty}^{\infty} |H(f)|^2 S_X(f)df \\
= \int_{f_c-\Delta f/2}^{f_c+\Delta f/2} S_X(f)df + \int_{-f_c-\Delta f/2}^{-f_c+\Delta f/2} S_X(f)df \\
\approx \Delta f \cdot [S_X(f_c) + S_X(-f_c)]
\]

The filter passes only those frequency components of the input random process \( X(t) \), which lie inside a *narrow frequency band* of width \( \Delta f \) centered about the frequency \( f_c \) and \( -f_c \).

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1.7 Properties of PSD

Property 0. Wiener-Khintchine-Einstein relation

Relation between autocorrelation function and PSD of a WSS process $X(t)$

\[
S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) \exp(-j2\pi f \tau) d\tau
\]

\[
R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) \exp(j2\pi f \tau) df
\]
1.7 Properties of PSD

Property 1. Power density at zero frequency

\[ S_X(0) \text{ [Watt/Hz]} = S_X(0) \text{ [Watt-Second]} = \int_{-\infty}^{\infty} R_X(\tau) \text{ [Watt]} \, d\tau \text{ [Second]} \]

Property 2: Average power

\[ E[X^2(t)] \text{ [Watt]} = \int_{-\infty}^{\infty} S_X(f) \text{ [Watt/Hz]} \, df \text{ [Hz]} \]
1.7 Properties of PSD

**Property 3:** PSD is an even function, i.e.,

$$S_X(f) = S_X(-f).$$

**Proof.**

$$S_X(-f) = \int_{-\infty}^{\infty} R_X(\tau)e^{-j2\pi(-f)\tau} \, d\tau$$

$$= \int_{-\infty}^{\infty} R_X(-s)e^{-j2\pi fs} \, ds, \quad s = -\tau$$

$$= \int_{-\infty}^{\infty} R_X(s)e^{-j2\pi fs} \, ds, \quad R_X(-s) = R_X(s)$$

$$= S_X(f)$$
1.7 Properties of PSD

Property 4: PSD is real.

Proof.

\[ R_x(\tau) \text{ real } \Rightarrow S_x(-f) = S_x^*(f) \]

Then with Property 3,

\[ S_x(f) = S_x(-f) = S_x^*(f). \]

Thus, \( S_X(f) \) is real.
1.7 Properties of PSD

**Property 5**: Non-negativity for WSS processes

\[ S_X(f) \geq 0 \]

**Proof**: Pass \( X(t) \) through a filter with impulse response \( h(\tau) = \cos(2\pi f_c \tau) \).

Then \( H(f) = (1/2)[\delta(f - f_c) + \delta(f + f_c)] \).
1.7 Properties of PSD

As a result,

\[ E[Y^2(t)] = \int_{-\infty}^{\infty} S_Y(f) df \]

\[ = \int_{-\infty}^{\infty} |H(f)|^2 S_X(f) df \]

\[ = \frac{1}{4} \left( \int_{-\infty}^{\infty} \delta(f - f_c) S_X(f) df + \int_{-\infty}^{\infty} \delta(f + f_c) S_X(f) df \right) \]

\[ = \frac{1}{4} (S_X(f_c) + S_X(-f_c)) \]

\[ = \frac{1}{2} S_X(f_c), \text{ since } S_X(f) = S_X(-f) \text{ from Property 3} \]

This step requires that \( S_X(f) \) is continuous, which is true in general.
1.7 Properties of PSD

Therefore, by passing through a proper filter,

\[ S_X(f_c) = 2E[Y^2(t)] \geq 0 \]

for any \( f_c \).
Example 1.5 (Continue from Example 1.2) 

Signal with Random Phase

Let \( X(t) = A \cos(2\pi f_c t + \Theta) \), where \( \Theta \) is uniformly distributed over \([-\pi, \pi)\).

\[
S_X(f) = \int_{-\infty}^\infty \frac{A^2}{2} \cos(2\pi f_c \tau) e^{-j2\pi f \tau} d\tau
\]

\[
= \frac{A^2}{4} \int_{-\infty}^\infty \left[ e^{j2\pi f_c \tau} + e^{-j2\pi f_c \tau} \right] e^{-j2\pi f \tau} d\tau
\]

\[
= \frac{A^2}{4} \left[ \int_{-\infty}^\infty e^{-j2\pi (f + f_c) \tau} d\tau + \int_{-\infty}^\infty e^{-j2\pi (f - f_c) \tau} d\tau \right]
\]

\[
R_X(\tau) = \frac{A^2}{2} \cos(2\pi f_c \tau).
\]

\[
= \frac{A^2}{4} \left( \delta(f + f_c) + \delta(f - f_c) \right)
\]
Example 1.5 (Continue from Example 1.2) Signal with Random Phase

\[ \frac{A^2}{4} \delta(f + f_c) \]

\[ \frac{A^2}{4} \delta(f - f_c) \]
Example 1.6 (Continue from Example 1.3) Random Binary Wave

Let

$$X(t) = \sum_{n=-\infty}^{\infty} A \cdot I_n \cdot p(t - nT - t_d)$$

where \(..., I_{-2}, I_{-1}, I_0, I_1, I_2, \ldots\) are independent, and each \(I_j\) is either \(-1\) or \(+1\) with equal probability, and

$$p(t) = \begin{cases} 1, & 0 \leq t < T \\ 0, & \text{otherwise} \end{cases}$$
Example 1.6 (continue from Example 1.3) Random Binary Wave

\[ R_X(\tau) = \begin{cases} 
A^2 \left( 1 - \frac{|\tau|}{T} \right), & |\tau| < T \\
0, & \text{otherwise}
\end{cases} \]

\[ S_X(f) = \int_{-T}^{T} A^2 \left( 1 - \frac{|\tau|}{T} \right) e^{-j2\pi f \tau} \, d\tau \]

\[ = A^2 \left( 1 - \frac{|\tau|}{T} \right) \left( \frac{1}{-j2\pi f} e^{-j2\pi f \tau} \right) \bigg|_{-T}^{T} - \int_{-T}^{T} A^2 \left( -\frac{1}{T} \text{sgn}(\tau) \right) \left( \frac{1}{-j2\pi f} e^{-j2\pi f \tau} \right) \, d\tau \]

\[ = -\frac{A^2}{j2\pi f T} \int_{-T}^{T} \text{sgn}(\tau) e^{-j2\pi f \tau} \, d\tau \]
(Continue from the previous slide.)

\[ S_x(f) = -\frac{A^2}{j2\pi fT} \int_{-T}^{T} \text{sgn}(\tau)e^{-j2\pi f\tau} d\tau \]

\[ = -\frac{A^2}{(j2\pi fT)(-j2\pi f)} \left( \int_{0}^{T} (-j2\pi f)e^{-j2\pi f\tau} d\tau - \int_{-T}^{0} (-j2\pi f)e^{-j2\pi f\tau} d\tau \right) \]

\[ = -\frac{A^2}{4\pi^2 f^2 T} \left( \left( e^{-j2\pi fT} \right)_{0}^{T} - \left( e^{-j2\pi fT} \right)_{-T}^{0} \right) \]

\[ = -\frac{A^2}{4\pi^2 f^2 T} \left( e^{-j2\pi fT} - 1 - 1 + e^{j2\pi fT} \right) \]

\[ = \frac{A^2}{4\pi^2 f^2 T} \left( 2 - 2\cos(2\pi fT) \right) \]

\[ = \frac{A^2}{\pi^2 f^2 T} \sin^2(\pi fT) = A^2 T \text{sinc}^2(fT) \]
1.7 Energy Spectral Density

- Energy of a (deterministic) function $p(t)$ is given by $\int_{-\infty}^{\infty} |p(t)|^2 \, dt$.
- Recall that the average power of $p(t)$ is given by
  $$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |p(t)|^2 \, dt.$$  
- Observe that
  $$\int_{-\infty}^{\infty} |p(t)|^2 \, dt = \int_{-\infty}^{\infty} p(t)p^*(t) \, dt$$
  $$= \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} p(f)e^{j2\pi ft} \, df \right) \left( \int_{-\infty}^{\infty} p(f')e^{j2\pi f't} \, df' \right)^* \, dt$$
(Continue from the previous slide.)

\[
\int_{-\infty}^{\infty} |p(t)|^2 \, dt = \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} p(f) e^{j2\pi ft} \, df \right) \left( \int_{-\infty}^{\infty} p^*(f') e^{-j2\pi f't} \, df' \right) dt \\
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(f) p^*(f') \left( \int_{-\infty}^{\infty} e^{-j2\pi(f-f')t} \, dt \right) \, df \, df' \\
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(f) p^*(f') \delta(f' - f) \, df \, df' \\
= \int_{-\infty}^{\infty} p(f) p^*(f) \, df \\
= \int_{-\infty}^{\infty} |p(f)|^2 \, df
\]

For the same reason as PSD, $|p(f)|^2$ is named energy spectral density (ESD) of $p(t)$. 
Example

- The ESD of a rectangular pulse of amplitude $A$ and duration $T$ is given by

$$E_g(f) = \left| \int_0^T A e^{-j2\pi ft} \, dt \right|^2 = A^2 T^2 \text{sinc}^2(fT)$$
Example 1.7 Mixing of a Random Process with a Sinusoidal Process

Let \( Y(t) = X(t) \cos(2\pi f_c t + \Theta) \), where \( \Theta \) is uniformly distributed over \([−\pi, \pi)\), and \( X(t) \) is WSS and independent of \( \Theta \).

\[
R_Y(t, u) = E[X(t)X(u)\cos(2\pi f_c t + \Theta)\cos(2\pi f_c u + \Theta)]
\]
\[
= E[X(t)X(u)]E[\cos(2\pi f_c t + \Theta)\cos(2\pi f_c u + \Theta)]
\]
\[
= R_X(t-u) \cos(2\pi f_c (t-u)) \frac{1}{2}
\]

\( \Rightarrow S_Y(f) = \frac{1}{4} [S_X(f - f_c) + S_X(f + f_c)] \)
1.7 How to Measure PSD?

