We migrate from *analog modulation* (continuous in both time and value) to *digital modulation* (discrete in both time and value) through *pulse modulation* (discrete in time but could be continuous in value).
3.1 Pulse Modulation

- Families of pulse modulation
  - Analog pulse modulation
    - A periodic pulse train is used as carriers (similar to sinusoidal carriers)
    - Some characteristic feature of each pulse, such as amplitude, duration, or position, is varied in a continuous matter in accordance with the sampled message signal.
  - Digital pulse modulation
    - Some characteristic feature of carriers is varied in a digital manner in accordance with the sampled, digitized message signal.
3.2 Sampling Theorem

\[ g_\delta(t) = \sum_{n=\infty}^{\infty} g(nT_s)\delta(t - nT_s) \]

\[ G_\delta(f) = \sum_{n=\infty}^{\infty} g(nT_s) \int_{-\infty}^{\infty} \delta(t - nT_s) \exp(-j2\pi ft) \, dt = \sum_{n=\infty}^{\infty} g(nT_s) \exp(-j2\pi nT_s f) \]

- \( T_s \) sampling period
- \( f_s = 1/T_s \) sampling rate
3.2 Sampling Theorem

- **Given:** \( G_\delta(f) = \sum_{n=-\infty}^{\infty} g(nT_s) \exp(-j2\pi nT_s f) \)

- **Claim:** \( G_\delta(f) = f_s \sum_{m=-\infty}^{\infty} G(f - mf_s) \)

In this figure, \( f_s = 2W \).
3.2 Spectrum of Sampled Signal

Proof:

Let \( L(f) = f_s \sum_{m=-\infty}^{\infty} G(f - mf_s) \), and notice that it is periodic with period \( f_s \).

\( \Rightarrow \) By Fourier Series Expansion,

\[
L(f) = \sum_{n=-\infty}^{\infty} c_n \exp\left( j2\pi \frac{n}{f_s} f \right), \quad \text{where} \quad c_n = \frac{1}{f_s} \int_{-f_s/2}^{f_s/2} L(f) \exp\left( -j2\pi \frac{n}{f_s} f \right) df
\]

\( \Rightarrow \)

\[
c_n = \frac{1}{f_s} \int_{-f_s/2}^{f_s/2} L(f) \exp\left( -j \frac{2\pi n}{f_s} f \right) df
\]

\[
= \int_{-f_s/2}^{f_s/2} \left( \sum_{m=-\infty}^{\infty} G(f - mf_s) \right) \exp\left( -j \frac{2\pi n}{f_s} f \right) df
\]
\[ c_n = \sum_{m=-\infty}^{\infty} \int_{-f_s/2}^{f_s/2} G(f-mf_s) \exp\left( -j \frac{2\pi n}{f_s} f \right) df, \quad s = f - mf_s \]

\[ = \sum_{m=-\infty}^{\infty} \int_{-f_s/2-mf_s}^{f_s/2-mf_s} G(s) \exp\left( -j \frac{2\pi n}{f_s} (s + mf_s) \right) ds \]

\[ = \sum_{m=-\infty}^{\infty} \int_{-f_s/2-mf_s}^{f_s/2-mf_s} G(s) \exp\left( -j \frac{2\pi n}{f_s} s \right) ds \]

\[ = \int_{-\infty}^{\infty} G(s) \exp\left( -j \frac{2\pi n}{f_s} s \right) ds \]

\[ = g(-nT_s) \]

\[ \Rightarrow L(f) = \sum_{n=-\infty}^{\infty} g(-nT_s) \exp\left( j2\pi \frac{n}{f_s} f \right) \]

\[ = \sum_{m=-\infty}^{\infty} g(mT_s) \exp\left( - j2\pi mT_s f \right), \text{ where } m = -n. \]

Q.E.D.
3.2 **First Important Conclusion from Sampling**

Uniform sampling at the time domain results in a periodic spectrum with a period equal to the sampling rate.

\[
g_\delta(t) = \sum_{n=-\infty}^{\infty} g(nT_s) \delta(t - nT_s) \Rightarrow G_\delta(f) = f_s \sum_{m=-\infty}^{\infty} G(f - mf_s)
\]

In this figure, \( f_s = 2W \).
3.2 Reconstruction from Sampling

Take $f_s = 2W$.