If $X(t)$ is not only (strictly) stationary but also ergodic, then any (deterministic) observation sample $x(t)$ in $[-T, T)$ satisfies:

$$
\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t) dt = E[X(t)] = \mu_X
$$

Sample average

Ensemble average

Similarly,

$$
\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t + \tau)x(t) dt = R_X(\tau)
$$
1.7 How to Measure PSD?

- Hence, we may use the time-limited Fourier transform of the time-averaged autocorrelation function:

\[
\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t + \tau)x(t)dt
\]

...to approximate the PSD.

...if \(x(t)\) complex...
(Notably, we only have the values of $x(t)$ for $t$ in $[-T, T]$.)

\[
S_X(f) = \int_{-T}^{T} \left[ \frac{1}{2T} \int_{-T}^{T} x(t + \tau)x^*(t)dt \right] \exp(-j2\pi f\tau)d\tau
\]

\[
= \frac{1}{2T} \int_{-T}^{T} x^*(t) \left( \int_{-T}^{T} x(t + \tau) \exp(-j2\pi f\tau)d\tau \right) dt
\]

\[
= \frac{1}{2T} \int_{-T}^{T} x^*(t) \left( \int_{-T+t}^{T+t} x(s) \exp(-j2\pi f(s - t))ds \right) dt, \quad s = t + \tau
\]

\[
= \frac{1}{2T} \int_{-T}^{T} x^*(t) \exp(j2\pi ft) \left( \int_{-T+t}^{T+t} x(s) \exp(-j2\pi fs)ds \right) dt
\]

\[
\approx \frac{1}{2T} \left( \int_{-T}^{T} x(t) \exp(-j2\pi ft)dt \right)^* \left( \int_{-T}^{T} x(s) \exp(-j2\pi fs)ds \right)
\]

\[
= \frac{1}{2T} |x_{2T}(f)|^2
\]
1.7 How to Measure PSD?

The estimate

\[ \frac{1}{2T} |x_{2T}(f)|^2 \]

is named the periodogram.

To summarize:

1. Observe \( x(t) \) for duration \([-T, T)\).
2. Calculate \( x_{2T}(f) = \int_{-T}^{T} x(t) \exp(-j2\pi ft)dt \).
3. Then \( S_X(f) \approx \frac{1}{2T} |x_{2T}(f)|^2 \).
1.7 Cross Spectral Density

Definition: For two (jointly WSS) random processes, \( X(t) \) and \( Y(t) \), their cross spectral densities are given by:

\[
\begin{align*}
S_{X,Y}(f) &= \int_{-\infty}^{\infty} R_{X,Y}(\tau) \exp(-j2\pi f \tau) d\tau \\
S_{Y,X}(f) &= \int_{-\infty}^{\infty} R_{Y,X}(\tau) \exp(-j2\pi f \tau) d\tau
\end{align*}
\]

where \( R_{X,Y}(t,u) = E[X(t)Y(u)] \) and \( R_{X,Y}(\tau) = R_{X,Y}(t-u) \),
and \( R_{Y,X}(t,u) = E[Y(t)X(u)] \) and \( R_{Y,X}(\tau) = R_{Y,X}(t-u) \)
(for \( \tau = t-u \)).
1.7 Cross Spectral Density

Property

\[ R_{X,Y}(\tau) = R_{Y,X}(-\tau) \Rightarrow S_{Y,X}(-f) = \int_{-\infty}^{\infty} R_{Y,X}(\tau) \exp(j2\pi f \tau) d\tau \]

\[ = \int_{-\infty}^{\infty} R_{Y,X}(-\tau) \exp(-j2\pi f \tau) d\tau \]

\[ = \int_{-\infty}^{\infty} R_{X,Y}(\tau) \exp(-j2\pi f \tau) d\tau \]

\[ = S_{X,Y}(f) \]
Example 1.8 PSD of Sum Process

Determine the PSD of sum process \( Z(t) = X(t) + Y(t) \) of two zero-mean WSS processes \( X(t) \) and \( Y(t) \).

Answer:

\[
R_Z(t,u) = E[Z(t)Z(u)] \\
= E[(X(t) + Y(t))(X(u) + Y(u))] \\
= E[X(t)X(u)] + E[X(t)Y(u)] + E[Y(t)X(u)] + E[Y(t)Y(u)] \\
= R_X(t,u) + R_{X,Y}(t,u) + R_{Y,X}(t,u) + R_Y(t,u).
\]
WSS implies that
\[ R_z(\tau) = R_x(\tau) + R_{x,y}(\tau) + R_{y,x}(\tau) + R_y(\tau). \]
Hence,
\[ S_z(f) = S_x(f) + S_{x,y}(f) + S_{y,x}(f) + S_y(f). \]
Q.E.D.

If \( X(t) \) and \( Y(t) \) are uncorrelated (and zero-mean), i.e., \( E[X(t + \tau)Y(t)] = E[X(t + \tau)]E[Y(t)] = 0 \),
\[ S_z(f) = S_x(f) + S_y(f). \]

The PSD of a sum process of zero-mean uncorrelated processes is equal to the sum of their individual PSDs.
Example 1.9

- Determine the CSD of output (complex) processes induced by separately passing jointly WSS (complex) inputs through a pair of stable (complex) LTI filters.
\[ R_{V,Z}(t, u) = E[V(t)Z^*(u)] \]
\[ = E \left[ \left( \int_{-\infty}^{\infty} h_1(\tau_1)X(t-\tau_1)d\tau_1 \right) \left( \int_{-\infty}^{\infty} h_2(\tau_2)Y(u-\tau_2)d\tau_2 \right)^\ast \right] \]
\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_1(\tau_1)h_2^*(\tau_2)E[X(t-\tau_1)Y^*(u-\tau_2)]d\tau_1d\tau_2 \]
\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_1(\tau_1)h_2^*(\tau_2)R_{X,Y}(t-u-\tau_1+\tau_2)d\tau_1d\tau_2 \]

\[ \Rightarrow R_{V,Z}(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_1(\tau_1)h_2^*(\tau_2)R_{X,Y}(\tau-\tau_1+\tau_2)d\tau_1d\tau_2 \]
\[ \Rightarrow S_{V,Z}(f) = H_1(f)H_2^*(f)S_{X,Y}(f) \]

Q.E.D.
1.8 Gaussian Process

**Definition.** A random variable is Gaussian distributed, if its pdf has the form

\[
f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_Y^2}} \exp\left[-\frac{(y - \mu_Y)^2}{2\sigma_Y^2}\right]
\]
1.8 Gaussian Process

Definition. An $n$-dimensional random vector is Gaussian distributed, if its pdf has the form

$$f_{\tilde{x}}(\tilde{x}) = \frac{1}{(2\pi |\Sigma|)^{n/2}} \exp\left(-\frac{1}{2} (\tilde{x} - \tilde{\mu})^T \Sigma^{-1} (\tilde{x} - \tilde{\mu})\right)$$

where $\tilde{\mu} = [E[X_1], E[X_2], \ldots, E[X_n]]^T$ is the mean vector, and

$$\Sigma = \begin{bmatrix}
\text{Cov}\{X_1, X_1\} & \text{Cov}\{X_1, X_2\} & \cdots \\
\text{Cov}\{X_2, X_1\} & \text{Cov}\{X_2, X_2\} & \cdots \\
\vdots & \vdots & \ddots \\
\end{bmatrix}_{n \times n}$$

is the covariance matrix.
1.8 Gaussian Process

For a Gaussian random vector, “uncorrelation” implies “independence.”

\[
\Sigma = \begin{bmatrix}
\text{Cov}\{X_1, X_1\} & 0 & \ldots \\
0 & \text{Cov}\{X_2, X_2\} & \ldots \\
\vdots & \vdots & \ddots
\end{bmatrix}_{n \times n} \Rightarrow f_{\tilde{x}}(\tilde{x}) = \prod_{i=1}^{n} f_{X_i}(x_i)
\]
1.8 Gaussian Process

**Definition.** A (real) random process $X(t)$ is said to be Gaussian distributed, if for every function $g(t)$, satisfying

$$\int_0^T \int_0^T g(t)g(u)R_X(t,u)\,dt\,du < \infty,$$

$Y = \int_0^T g(t)X(t)\,dt$ is a Gaussian random variable.

Notably, $E[Y^2] = \int_0^T \int_0^T g(t)g(u)R_X(t,u)\,dt\,du$. 
1.8 Central Limit Theorem

Theorem (Central Limit Theorem) For a sequence of independent and identically distributed (i.i.d.) random variables $X_1, X_2, X_3, \ldots$

$$\lim_{n \to \infty} \Pr \left[ \frac{(X_1 - \mu_X) + \cdots + (X_n - \mu_X)}{\sigma_X \sqrt{n}} \leq y \right] = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$$

where $\mu_X = E[X_j]$ and $\sigma_X^2 = E[X_j^2]$. 
1.8 Properties of Gaussian processes

Property 1. The output of a stable linear filter is a Gaussian process if the input is a Gaussian process. (This is self-justified by the definition of Gaussian processes.)

Property 2. A finite number of samples of a Gaussian process forms a multi-dimensional Gaussian vector. (No proof. Some books use this as the definition of Gaussian processes.)
1.8 Properties of Gaussian processes

Property 3. A WSS Gaussian process is also strictly stationary.

(An immediate consequence of Property 2.)
1.9 Noise

- Noise
  - An unwanted signal that will disturb the transmission, or processing, of signals in communication systems.

- Types
  - Shot noise
  - Thermal noise
  - … etc.
1.9 Shot Noise

A noise arises from the discrete nature of diodes and transistors.

E.g., a current pulse is generated every time an electron is emitted by the cathode.

Mathematical model

\[ X_{\text{Shot}}(t) = \sum_{k=-\infty}^{\infty} p(t - \tau_k) \]

where \( \{\tau_k\}_{k=-\infty}^{\infty} \) are the sequence of time a pulse is generated, and \( p(t) \) is a pulse shape with finite duration.
1.9 Shot Noise

- $X_{\text{Shot}}(t)$ is called the shot noise.

- A more useful model is to count the number of electrons emitted in the time interval $(0, t]$.

\[
N(t) = \max\{k : \tau_k \leq t\}
\]
1.9 Shot Noise

- $N(t)$ behaves like a Poisson Counting Process.

**Definition** (Poisson counting process) A Poisson counting process with parameter $\lambda$ is a process $\{N(t), t \geq 0\}$ with $N(0) = 0$ and stationary independent increments satisfying that for $0 < t_1 < t_2$, $N(t_2) - N(t_1)$ is Poisson distributed with mean $\lambda(t_2 - t_1)$. In other words,

$$\Pr\{N(t_2) - N(t_1) = k\} = \frac{[\lambda(t_2 - t_1)]^k}{k!} \exp[-\lambda(t_2 - t_1)]$$
1.9 Shot Noise

- A detailed statistical characterization of the shot-noise process $X(t)$ is in general hard.

- Some properties are quoted below.
  - $X_{Shot}(t)$ is strictly stationary.
  - Mean
    \[ \mu_{X_{Shot}} = \lambda \int_{-\infty}^{\infty} p(t)dt \]
    (See the next slide.)
  - Autocovariance function
    \[ C_{X_{Shot}}(\tau) = \lambda \int_{-\infty}^{\infty} p(t + \tau)p(t)dt \]
1.9 Shot Noise

For your reference

\[ X_{\text{Shot}}(t) = \frac{dN(t)}{dt} \ast p(t) \]
\[ \Rightarrow \mu X_{\text{Shot}}(t) = \int_{-\infty}^{\infty} p(\tau) \frac{d\mu_N(s)}{ds} \bigg|_{s=t-\tau} d\tau \]
\[ = \int_{-\infty}^{\infty} p(\tau) \frac{d(\lambda s)}{ds} \bigg|_{s=t-\tau} d\tau \]

\[ C_{X_{\text{Shot}}}(t + \tau, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(\tau_1)p^*(\tau_2) \frac{d^2(C_N(t_1, t_2))}{dt_1 dt_2} \bigg|_{t_1=t+\tau-t_1,t_2=t-t_2} d\tau_1 d\tau_2 \]
\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(\tau_1)p^*(\tau_2) \frac{d^2(\lambda \cdot \min\{t_1, t_2\})}{dt_1 dt_2} \bigg|_{t_1=t+\tau-t_1,t_2=t-t_2} d\tau_1 d\tau_2 \]
\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(\tau_1)p^*(\tau_2) \frac{d(\lambda \cdot u(t_1 - t_2))}{dt_1} \bigg|_{t_1=t+\tau-t_1,t_2=t-t_2} d\tau_1 d\tau_2 \]
\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(\tau_1)p^*(\tau_2) \lambda \cdot \delta(t_1 - t_2) \bigg|_{t_1=t+\tau-t_1,t_2=t-t_2} d\tau_1 d\tau_2 \]
\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(\tau_1)p^*(\tau_2) \lambda \cdot \delta(\tau - \tau_1 + \tau_2) d\tau_1 d\tau_2 \]
\[ = \lambda \int_{-\infty}^{\infty} p(\tau_2 + \tau)p^*(\tau_2) d\tau_2 \]

Note that \( u(t) \) is the unit step function.
1.9 Shot Noise

Example. \( p(t) \) is a rectangular pulse of amplitude \( A \) and duration \( T \).

\[
\mu_{X_{\text{Shot}}} = \lambda \int_{-\infty}^{\infty} p(t)dt = \lambda \int_{0}^{T} Adt = \lambda AT
\]

\[
C_{X_{\text{Shot}}} (\tau) = \begin{cases} 
\lambda A^2 (T - |\tau|), & \text{if } |\tau| < T \\
0, & \text{otherwise}
\end{cases}
\]
1.9 Thermal Noise

☐ A noise arises from the random motion of electrons in a conductor.

☐ Mathematical model

- Thermal noise voltage $V_{TN}$ that appears across the terminals of a resistor, measured in a bandwidth of $\Delta f$ Herz, is zero-mean Gaussian distributed with variance $E[V_{TN}^2] = 4kTR \cdot \Delta f$ [volts$^2$]

where $k = 1.38 \times 10^{-23}$ joules per degree Kelvin is the Boltzmann's constant, $R$ is the resistance in ohms, and $T$ is the absolute temperature in degrees Kelvin.
1.9 Thermal Noise

- Model of a noisy resistor

\[ E[V_{TN}^2] = E[I_{TN}^2]R^2 \]
1.9 White Noise

- A (often implicitly, zero-mean) noise is white if its PSD equals constant for all frequencies.
  - It is often defined as: \( S_W(f) = \frac{N_0}{2} \)

- Impracticability
  - The noise has infinite power

\[
E[W^2(t)] = \int_{-\infty}^{\infty} S_W(f) df = \int_{-\infty}^{\infty} \frac{N_0}{2} df = \infty.
\]
1.9 White Noise

☐ Another impracticability

- No matter how closely in time two samples are, they are uncorrelated!

☐ So impractical, why white noise is so popular in the analysis of communication system?

- There do exist noise sources that have a flat power spectral density over a range of frequencies that is much larger than the bandwidths of subsequent filters (or measurement devices).
1.9 White Noise

Some physically measurements have shown that the PSD of (a certain kind of) noise has the form

\[ S_w(f) = kTR \frac{2\alpha^2}{\alpha^2 + (2\pi f)^2} \]

where \( k \) is the Boltzmann’s constant, \( T \) is the absolute temperature, \( \alpha \) and \( R \) are the parameters of physical medium.

When \( f \ll \alpha \),

\[ S_w(f) = kTR \frac{2\alpha^2}{\alpha^2 + (2\pi f)^2} \approx 2kTR = \frac{N_0}{2} \]
Example 1.10 Ideal Low-Pass Filtered White Noise

After passing through a filter, the PSD of the zero-mean white noise becomes:

\[ S_{FW}(f) = |H(f)|^2 \]
\[ S_W(f) = \begin{cases} 
\frac{N_0}{2}, & |f| < B \\
0, & \text{otherwise} 
\end{cases} \]

\[ R_{FW}(\tau) = \int_{-B}^{B} \frac{N_0}{2} \exp(j2\pi f \tau)df = N_0 B \text{sinc}(2B \tau) \]

\[ \Rightarrow \tau = \pm k / (2B) \quad \text{for non-zero integer } k \text{ implies } R_{FW}(\tau) = 0, \text{ i.e., uncorrelated.} \]
Example 1.10 Ideal Low-Pass Filtered White Noise

So if we sample the noise at rate of $2B$ times per second, the resultant noise samples are uncorrelated!
Example 1.11

Channel Encoder

Modulator

\[ m(t), m(t), m(t), -m(t) \]

Carrier wave

\[ \cos(2\pi f_c t) \]

Local carrier

\[ \sqrt{2/T}\cos(2\pi f_c t) \]

Correlator

\[ \int_0^T x(t) \, dt \]

0110...

0

N

\[ w(t) \]

\[ s(t) \]
Example 1.11

In the previous figure, a scaling factor $\sqrt{2/T}$ is added to the local carrier to normalize the signal energy.

Signal Energy = $\int_0^T \left( \sqrt{\frac{2}{T}} \cos(2\pi f_c t) \right)^2 dt$

$= \int_0^T \frac{2}{T} \cos^2(2\pi f_c t) dt$

$= \int_0^T 1 - \cos(4\pi f_c t) \frac{dt}{T}$

$= 1.$
Example 1.11

Noise \( N = \int_{0}^{T} w(t) \sqrt{\frac{2}{T}} \cos(2\pi f_c t) dt \)

\[
\mu_N = E \left[ \int_{0}^{T} w(t) \sqrt{\frac{2}{T}} \cos(2\pi f_c t) dt \right] = \int_{0}^{T} E[w(t)] \sqrt{\frac{2}{T}} \cos(2\pi f_c t) dt = 0.
\]

\[
\sigma_N^2 = E \left[ \int_{0}^{T} w(t) \sqrt{\frac{2}{T}} \cos(2\pi f_c t) dt \cdot \int_{0}^{T} w(s) \sqrt{\frac{2}{T}} \cos(2\pi f_c s) ds \right]
\]

\[
= \frac{2}{T} \int_{0}^{T} \int_{0}^{T} E[w(t)w(s)] \cos(2\pi f_c t) \cos(2\pi f_c s) ds dt
\]
(Continue from the previous slide.)

\[
\sigma_N^2 = \frac{2}{T} \int_0^T \int_0^T \frac{N_0}{2} \delta(t - s) \cos(2\pi f_c t) \cos(2\pi f_c s) ds dt
\]

\[
= \frac{N_0}{T} \int_0^T \cos^2(2\pi f_c s) ds
\]

\[
= \frac{N_0}{2}.
\]

If \( w(t) \) is white Gaussian, then the pdf of \( N \) is uniquely determined by the first and second moments.
1.10 Narrowband Noise

- In general, the receiver of a communication system includes a *narrowband filter* whose bandwidth is just large enough to pass the modulated component of the received signal.
- The noise is therefore also filtered by this *narrowband filter*.
- So the noise’s PSD after being filtered may look like the figures in the next slide.
1.10 Narrowband Noise

The analysis on narrowband noise will be covered in subsequent sections.
A2.2 Bandwidth

- The bandwidth is the width of the frequency range outside which the power is essentially negligible.
  - E.g., the bandwidth of a (strictly) band-limited signal shown below is $B$. 

![Diagram showing a band-limited signal with bandwidth $B$]
A2.2 Null-to-Null Bandwidth

☑ Most signals of practical interest are not *strictly* band-limited.

- Therefore, there may *not* be a universally accepted definition of bandwidth for such signals.
- In such case, people may use *null-to-null bandwidth*.
  - The width of the main *spectral lobe* that lies inside the positive frequency region ($f > 0$).
A2.2 Null-to-Null Bandwidth

\[ X(t) = \sum_{n=-\infty}^{\infty} A \cdot I_n \cdot p(t - nT - t_d) \], where \( p(t) \) is a rectangular pulse of duration \( T \) and amplitude \( A \).

\[ S_N(f) \]

\[ \Rightarrow S_X(f) = A^2 T \text{sinc}^2(fT) \]

The null-to-null bandwidth is \( 1/T \).
A2.2 Null-to-Null Bandwidth

The *null-to-null bandwidth* in this case is $2B$. 