Ideal lowpass filter

\[ G_\delta(f) \]

\[ G(f) = \frac{1}{2W} G_\delta(f) \text{ for } |f| \leq W. \]
3.2 Aliasing due to Sampling

When \( f_s < 2W \), \( G(f) \) cannot be reconstructed by undersampled samples.
3.2 Second Important Conclusion from Sampling

- A band-limited signal of finite energy with bandwidth $W$ can be completely described by its samples of sampling rate $f_s \geq 2W$.
  - $2W$ is commonly referred to as the *Nyquist rate*.

- How to reconstruct a band-limited signal from its samples?
\[ g(t) = \int_{-\infty}^{\infty} G(f) \exp(j2\pi ft) df \]
\[ = \int_{-W}^{W} G(f) \exp(j2\pi ft) df \]
\[ = \frac{1}{f_s} \int_{-W}^{W} \left( \sum_{n=-\infty}^{\infty} g(nT_s) \exp(-j2\pi nf_s) \right) \exp(j2\pi ft) df \]
\[ = \frac{1}{f_s} \sum_{n=-\infty}^{\infty} g(nT_s) \int_{-W}^{W} \exp(j2\pi (t-nT_s)f) df \]
\[ = \frac{2W}{f_s} \sum_{n=-\infty}^{\infty} g(nT_s) \frac{\sin[2\pi W (t-nT_s)]}{2\pi W (t-nT_s)} \]
\[ = \sum_{n=-\infty}^{\infty} g(nT_s) \left( 2WT_s \text{sinc}[2W(t-nT_s)] \right) \]

\[ 2WT_s \text{sinc}[2W(t-nT_s)] \] plays the role of an interpolation function for samples.
3.2 Band-Unlimited Signals

- The signal encountered in practice is often not strictly band-limited.
- Hence, there is always “aliasing” after sampling.
- To combat the effects of aliasing, a low-pass anti-aliasing filter is used to attenuate the frequency components outside $[-f_s, f_s]$.
- In this case, the signal after passing the anti-aliasing filter is often treated as bandlimited with bandwidth $f_s/2$ (i.e., $f_s = 2W$). Hence,

$$g(t) = \sum_{n=-\infty}^{\infty} g(nT_s) \text{sinc} \left( \frac{t}{T_s} - n \right)$$
3.2 Interpolation in terms of Filtering

Observe that

\[ g(t) = \sum_{n=-\infty}^{\infty} g(nT_s) \text{sinc} \left( \frac{t}{T_s} - n \right) \]

is indeed a convolution between \( g_\delta(t) \) and \( \text{sinc}(t/T_s) \).

\[
g_\delta(t) * \text{sinc} \left( \frac{t}{T_s} \right) = \int_{-\infty}^{\infty} g_\delta(\tau) \text{sinc} \left( \frac{t-\tau}{T_s} \right) d\tau
\]

\[
= \int_{-\infty}^{\infty} \left( \sum_{n=-\infty}^{\infty} g(nT_s) \delta(\tau-nT_s) \right) \text{sinc} \left( \frac{t-\tau}{T_s} \right) d\tau
\]

\[
= \sum_{n=-\infty}^{\infty} g(nT_s) \int_{-\infty}^{\infty} \delta(\tau-nT_s) \text{sinc} \left( \frac{t-\tau}{T_s} \right) d\tau
\]
(Continue from the previous slide.)

\[ g_\delta(t) \ast \text{sinc} \left( \frac{t}{T_s} \right) = \sum_{n=-\infty}^{\infty} g(nT_s) \text{sinc} \left( \frac{t}{T_s} - n \right) \]

⇒ Reconstruction filter (interpolation filter) \( h(t) = \text{sinc} \left( \frac{t}{T_s} \right) \)

⇒ \( H(f) = T_s \text{rect}(T_s f) \)
3.2 Physical Realization of Reconstruction Filter