![Diagram showing null-to-null bandwidth](image)
A2.2 3-dB Bandwidth

A 3-dB bandwidth is the displacement between the two (positive) frequencies, at which the magnitude spectrum of the signal reaches its maximum value, and at which the magnitude spectrum of the signal drops to $\frac{1}{\sqrt{2}}$ of the peak spectrum value.

**Drawback:** A small 3-dB bandwidth does not necessarily indicate that most of the power will be confined within a small range. (E.g., the signal may have slowly decreasing tail.)
A2.2 3-dB Bandwidth

\[
\frac{S_X(f)}{S_X(0)} = \frac{A^2 T \text{sinc}^2(fT)}{A^2 T} = \frac{1}{\sqrt{T}}
\]

The 3-dB bandwidth is 0.3
A2.2 Root-Mean-Square Bandwidth

- rms bandwidth

\[ B_{\text{rms}} = \left( \frac{\int_{-\infty}^{\infty} f^2 S_X(f) df}{\int_{-\infty}^{\infty} S_X(f) df} \right)^{1/2} \]

- Disadvantage: Sometimes,

\[ \int_{-\infty}^{\infty} f^2 S_X(f) df = \infty \]

even if

\[ \int_{-\infty}^{\infty} S_X(f) df < \infty. \]
A2.2 Bandwidth of Deterministic Signals

The previous definitions can also be applied to Deterministic Signals, where PSD is replaced by ESD.

For example, a deterministic signal with spectrum \( G(f) \) has rms bandwidth:

\[
B_{rms} = \left( \frac{\int_{-\infty}^{\infty} f^2 |G(f)|^2 df}{\int_{-\infty}^{\infty} |G(f)|^2 df} \right)^{1/2}
\]
A2.2 Noise Equivalent Bandwidth

- An important consideration in communication system is the *noise power* at a linear filter output due to a white noise input.
  - We can characterize the *noise-resistant ability* of this filter by its *noise equivalent bandwidth*.
  - Noise equivalent bandwidth = The bandwidth of an *ideal low-pass filter* through which the same output filter noise power is resulted.
A2.2 Noise Equivalent Bandwidth

\[ B_{NE} = \frac{\int_{-\infty}^{\infty} |H(f)|^2 \, df}{2 |H(0)|^2} \]
A2.2 Noise Equivalent Bandwidth

- Output noise power for a general linear filter

\[ \int_{-\infty}^{\infty} S_W(f) |H(f)|^2 \, df = \frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 \, df \]

- Output noise power for an ideal low-pass filter of bandwidth \( B \) and the same amplitude as the general linear filter at \( f = 0 \).

\[ \int_{-\infty}^{\infty} S_W(f) |H(f)|^2 \, df = \frac{N_0}{2} \int_{-B}^{B} |H(0)|^2 \, df = BN_0 |H(0)|^2 \]

\[ B_{NE} = \frac{\int_{-\infty}^{\infty} |H(f)|^2 \, df}{2 |H(0)|^2} \]
A2.2 Time-Bandwidth Product

- Time-Scaling Property of Fourier Transform

  - Reducing the time-scale by a factor of \( a \) extends the bandwidth by a factor of \( a \).

  \[
  g(t) \rightarrow G(f) \Leftrightarrow g(at) \rightarrow \frac{1}{|a|}G\left(\frac{f}{a}\right)
  \]

  - This hints that the product of time- and frequency-parameters should remain constant, which is named the time-bandwidth product or bandwidth-duration product.
Since there are various definitions of time-parameter (e.g., duration of a signal) and frequency-parameter (e.g., bandwidth), the time-bandwidth product constant may change for different definitions.

E.g., *rms duration* and *rms bandwidth* of a pulse $g(t)$

$$T_{rms} = \left( \frac{\int_{-\infty}^{\infty} t^2 |g(t)|^2 \, dt}{\int_{-\infty}^{\infty} |g(t)|^2 \, dt} \right)^{1/2}$$

$$B_{rms} = \left( \frac{\int_{-\infty}^{\infty} f^2 |G(f)|^2 \, df}{\int_{-\infty}^{\infty} |G(f)|^2 \, df} \right)^{1/2}$$

Then $T_{rms} B_{rms} \geq \frac{1}{4\pi} = 0.07957...$
A2.2 Time-Bandwidth Product

Example: \( g(t) = \exp(-\pi t^2) \). Then \( G(f) = \exp(-\pi f^2) \).

\[
T_{rms} = B_{rms} = \left( \frac{\int_{-\infty}^{\infty} t^2 e^{-2\pi^2 t^2} dt}{\int_{-\infty}^{\infty} e^{-2\pi^2 t^2} dt} \right)^{1/2} = \frac{1}{2\sqrt{\pi}}. \quad \text{Then} \quad T_{rms} B_{rms} = \frac{1}{4\pi}.
\]

Example: \( g(t) = \exp(-|t|) \). Then \( G(f) = 2/(1+4\pi^2 f^2) \).

\[
T_{rms} B_{rms} = \left( \frac{\int_{-\infty}^{\infty} t e^{-2|t|} dt}{\int_{-\infty}^{\infty} e^{-2|t|} dt} \right)^{1/2} \left( \frac{\int_{-\infty}^{\infty} \frac{f^2}{(1 + 4\pi^2 f^2)^2} df}{\int_{-\infty}^{\infty} \frac{1}{(1 + 4\pi^2 f^2)^2} df} \right)^{1/2} = \frac{1}{\sqrt{2}} \times \frac{1}{2\pi} \geq \frac{1}{4\pi}.
\]
A2.3 Hilbert Transform

- Let \( G(f) \) be the spectrum of a real function \( g(t) \).
  - By convention, denote by \( u(f) \) the unit step function, i.e.,
    \[
    u(f) = \begin{cases} 
    1, & f > 0 \\ 
    1/2, & f = 0 \\ 
    0, & f < 0 
    \end{cases}
    \]

- Put \( g_+(t) \) to be the function corresponding to \( 2u(f)G(f) \).

Multiply 2 to unchange the area.
A2.3 Hilbert Transform

- How to obtain $g_+(t)$?
- Answer: Hilbert Transformer.

*Proof:* Observe that

$$2u(f) = 1 + \text{sgn}(f), \text{ where } \text{sgn}(f) = \begin{cases} 
1, & f > 0 \\
0, & f = 0 \\
-1, & f < 0
\end{cases}$$

Then by the next slide (also Slide 1-109), we learn that

$$2u(f) \overset{\text{Inverse Fourier}}{\rightarrow} \delta(t) + j\frac{1}{\pi t} \cdot 1\{t \neq 0\}$$
By extended Fourier transform,

\[ \int_{-\infty}^{\infty} \text{sgn}(f)e^{-a|f|+j2\pi ft}df = \int_{0}^{\infty} e^{-a|f|+j2\pi ft}df - \int_{-\infty}^{0} e^{-a|f|+j2\pi ft}df \]

\[ = \int_{0}^{\infty} e^{-(a-j2\pi t)f}df - \int_{-\infty}^{0} e^{(a+j2\pi t)f}df \]

\[ = \frac{1}{a-j2\pi t} - \frac{1}{a+j2\pi t} \]

\[ = \frac{j4\pi t}{a^2 + 4\pi^2 t^2} \]

\[ \text{sgn}(f) \xrightarrow{\text{Inverse Fourier}} \lim_{a \downarrow 0} j \frac{4\pi t}{a^2 + 4\pi^2 t^2} = \begin{cases} \frac{j}{\pi t}, & t \neq 0 \\ 0, & t = 0 \end{cases} \]

\[ 2u(f) = 1 + \text{sgn}(f) \xrightarrow{\text{Inverse Fourier}} \delta(t) + \frac{j}{\pi t} \cdot 1\{t \neq 0\} \]
A2.3 Hilbert Transform

\[ g_+(t) = \text{Fourier}^{-1}\{2u(f)G(f)\} \]

\[ = \text{Fourier}^{-1}\{2u(f)\} * \text{Fourier}^{-1}\{G(f)\} \]

\[ = \left( \delta(t) + j \frac{1}{\pi t} \right) * g(t) \]

\[ = g(t) + j \frac{1}{\pi t} \cdot 1\{t \neq 0\} * g(t) \]

\[ = g(t) + j \hat{g}(t), \]

where \( \hat{g}(t) = \int_{-\infty}^{\infty} \frac{g(\tau)}{\pi(t-\tau)} d\tau \) is named the Hilbert Transform of \( g(t) \).
A2.3 Hilbert Transform

\[ h(\tau) = \frac{1}{\pi \tau} \]

\[ g(t) \xrightarrow{h(\tau)} \hat{g}(t) \]

\[ h(\tau) = \frac{1}{\pi \tau} \rightarrow H(f) = -j \text{sgn}(f), \text{ where } \text{sgn}(f) = \begin{cases} 1, & f > 0 \\ 0, & f = 0 \\ -1, & f < 0 \end{cases} \]

\[ \Rightarrow \hat{G}(f) = -j \text{sgn}(f) \cdot G(f) = \begin{cases} |G(f)| \exp\{j[\angle G(f) - \pi / 2]\}, & f > 0 \\ 0, & f = 0 \\ |G(f)| \exp\{j[\angle G(f) + \pi / 2]\}, & f < 0 \end{cases} \]
A2.3 Hilbert Transform

- Hence, Hilbert Transform is basically a 90 degree phase shifter.
A2.3 Hilbert Transform

Hilbert Transform Pair:

\[
\begin{align*}
\hat{g}(t) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(\tau)}{t - \tau} d\tau \\
g(t) &= -\frac{1}{\pi} \int_{-\infty}^{\infty} \hat{g}(\tau) \frac{d\tau}{t - \tau}
\end{align*}
\]
A2.3 Hilbert Transform

An important property of Hilbert Transform is that:

\[ g(t) \text{ and } \hat{g}(t) \text{ are orthogonal in the sense of Integration.} \]

In other words, \( \int_{-\infty}^{\infty} g(t)\hat{g}(t)dt = 0. \)

(See the proof in the next slide.)

The real and imaginary parts of \( g_+(t) = g(t) + j\hat{g}(t) \) are orthogonal to each other.