- An ideal lowpass filter is not physically realizable.
- Instead, we can use an anti-aliasing filter of bandwidth $W$, and use a sampling rate $f_s > 2W$. Then, the spectrum of a reconstruction filter can be shaped like:
Signal spectrum with bandwidth $W$

Signal spectrum after sampling with $f_s > 2W$

The physically realizable reconstruction filter

$g_\delta(t) * h_{\text{realizable}}(t) = G_\delta(f)H_{\text{realizable}}(f) \equiv G_\delta(f)H_{\text{ideal}}(f) \equiv g_\delta(t) * h_{\text{ideal}}(t)$
3.3 Pulse-Amplitude Modulation (PAM)

- **PAM**
  - The amplitude of regularly spaced pulses is varied in proportion to the corresponding sample values of a continuous message signal.

  Notably, the top of each pulse is maintained flat. So, this is PAM, not natural-sampling for which the message signal is directly multiplied by a periodic train of rectangular pulses.
3.3 Pulse-Amplitude Modulation (PAM)

- The operation of generating a PAM modulated signal is often referred to as “sample and hold.”

- This “sample and hold” process can also be analyzed through “filtering technique.”

\[
s(t) = \sum_{n=-\infty}^{\infty} m(nT_s)h(t - nT_s) = m_\delta(t) * h(t)
\]

where \( h(t) = \begin{cases} 
1, & 0 < t < T \\
1/2, & t = 0, t = T \\
0, & \text{otherwise}
\end{cases} \) and \( m_\delta(t) = \sum_{n=-\infty}^{\infty} m(nT_s)\delta(t - nT_s). \)
3.3 Pulse-Amplitude Modulation (PAM)

By taking “filtering” standpoint, the spectrum of $S(f)$ can be derived as:

$$S(f) = M_\delta(f)H(f)$$

$$= \left( f_s \sum_{k=-\infty}^{\infty} M(f - kf_s) \right)H(f)$$

$$= f_s \sum_{k=-\infty}^{\infty} M(f - kf_s)H(f)$$

- $M(f)$ is the message signal with bandwidth $W$ (or having experienced an anti-aliasing filter of bandwidth $W$).
- $f_s \geq 2W$. 
3.3 Pulse-Amplitude Modulation (PAM)

PAM signal \( s(t) \) → Reconstruction filter → Equalizer → Message signal \( m(t) \)

\[
S(f) = f_s \sum_{k=-\infty}^{\infty} M(f - kf_s)H(f)
\]

\[
= f_s M(f)H(f) + f_s \sum_{|k| \neq 1} M(f - kf_s)H(f)
\]

Reconstruction Filter \( \rightarrow M(f)H(f) \) → Equalizer \( \rightarrow M(f) \)
3.3 Feasibility of Equalizer Filter

- The distortion of $M(f)$ is due to $M(f)H(f)$,

\[
M(f)H(f) = \begin{cases} 
1, & 0 < t < T \\
1/2, & t = 0, t = T \text{ or } H(f) = T \text{sinc}(fT) \exp(-j\pi fT) \\
0, & \text{otherwise}
\end{cases}
\]

where $h(t) = \begin{cases} 
1, & 0 < t < T \\
1/2, & t = 0, t = T \text{ or } H(f) = T \text{sinc}(fT) \exp(-j\pi fT) \\
0, & \text{otherwise}
\end{cases}$

\[
E(f) = \begin{cases} 
1/H(f) = 1/T \text{sinc}(fT) \exp(j\pi fT), & |f| \leq W \\
0, & \text{otherwise}
\end{cases}
\]

**Question:** Is the above $E(f)$ feasible or realizable?
\[
\frac{1}{T} > \frac{1}{T_s} = f_s > 2W.
\]

\[
\tilde{E}(f) = \begin{cases} 
\frac{1}{T \text{sinc}(fT)}, & |f| \leq W \\
0, & \text{otherwise}
\end{cases}
\]

E.g., \( T = 1, \ W = 1/8. \)

This gives an equalizer:

\[
i(t) \xrightarrow{\tilde{E}(f)} o_1(t) \xrightarrow{\delta(t + T/2) \text{ or } \exp(j\pi fT)} o(t)
\]

A lowpass filter non-realizable! Why?