(Examples of Hilbert Transform Pairs can be found in Table A6.4.)
\[
\int_{-\infty}^{\infty} g(t)\hat{g}(t)dt = \int_{-\infty}^{\infty} g(t)\left(\int_{-\infty}^{\infty} \hat{G}(f)e^{i2\pi ft} df\right)dt
\]
\[
= \int_{-\infty}^{\infty} \hat{G}(f)\left(\int_{-\infty}^{\infty} g(t)e^{i2\pi ft} dt\right)df
\]
\[
= \int_{-\infty}^{\infty} \hat{G}(f)G(-f)df
\]
\[
= -j\int_{-\infty}^{\infty} \text{sgn}(f)G(f)G(-f)df
\]
\[
= -j\left(\int_{0}^{\infty} G(f)G(-f)df - \int_{-\infty}^{0} G(f)G(-f)df\right)
\]
\[
= -j\left(\int_{0}^{\infty} G(f)G(-f)df - \int_{0}^{\infty} G(-f)G(f)df\right)
\]
\[
= 0, \text{ if } \int_{0}^{\infty} G(f)G(-f)df < \infty.
\]
A2.4 Complex Representation of Signals and Systems

- $g_+(t)$ is named the *pre-envelope*, or *analytical signal*, of $g(t)$.
- We can similarly define

$$g_-(t) = g(t) - j\hat{g}(t)$$

\[2(1-u(f))G(f)\]
A2.4 Canonical Representation of Band-Pass Signals

Now let \( G(f) \) be a narrow-band signal for which \( 2W \ll f_c \).

Then we can obtain its pre-envelope \( G_+(f) \).

Afterwards, we can shift the pre-envelope to its low-pass signal \( \tilde{G}(f) = G_+(f + f_c) \).
A2.4 Canonical Representation of Band-Pass Signal

These steps give the relation between the complex lowpass signal (baseband signal) and the real bandpass signal (passband signal).

\[ g(t) = \text{Re}(g_+(t)) = \text{Re}(\tilde{g}(t) \exp(j2\pi f_c t)) \]

Quite often, the real and imaginary parts of complex lowpass signal are respectively denoted by \( g_I(t) \) and \( g_Q(t) \).
A2.4 Canonical Representation of Band-Pass Signal

In terminology,

\[
\begin{align*}
g_+(t) & \quad \text{pre-envelope} \\
\tilde{g}(t) & \quad \text{complex envelope} \\
g_I(t) & \quad \text{in-phase component of the band-pass signal } g(t) \\
g_Q(t) & \quad \text{quadrature component of the band-pass signal } g(t)
\end{align*}
\]

This leads to a canonical, or standard, expression for \( g(t) \).

\[
g(t) = \text{Re}\left\{ (g_I(t) + jg_Q(t)) \exp(j2\pi f_c t) \right\}
\]

\[
= g_I(t) \cos(2\pi f_c t) - g_Q(t) \sin(2\pi f_c t)
\]
Rotate at the rate $2\pi f_c$

$$(g_I(t) + jg_Q(t))\exp(j2\pi f_c t)$$
A2.4 Canonical Representation of Band-Pass Signal

Canonical transmitter

\[ g(t) = g_I(t) \cos(2\pi f_c t) - g_Q(t) \sin(2\pi f_c t) \]
A2.4 Canonical Representation of Band-Pass Signal

**Canonical receiver**

\[
2g(t) \cos(2\pi f_c t) = 2g_I(t) \cos^2(2\pi f_c t) - 2g_Q(t) \sin(2\pi f_c t) \cos(2\pi f_c t) = g_I(t) + g_I(t) \cos(4\pi f_c t) - g_Q(t) \sin(4\pi f_c t)
\]

\[
-2g(t) \sin(2\pi f_c t) = -2g_I(t) \sin(2\pi f_c t) \cos(2\pi f_c t) + 2g_Q(t) \sin^2(2\pi f_c t) = g_Q(t) - g_Q(t) \cos(4\pi f_c t) - g_I(t) \sin(4\pi f_c t)
\]
A2.4 More Terminology

\[
\begin{align*}
g_+ (t) & \quad \text{pre-envelope} \\
\tilde{g} (t) & \quad \text{complex envelope} \\
g_I (t) & \quad \text{in-phase component of the band-pass signal } g(t) \\
g_Q (t) & \quad \text{quadrature component of the band-pass signal } g(t)
\end{align*}
\]

\[
\begin{align*}
a(t) &= |g_+ (t)| = |\tilde{g} (t)| = \sqrt{g_I^2 (t) + g_Q^2 (t)} \quad \text{natural envelope or envelope of } g(t) \\
\phi(t) &= \tan^{-1} \left( \frac{g_Q (t)}{g_I (t)} \right) \quad \text{phase of } g(t)
\end{align*}
\]
A2.4 Bandpass System

Consider the case of passing a band-pass signal $x(t)$ through a real LTI filter $h(\tau)$ to yield an output $y(t)$.

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

Can we have a low-pass equivalent system for the bandpass system?
Similar to the previous analysis, we have:

**Assumption**: The spectrum of $x(t)$ is limited to within $\pm W$ Hz of the carrier frequency $f_c$, and $W < f_c$.

\[
x(t) = \text{Re}\left\{ \tilde{x}(t)e^{j2\pi f_c t} \right\} = x_I(t) \cos(2\pi f_c t) - x_Q(t) \sin(2\pi f_c t)
\]

\[
\tilde{x}(t) = x_I(t) + jx_Q(t)
\]

**Assumption**: The spectrum of $h(\tau)$ is limited to within $\pm B$ Hz of the carrier frequency $f_c$, and $B < f_c$.

\[
h(\tau) = \text{Re}\left\{ \tilde{h}(\tau)e^{j2\pi f_c \tau} \right\} = h_I(\tau) \cos(2\pi f_c \tau) - h_Q(\tau) \sin(2\pi f_c \tau)
\]

\[
\tilde{h}(\tau) = h_I(\tau) + jh_Q(\tau) \quad \text{complex impulse response}
\]
Now, is the filter output $y(t)$ also a bandpass signal?

$$Y(f) = X(f)H(f)$$

$\Rightarrow$ The spectrum of $y(t)$ is limited to within $\pm \min\{W, B\}$ Hz of the carrier frequency $f_c$, provided that $\max\{W, B\} < f_c$

$$\begin{cases} y(t) = \text{Re}\{\tilde{y}(t)e^{j2\pi f_c t}\} = y_I(t)\cos(2\pi f_c t) - y_Q(t)\sin(2\pi f_c t) \\ \tilde{y}(t) = y_I(t) + j y_Q(t) \end{cases}$$
A2.4 Bandpass System

- Question: Is the following system valid?

\[ \tilde{y}(t) = \int_{-\infty}^{\infty} \tilde{h}(\tau) \tilde{x}(t - \tau) d\tau \]

- The advantage of the above equivalent system is that there is no need to deal with the *carrier frequency* in the system analysis.

- The answer to the question is YES (with some modification)! It will be substantiated in the sequel.
It suffices to show that $\tilde{Y}(f) = \tilde{X}(f)\tilde{H}(f)$.

Observe that
\[
\begin{align*}
Y_+(f) &= 2u(f)Y(f) \\
X_+(f) &= 2u(f)X(f) \\
H_+(f) &= 2u(f)H(f)
\end{align*}
\]
and
\[
\begin{align*}
\tilde{Y}(f) &= Y_+(f + f_c) \\
\tilde{X}(f) &= X_+(f + f_c) \\
\tilde{H}(f) &= H_+(f + f_c)
\end{align*}
\]
Consequently,
\[
\begin{align*}
\tilde{X}(f)\tilde{H}(f) &= X_+(f + f_c)H_+(f + f_c) \\
&= (2u(f + f_c)X(f + f_c))(2u(f + f_c)H(f + f_c)) \\
&= 4u(f + f_c)X(f + f_c)H(f + f_c) \\
&= 4u(f + f_c)Y(f + f_c) \\
&= 2Y_+(f + f_c) \\
&= 2\tilde{Y}(f)
\end{align*}
\]
There is an additional multiplicative constant $2$ at the output!
A2.4 Bandpass System

Conclusion:

\[ \tilde{y}(t) = \frac{1}{2} \int_{-\infty}^{\infty} \tilde{h}(\tau) \tilde{x}(t - \tau) d\tau \]
A2.4 Bandpass System

Final note on bandpass system

- Some books define $H_+(f) = u(f)H(f)$ for a filter, instead of $H_+(f) = 2u(f)H(f)$ as for a signal.
- As a result of this definition (i.e., $H_+(f) = u(f)H(f)$),

\[
h(\tau) = 2 \text{Re}\left\{\tilde{h}(\tau)e^{j2\pi\tau}\right\} \text{ and } \tilde{y}(t) = \int_{-\infty}^{\infty} \tilde{h}(\tau)\tilde{x}(t-\tau)d\tau
\]

- It is justifiable to remove 2 in $H_+(f) = u(f)H(f)$, because a filter is used to filter out the signal; hence, it is not necessary to make the total area constant.
1.11 Representation of Narrowband Noise in terms of In-Phase and Quadrature Components

- In Appendix 2.4, the bandpass system representation is discussed based on deterministic signals.

- How about a random process? Can we have a low-pass isomorphism system to a bandpass random process.

- Take the noise process $N(t)$ as an example.

\[ Y(t) = \int_{-\infty}^{\infty} h(\tau)N(t-\tau)d\tau \]
1.11 Representation of Narrowband Noise in Terms of In-Phase and Quadrature Components

- A WSS real-valued zero-mean noise process \( N(t) \) is a bandpass process if its PSD \( S_N(f) \neq 0 \) only for \( |f - f_c| < B \) and \( |f + f_c| < B \), and also \( B < f_c \).

  - Similar to the analysis for deterministic signals, let

    \[
    \begin{align*}
    N(t) &= N_I(t)\cos(2\pi f_c t) - N_Q(t)\sin(2\pi f_c t) \\
    \tilde{N}(t) &= N_I(t) + jN_Q(t)
    \end{align*}
    \]

    for some joint zero-mean WSS of \( N_I(t) \) and \( N_Q(t) \).

  - Notably, the joint WSS of \( N_I(t) \) and \( N_Q(t) \) immediately imply WSS of \( \tilde{N}(t) \).
1.11 PSD of $N_I(t)$ and $N_Q(t)$

First, we note that joint WSS of $N_I(t)$ and $N_Q(t)$ and the WSS of $N(t)$ imply:

\[
\begin{align*}
R_{N_I}(\tau) &= R_{N_Q}(\tau) \\
R_{N_{I,N_Q}}(\tau) &= -R_{N_{Q,N_I}}(\tau)
\end{align*}
\]

(See the proof in the sequel.)
\[ R_N(\tau) = E[N(t + \tau)N(t)] \]
\[ = E[(N_I(t + \tau) \cos(2\pi f_c(t + \tau)) - N_Q(t + \tau) \sin(2\pi f_c(t + \tau))) \times (N_I(t) \cos(2\pi f_c t) - N_Q(t) \sin(2\pi f_c t))] \]
\[ = R_{N_I}(\tau) \cos(2\pi f_c(t + \tau)) \cos(2\pi f_c t) \]
\[ + R_{N_Q}(\tau) \sin(2\pi f_c(t + \tau)) \sin(2\pi f_c t) \]
\[ - R_{N_I,N_Q}(\tau) \cos(2\pi f_c(t + \tau)) \sin(2\pi f_c t) \]
\[ - R_{N_Q,N_I}(\tau) \sin(2\pi f_c(t + \tau)) \cos(2\pi f_c t) \]
\[ = R_{N_I}(\tau) \frac{\cos(2\pi f_c \tau) + \cos(2\pi f_c(2t + \tau))}{2} \]
\[ + R_{N_Q}(\tau) \frac{\cos(2\pi f_c \tau) - \cos(2\pi f_c(2t + \tau))}{2} \]
\[ - R_{N_I,N_Q}(\tau) \frac{\sin(2\pi f_c(2t + \tau)) - \sin(2\pi f_c \tau)}{2} \]
\[ - R_{N_Q,N_I}(\tau) \frac{\sin(2\pi f_c(2t + \tau)) + \sin(2\pi f_c \tau)}{2} \]
These two terms must equal zero, since $R_N(\tau)$ is not a function of $t$. 

$$R_N(\tau) = \frac{1}{2} [R_{N_1}(\tau) + R_{N_Q}(\tau)] \cos(2\pi f_c \tau)$$

$$+ \frac{1}{2} [R_{N_{1N_Q}}(\tau) - R_{N_{N_1},N_Q}(\tau)] \sin(2\pi f_c \tau)$$

$$+ \frac{1}{2} [R_{N_1}(\tau) - R_{N_Q}(\tau)] \cos(2\pi f_c (2t + \tau))$$

$$- \frac{1}{2} [R_{N_{1N_Q}}(\tau) + R_{N_{N_1},N_Q}(\tau)] \sin(2\pi f_c (2t + \tau))$$

$$\Rightarrow \begin{cases} R_{N_1}(\tau) = R_{N_Q}(\tau) \\ R_{N_{1N_Q}}(\tau) = -R_{N_{N_1},N_Q}(\tau) \end{cases}$$

(Property 1)

$$\Rightarrow R_N(\tau) = R_{N_1}(\tau) \cos(2\pi f_c \tau) - R_{N_{Q,N_1}}(\tau) \sin(2\pi f_c \tau)$$

(Property 2)
1.11 PSD of $N_I(t)$ and $N_Q(t)$

Some other properties

\[
R_{\tilde{N}}(\tau) = E[\tilde{N}(t + \tau)\tilde{N}^*(t)] \\
= E[(N_I(t + \tau) + jN_Q(t + \tau)) \times (N_I(t) - jN_Q(t))] \\
= R_{N_I}(\tau) + R_{N_Q}(\tau) + j[R_{N_Q,N_I}(\tau) - R_{N_I,N_Q}(\tau)] \\
= 2R_{N_I}(\tau) + j2R_{N_Q,N_I}(\tau) \quad \text{(Property 3)}
\]
1.11 PSD of $N_I(t)$ and $N_Q(t)$

Properties 2 and 3 jointly imply

$$R_N(\tau) = \frac{1}{2} \text{Re}\{R_{\tilde{N}}(\tau) \exp(j2\pi f_c \tau)\}$$ (Property 4)

- As in Slide 1-217,
  - some books define
    $$\begin{align*}
    R_{\tilde{X}}(\tau) &= \frac{1}{2} \text{E}[\tilde{X}(t+\tau)\tilde{X}^*(t)] \quad \text{for complex } \tilde{X}(t); \\
    R_X(\tau) &= \text{E}[X(t+\tau)X(t)] \quad \text{for real } X(t)
    \end{align*}$$
  - As a result of the two “inconsistent” definitions, a “simpler” relation is obtained: $R_N(\tau) = \text{Re}\{R_{\tilde{N}}(\tau) \exp(j2\pi f_c \tau)\}$
  - For consistence, we let $R_{\tilde{X}}(\tau) = \text{E}[\tilde{X}(t+\tau)\tilde{X}^*(t)]$ for complex $\tilde{X}(t)$
1.11 Summary of Spectrum Properties

1. \( R_{N_I}(\tau) = R_{N_Q}(\tau) \) and \( R_{N_I,N_Q}(\tau) = -R_{N_Q,N_I}(\tau) \)

2. \( R_N(\tau) = R_{N_I}(\tau) \cos(2\pi f_c \tau) - R_{N_Q,N_I}(\tau) \sin(2\pi f_c \tau) \)

3. \( R_{\tilde{N}}(\tau) = 2 \cdot R_{N_I}(\tau) + j2 \cdot R_{N_Q,N_I}(\tau) \) \( \Rightarrow R_N(0) = R_{N_I}(0) = R_{N_Q}(0) \)

4. \( R_N(\tau) = \frac{1}{2} \cdot \text{Re}\{R_{\tilde{N}}(\tau) \exp(j2\pi f_c \tau)\} \)

5. \( S_{\tilde{N}}(f) \) is real-valued. By \( R_{\tilde{N}}(\tau) = R_{\tilde{N}}^*(-\tau) \).

6. \( S_{N_I}(f) = S_{N_Q}(f) \) and \( S_{N_I,N_Q}(f) = -S_{N_Q,N_I}(f) \) \[From 1.\]

7. \( S_N(f) = \frac{1}{4} (S_{\tilde{N}}(f - f_c) + S_{\tilde{N}}(-f - f_c)) \) \[From 4. See the next slide.]
\[ S_N(f) = \int_{-\infty}^{\infty} R_N(\tau)e^{-j2\pi f \tau} d\tau \]

\[ = \int_{-\infty}^{\infty} \frac{1}{2} \text{Re}\{ R_\tilde{N}(\tau)e^{j2\pi f_c \tau} e^{-j2\pi f \tau} \} d\tau \]

\[ = \int_{-\infty}^{\infty} \frac{1}{4} \left( R_\tilde{N}(\tau)e^{j2\pi f_c \tau} + (R_\tilde{N}(\tau)e^{j2\pi f_c \tau})^* \right) e^{-j2\pi f \tau} d\tau \]

\[ = \int_{-\infty}^{\infty} \frac{1}{4} \left( R_\tilde{N}(\tau)e^{j2\pi f_c \tau} + R_\tilde{N}^*(\tau)e^{-j2\pi f_c \tau} \right) e^{-j2\pi f \tau} d\tau \]

\[ = \frac{1}{4} \int_{-\infty}^{\infty} R_\tilde{N}(\tau)e^{-j2\pi (f-f_c) \tau} d\tau + \frac{1}{4} \left( \int_{-\infty}^{\infty} R_\tilde{N}(\tau)e^{-j2\pi (-f-f_c) \tau} d\tau \right)^* \]

\[ = \frac{1}{4} \left( S_\tilde{N}(f-f_c) + S_\tilde{N}^*(-f-f_c) \right) \]

\[ = \frac{1}{4} \left( S_\tilde{N}(f-f_c) + S_\tilde{N}(-f-f_c) \right) \]
8. $N_I(t)$ and $N_Q(t)$ are uncorrelated, since they have zero means.

$$
\begin{align*}
R_{N_I,N_Q}(\tau) &= -R_{N_Q,N_I}(\tau) \\
R_{N_I,N_Q}(-\tau) &= E[N_I(t)N_Q(t+\tau)] = R_{N_Q,N_I}(\tau)
\end{align*}
\Rightarrow R_{N_I,N_Q}(\tau) = -R_{N_I,N_Q}(-\tau) \\
\Rightarrow R_{N_I,N_Q}(0) = -R_{N_I,N_Q}(0) \\
\Rightarrow R_{N_I,N_Q}(0) = E[N_I(t)N_Q(t)] = 0.
$$
\[ N(t) = N_I(t) \cos(2\pi f_c t) - N_Q(t) \sin(2\pi f_c t) \]

\[ 2 \cos(2\pi f_c t) \]

\[ 2 \sin(2\pi f_c t) \]

\[ V_I(t) \]

\[ V_Q(t) \]

\[ N_I(t) \]

\[ N_Q(t) \]

\[ R_{V_I}(t,u) = E[V_I(t)V_I(u)] \]

\[ = 4E[N(t)\cos(2\pi f_c t)N(u)\cos(2\pi f_c u)] \]

\[ = 4R_N(t,u)\cos(2\pi f_c t)\cos(2\pi f_c u) \]

Notably, \( V_I(t) \) is not WSS.
\[
\overline{R}_{V_i}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} R_{V_i}(t + \tau, t) dt
\]
\[
= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} 4R_N(\tau) \cos(2\pi f_c (t + \tau)) \cos(2\pi f_c t) dt
\]
\[
= R_N(\tau) \lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T} [\cos(2\pi f_c (2t + \tau)) + \cos(2\pi f_c \tau)] dt
\]
\[
= R_N(\tau) \lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T} \cos(2\pi f_c (2t + \tau)) dt + 2R_N(\tau) \cos(2\pi f_c \tau)
\]
\[
= 2R_N(\tau) \cos(2\pi f_c \tau)
\]

\[
\overline{S}_{V_i}(f) = S_N(f - f_c) + S_N(f + f_c)
\]

\[
\overline{S}_{N_i}(f) = |H(f)|^2 \overline{S}_{V_i}(f), \text{ where } |H(f)|^2 = \begin{cases} 
1, & |f| \leq B \\
0, & f > B
\end{cases}
\]
\[
S_{N_I}(f) = \overline{S}_{N_I}(f) = |S_N(f - f_c) + S_N(f + f_c)|^2 H(f)
\]

Similarly,

\[
S_{N_Q}(f) = \begin{cases} 
S_N(f - f_c) + S_N(f + f_c), & |f| < B \\
0, & \text{otherwise}
\end{cases}
\]

This result (namely, \(S_{N_I}(f) = S_{N_Q}(f)\)) coincides with Property 1, for which \(R_{N_I}(\tau) = R_{N_Q}(\tau)\).
Next, we turn to $R_{N_I,N_Q}(\tau)$.

\[
N(t) = N_I(t) \cos(2\pi f_c t) - N_Q(t) \sin(2\pi f_c t)
\]

\[
R_{V_I,V_Q}(t,u) = E[V_I(t)V_Q(u)]
\]

\[
= -4 E[N(t) \cos(2\pi f_c t)N(u)\sin(2\pi f_c u)]
\]

\[
= -4 R_N(t,u) \cos(2\pi f_c t)\sin(2\pi f_c u)
\]

\[
R_{V_I,V_Q}(t+\tau,t) = -4 R_N(\tau) \cos(2\pi f_c (t+\tau))\sin(2\pi f_c t)
\]
\[
\bar{R}_{V_I,V_Q}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} R_{V_I,V_Q}(t + \tau, t) dt \\
= -R_N(\tau) \lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T} [\sin(2\pi f_c (2t + \tau)) - \sin(2\pi f_c \tau)] dt \\
= 2R_N(\tau) \sin(2\pi f_c \tau)
\]

\[
\bar{S}_{V_I,V_Q}(f) = j[S_N(f + f_c) - S_N(f - f_c)]
\]

\[
R_{N_I,N_Q}(t,u) = E[N_I(t)N_Q(u)] \\
= E\left[\int_{-\infty}^{\infty} h_I(\tau_1)V_I(t - \tau_1) d\tau_1 \cdot \int_{-\infty}^{\infty} h_Q(\tau_2)V_Q(u - \tau_2) d\tau_2\right] \\
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_I(\tau_1)h_Q(\tau_2)E[V_I(t - \tau_1)V_Q(u - \tau_2)] d\tau_1 d\tau_2 \\
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_I(\tau_1)h_Q(\tau_2)R_{V_I,V_Q}(t - \tau_1, u - \tau_2) d\tau_1 d\tau_2
\]
\( \bar{R}_{NI,NQ}(\tau) \)

\[
\begin{align*}
&= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} R_{NI,NQ}(t + \tau, t) dt \\
&= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_I(\tau_1)h_Q(\tau_2)R_{V_I,V_Q}(t + \tau - \tau_1, t - \tau_2) d\tau_1 d\tau_2 dt \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_I(\tau_1)h_Q(\tau_2) \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} R_{V_I,V_Q}(t + \tau - \tau_1, t - \tau_2) dt d\tau_1 d\tau_2 \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_I(\tau_1)h_Q(\tau_2) \bar{R}_{V_I,V_Q}(\tau - \tau_1 + \tau_2) d\tau_1 d\tau_2
\end{align*}
\]
\[ \overline{S}_{N_1, N_Q}(f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_1(\tau_1) h_Q(\tau_2) R_{\nu_1, \nu_Q}(\tau_2 + \tau - \tau_1) e^{-j2\pi f \tau} d\tau d\tau_1 d\tau_2 \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_1(\tau_1) h_Q(\tau_2) R_{\nu_1, \nu_Q}(u) e^{-j2\pi f (u - \tau_2 + \tau_1)} dud\tau_1 d\tau_2, \]

(Let \( u = \tau_2 + \tau - \tau_1 \).)

\[ = H_I(f) H_Q(-f) \overline{S}_{\nu_1, \nu_Q}(f) \]

Here, \( H_Q(-f) = H^*_Q(f) \).

Take \( H_I(f) = H_Q(f) = H(f) \).

Then \( H_I(f) H_Q(-f) = H(f) H(-f) = H(f) H^*(f) = |H(f)|^2 \).

\[ S_{N_1, N_Q}(f) = \overline{S}_{N_1, N_Q}(f) \]

\[ = j[S_N(f + f_c) - S_N(f - f_c)] |H(f)|^2 \]

\[ = \begin{cases} 
  j[S_N(f + f_c) - S_N(f - f_c)], & |f| < B \\
  0, & \text{otherwise}
\end{cases} \]
Finally,

\[ R_{N}(\tau) = 2 \cdot R_{N_I}(\tau) + j2 \cdot R_{N,Q,N_I}(\tau) = 2 \cdot R_{N_I}(\tau) - j2 \cdot R_{N_I,N_Q}(\tau) \]

implies that

\[
S_{N}(f) = 2 \cdot S_{N_I}(f) - j2 \cdot S_{N_I,N_Q}(f) \\
= \begin{cases} 
2[S_N(f - f_c) + S_N(f + f_c)] + 2[S_N(f + f_c) - S_N(f - f_c)], & |f| < B; \\
0, & \text{otherwise}
\end{cases}
\]

\[
= \begin{cases} 
4 \cdot S_N(f + f_c), & |f| < B; \\
0, & \text{otherwise}
\end{cases}
\]

\[
= \begin{cases} 
2 \cdot S_{N_+}(f + f_c), & |f| < B; \\
0, & \text{otherwise}
\end{cases}
\]

\[= 2 \cdot S_{N_+}(f + f_c), \text{ provided } B > W.\]

where \( S_{N_+}(f) = 2u(f)S_N(f) \).
Example 1.12 Ideal Band-Pass Filtered White Noise

\[
R_N(\tau) = \int_{-f_c-B}^{-f_c+B} \frac{N_0}{2} e^{j2\pi f \tau} df + \int_{f_c-B}^{f_c+B} \frac{N_0}{2} e^{j2\pi f \tau} df
\]

\[
= 2BN_0 \text{sinc}(2B\tau) \cos(2\pi f_c \tau)
\]

\[
R_{N_I}(\tau) = 2BN_0 \text{sinc}(2B\tau)
\]

\[
S_{N_I}(f) = S_{N_Q}(f)
\]
1.12 Representation of Narrowband Noise in Terms of Envelope and Phase Components

Now we turn to *envelope* \( R(t) \) and *phase* \( \Psi(t) \) components of a random process of the form

\[
N(t) = N_I(t) \cos(2\pi f_c t) - N_Q(t) \sin(2\pi f_c t) \\
= R(t) \cos[2\pi f_c t + \Psi(t)]
\]

where \( R(t) = \sqrt{N_I^2(t) + N_Q^2(t)} \) and \( \Psi(t) = \tan^{-1}[N_Q(t) / N_I(t)] \).
1.12 Pdf of $R(t)$ and $\Psi(t)$

- Assume that $N(t)$ is a white Gaussian process with two-sided PSD $\sigma^2 = N_0/2$.
- For convenience, let $N_I$ and $N_Q$ be snapshot samples of $N_I(t)$ and $N_Q(t)$.
- Then

$$f_{N_I,N_Q}(n_I,n_Q) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{n_I^2 + n_Q^2}{2\sigma^2}\right)$$
1.12 Pdf of $R(t)$ and $\Psi(t)$

By $n_I = r \cos(\psi)$ and $n_Q = r \sin(\psi)$,

$$
\int_{A(n_I, n_Q)} \frac{1}{2\pi\sigma^2} \exp \left( - \frac{n_I^2 + n_Q^2}{2\sigma^2} \right) dn_I dn_Q = \int_{A(r, \psi)} \frac{1}{2\pi\sigma^2} \exp \left( - \frac{r^2}{2\sigma^2} \right) \left| \begin{array}{cc} \frac{dn_I}{dr} \\ \frac{dn_I}{d\psi} \end{array} \right| \left| \begin{array}{cc} \frac{dn_Q}{dr} \\ \frac{dn_Q}{d\psi} \end{array} \right| dr d\psi
$$

$$
= \int_{A(r, \psi)} \frac{1}{2\pi\sigma^2} \exp \left( - \frac{r^2}{2\sigma^2} \right) r dr d\psi
$$

So $f_{R, \Psi}(r, \psi) = \frac{r}{2\pi\sigma^2} \exp \left( - \frac{r^2}{2\sigma^2} \right) = \frac{1}{2\pi} \times \frac{r}{\sigma^2} \exp \left( - \frac{r^2}{2\sigma^2} \right)$. 
1.12 Pdf of $R(t)$ and $\Psi(t)$

- $R$ and $\Psi$ are therefore independent.

\[
\begin{align*}
  f_R(r) &= \frac{r}{\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right) \quad \text{for } r \geq 0. \quad \text{Rayleigh distribution.} \\
  f_\Psi(\psi) &= \frac{1}{2\pi} \quad \text{for } 0 \leq \psi < 2\pi.
\end{align*}
\]

*Normalized Rayleigh distribution with $\sigma^2 = 1$.*
Now suppose the previous Gaussian white noise is added to a sinusoid of amplitude $A$.

Then

$$x(t) = A\cos(2\pi f_c t) + n(t)$$

$$= A\cos(2\pi f_c t) + n_I(t)\cos(2\pi f_c t) - n_Q\sin(2\pi f_c t)$$

$$= x_I(t)\cos(2\pi f_c) - x_Q(t)\sin(2\pi f_c)$$

Uncorrelation for Gaussian $n_I(t)$ and $n_Q(t)$ implies their independence.
1.13 Sine Wave Plus Narrowband Noise

This gives the pdf of \( x_I(t) \) and \( x_Q(t) \) as:

\[
f_{x_I,x_Q}(x_I, x_Q) = \frac{1}{2\pi\sigma^2} \exp\left[-\frac{(x_I - A)^2 + x_Q^2}{2\sigma^2}\right]
\]

By \( x_I = r \cos(\psi) \) and \( x_Q = r \sin(\psi) \),

\[
f_{R,\psi}(r, \psi) = \frac{1}{2\pi\sigma^2} \exp\left[-\frac{(r \cos(\psi) - A)^2 + r^2 \sin^2(\psi)}{2\sigma^2}\right]
\]

\[
= \frac{r}{2\pi\sigma^2} \exp\left(-\frac{r^2 + A^2 - 2Ar \cos(\psi)}{2\sigma^2}\right)
\]
1.13 Sine Wave Plus Narrowband Noise

- Notably, in this case, $R$ and $\Psi$ are no longer independent.
- We are more interested in the marginal distribution of $R$.

$$f_r(r) = \int_0^{2\pi} f_{r,\Psi}(r, \Psi) d\Psi$$

$$= \int_0^{2\pi} \frac{r}{2\pi \sigma^2} \exp\left( - \frac{r^2 + A^2 - 2Ar \cos(\Psi)}{2\sigma^2} \right) d\Psi$$
1.13 Sine Wave Plus Narrowband Noise

\[
f_R (r) = \frac{r}{2\pi\sigma^2} \exp \left( - \frac{r^2 + A^2}{2\sigma^2} \right) \int_0^{2\pi} \exp \left( \frac{2Ar \cos(\psi)}{2\sigma^2} \right) d\psi
\]

\[
= \frac{r}{\sigma^2} \exp \left( - \frac{r^2 + A^2}{2\sigma^2} \right) I_0 \left( \frac{Ar}{\sigma^2} \right),
\]

where \( I_0(x) = \frac{1}{2\pi} \int_0^{2\pi} \exp(x \cos(\psi)) d\psi \) is the modified Bessel function of the first kind of zero order.

This distribution is named the \textit{Rician distribution}.
1.13 Normalized Rician distribution

Let \( v = \frac{r}{\sigma} \) and \( a = \frac{A}{\sigma} \).

\[
f_v(v) = v \cdot \exp \left( -\frac{v^2 + a^2}{2} \right) I_0(av),
\]

\( I_0 \) is the modified Bessel function of the first kind.
A3.1 Bessel Functions

- Bessel’s equation of order \( n \)
  \[
  x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2) y = 0
  \]

- Its solution \( J_n(x) \) is the **Bessel function of the first kind of order** \( n \).
  \[
  J_n(x) = \frac{1}{\pi} \int_0^\pi \cos(x \sin(\theta) - n\theta) \, d\theta
  \]
  
  \[
  = \frac{1}{2\pi} \int_{-\pi}^\pi \exp(jx \sin \theta - jn\theta) \, d\theta = \sum_{m=0}^\infty \frac{(-1)^m (x/2)^{n+2m}}{m!(n+m)!}
  \]
### A3.2 Properties of the Bessel Function

1. \( J_n(x) = (-1)^n J_{-n}(x) = (-1)^n J_n(-x) \)

2. \( J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x) \)

3. When \( x \) small, \( J_n(x) \approx \frac{x^n}{2^n n!} \).

4. When \( x \) large, \( J_n(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4} - \frac{n\pi}{2}\right) \)

5. \( \lim_{n \to \infty} J_n(x) = 0 \)

6. \( \sum_{n=-\infty}^{\infty} J_n(x) \exp(jn\phi) = \exp(jx \sin(\phi)) \)

7. \( \sum_{n=-\infty}^{\infty} J_n^2(x) = 1 \).
A3.3 Modified Bessel Function

- Modified Bessel’s equation of order \( n \)
  \[
  x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (j^2 x^2 - n^2)y = 0 \quad (\text{i.e., } x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - (x^2 + n^2) = 0)
  \]

- Its solution \( I_n(x) \) is the **Modified Bessel function of the first kind of order \( n \)**.
  \[
  I_n(x) = j^{-n}J_n(jx) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(x \cos(\theta))\cos(n\theta) d\theta
  \]
  \[
  = \sum_{m=0}^{\infty} \frac{(x/2)^{n+2m}}{m!(n+m)!}
  \]

- The modified Bessel function is monotonically increasing in \( x \) for all \( n \).
A3.3 Properties of Modified Bessel Function

3'. \( \lim_{x \to 0} I_n(x) = \begin{cases} 1, & n = 0 \\ 0, & n \geq 1 \end{cases} \)

4'. When \( x \) large, \( I_n(x) \approx \frac{\exp(x)}{\sqrt{2\pi x}} \).

6'. \( \sum_{n=-\infty}^{\infty} I_n(x) \exp(jn\phi) = \exp(x \cos(\phi)) \)
1.14 Computer Experiments: Flat-Fading Channel

- Model of a multi-path channel

\[ A \cos(2\pi f_c t) \]

Assume \( N \) paths.

\[ \sum_{k=1}^{N} A_k \cos(2\pi f_c t + \Theta_k) \]
1.14 Computer Experiments: Flat-Fading Channel

\[ Y(t) = \sum_{k=1}^{N} A_k \cos(2\pi f_c t + \Theta_k) = Y_I \cos(2\pi f_c t) - Y_Q \sin(2\pi f_c t) \]

where \( Y_I = \sum_{k=1}^{N} A_k \cos(\Theta_k) \) and \( Y_Q = \sum_{k=1}^{N} A_k \sin(\Theta_k) \).

\[ \Rightarrow \text{Input } \tilde{X}(t) = A \text{ induces output } \tilde{Y}(t) = Y_I + jY_Q, \]

which is independent of \( t \).

Assume \( \{(A_k, \Theta_k)\} \) i.i.d., and \( A_k \) uniform over \([-1,+1)\) and \( \Theta_k \) uniform over \([0,2\pi)\).
1.14 Experiment 1

By Central Limit Theorem,

\[
\frac{Y_i}{\sqrt{N/6}} = \frac{A_1 \cos(\Theta_1) + \cdots + A_N \cos(\Theta_N)}{\sqrt{N/6}} \rightarrow \text{Normal}(0,1)
\]

So \( Y_i \) is approximately Gaussian distributed with mean 0 and variance \( (N/6) \).

For any Gaussian random variable \( G \), we have

\[
\sqrt{\beta_1} = \frac{E[(G - \mu)^3]}{E^{3/2}[(G - \mu)^2]} = 0 \quad \text{and} \quad \beta_2 = \frac{E[(G - \mu)^4]}{E^{2}[(G - \mu)^2]} = 3.
\]
1.14 Experiment 1

\( \sqrt{\beta_1} \) skewness
\( \beta_2 \) kurtosis

\( \begin{cases} \sqrt{\beta_1} < 0 \\
\beta_2 = 3 \end{cases} \)

\( \begin{cases} \sqrt{\beta_1} > 0 \\
\beta_2 = 3 \end{cases} \)
1.14 Experiment 1

- Normality Test: We can therefore use $\beta_1$ and $\beta_2$ to examine the degree of resemblance to Gaussian for a random variable.

<table>
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<th>$Y_I$</th>
<th>$N$</th>
<th>10</th>
<th>100</th>
<th>1000</th>
<th>10000</th>
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<td>0.2443</td>
<td>0.0255</td>
<td>0.0065</td>
<td>0.0003</td>
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<tr>
<td>$\beta_2$</td>
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<table>
<thead>
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<th>$Y_Q$</th>
<th>$N$</th>
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<th>100</th>
<th>1000</th>
<th>10000</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.0874</td>
<td>0.0017</td>
<td>0.0004</td>
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<td>2.7109</td>
<td>3.1663</td>
<td>3.0135</td>
</tr>
</tbody>
</table>

$S_N = (U_1 + \cdots + U_N) / a_N$ for zero-mean i.i.d. $\{U_k\}$

$\Rightarrow \bar{\beta}_2 = \frac{E[S_N^4]}{E^2[S_N^2]} = \frac{E[(U_1 + \cdots + U_N)^4]}{E^2[(U_1 + \cdots + U_N)^2]} = \frac{NE[U^4] + 3N(N-1)E^2[U^2]}{N^2E^2[U^2]} = 3 + \frac{E[U^4]}{E^2[U^2]} - \frac{3}{N}$

$E[\cos(\Theta)] = E[\sin(\Theta)] = 0$

$E[A^2 \cos^2(\Theta)] = E[A^2 \sin^2(\Theta)] = 1/6$

$E[A^4 \cos^4(\Theta)] = E[A^4 \sin^4(\Theta)] = 3/40$

$N = 10\quad 100\quad 1000\quad 10000$

$2.97\quad 2.997\quad 2.9997\quad 2.99997$
1.14 Experiment 2

- We can re-formulate $Y(t)$ as:

$$Y(t) = R \cos(2\pi f_c t + \Psi),$$

where $R = \sqrt{Y_I^2 + Y_Q^2}$ and $\Psi = \tan^{-1}(Y_Q / Y_I)$.

- If $Y_I$ and $Y_Q$ approach independent Gaussian, then $R$ and $\Psi$ respectively approach Rayleigh and uniform distributions.
1.14 Experiment 2

\[ N = 10,000 \]

The experimental curve is obtained with 100 histograms and 100 ensemble averages being computed.

\[ \int_1^{1.05} re^{-r^2/2} dr = 0.0303016 \quad \text{and} \quad \frac{303}{10000} \cdot 0.05 = 0.606 \]

See Fig. 1.25 in textbook.
1.14 Experiment 2

Input = \sin(2\pi \times 10^6 t).

The corresponding Rayleigh channel output

\[ \sum_{k=1}^{10000} A_k \sin(2\pi \times 10^6 t + \Theta_k) \]

See Fig. 1.26 in textbook.
1.15 Summary and Discussion

- Definition of Probability System and Probability Measure
- Random variable, random vector and random process
- Autocorrelation and crosscorrelation
- Definition of WSS
- Why ergodicity?
  - Time average as a good “estimate” of ensemble average
- Characteristic function and Fourier transform
  - Dirichlet’s condition
  - Dirac delta function
  - Fourier series and its relation to Fourier transform
1.15 Summary and Discussion

- Power spectral density and its properties
- Energy spectral density
- Cross spectral density
- Stable LTI filter
  - Linearity and convolution
- Narrowband process
  - Canonical low-pass isomorphism
  - In-phase and quadrature components
  - Hilbert transform
  - Bandpass system
1.15 Summary and Discussion

- **Bandwidth**
  - Null to null, 3dB, rms, noise-equivalent
  - Time-bandwidth product

- **Noise**
  - Shot noise, thermal noise, and white noise

- **Gaussian, Rayleigh and Rician**
  - Central Limit Theorem