Because "\( o_1(t) = 0 \) for \( t < 0 \)" does not imply "\( o(t) = 0 \) for \( t < 0 \)."
3.3 Feasibility of Equalizer Filter

- Causal

A reasonable assumption for a feasible linear filter system is that:

For any \( i(t) \) satisfying \( i(t) = 0 \) for \( t < 0 \), we have \( o(t) = 0 \) for \( t < 0 \).

A necessary and sufficient condition for the above assumption to hold is that \( h(t) = 0 \) for \( t < 0 \).
Simplified Proof:

\[
\begin{align*}
\begin{cases}
    h(t) = 0 \text{ for } t < 0 \\
    i(t) = 0 \text{ for } t < 0
\end{cases}
\Rightarrow o(t) &= \int_{-\infty}^{\infty} h(\tau)i(t-\tau)\,d\tau = \int_{0}^{t} h(\tau)i(t-\tau)\,d\tau \\
\Rightarrow o(t) &= 0 \text{ for } t < 0
\end{align*}
\]

If \( \int_{-\infty}^{-a} h(t)\,dt \neq 0 \) for some \( a > 0 \), then take \( i(t) = \begin{cases} 
    0, & \text{for } t < 0; \\
    1, & \text{for } t \geq 0.
\end{cases} \)

\( \Rightarrow o(-a) = \int_{-\infty}^{-a} h(\tau)\,d\tau \neq 0 \), which means that there will be a nonzero output due to completely zero input! Therefore, \( \int_{-\infty}^{-a} h(\tau)\,d\tau = 0 \) for every \( a > 0 \).
3.3 Aperture Effect

- The distortion of $M(f)$ due to $M(f)H(f)$

\[ h(t) = \begin{cases} 
1, & 0 < t < T \\
1/2, & t = 0, t = T \\
0, & \text{otherwise}
\end{cases} \]

where $H(f) = T \text{sinc}(fT) \exp(-j\pi fT)$

is very similar to the distortion caused by the finite size of the scanning aperture in television. So, it is named the aperture effect.

- If $T/T_s \leq 0.1$, the amplitude distortion is less than 0.5%; hence, the equalizer may not be necessary.
\[ \tilde{E}(f) = \begin{cases} \frac{1}{T \text{sinc}(fT)}, & |f| \leq W \text{ and } \frac{1}{T} > \frac{1}{T_s} = f_s > 2W, \\ 0, & \text{otherwise} \end{cases} \]

\[ \Rightarrow \tilde{E}(f) = \begin{cases} \frac{1}{\text{sinc}(f)}, & |f| \leq 0.04 \text{ for } T = 1, T_s = 10, W = 0.04, \\ 0, & \text{otherwise} \end{cases} \]
3.3 Pulse-Amplitude Modulation

- Final notes on PAM
  - PAM is rather stringent in its system requirement, such as short duration of pulse.
  - Also, the noise performance of PAM may not be sufficient for long distance transmission.
  - Accordingly, PAM is often used as a mean of message processing for time-division multiplexing, from which conversion to some other form of pulse modulation is subsequently made. Details will be discussed in Section 3.9.
3.4 Other Forms of Pulse Modulation

- **Pulse-Duration Modulation (or Pulse-Width Modulation)**
  - Samples of the message signal are used to vary the duration of the pulses.

- **Pulse-Position Modulation**
  - The position of a pulse relative to its unmodulated time of occurrence is varied in accordance with the message signal.
Pulse trains

PDM

PPM
3.4 Other Forms of Pulse Modulation

- Comparisons between PDM and PPM
  - PPM is more power efficient because excessive pulse duration consumes considerable power.

- Final note
  - It is expected that PPM is immune to additive noise, since additive noise only perturbs the amplitude of the pulses rather than the positions.
  - However, since the pulse cannot be made perfectly rectangular in practice (namely, there exists a non-zero transition time in pulse edge), the detection of pulse positions is somehow still affected by additive noise.
3.5 Bandwidth-Noise Trade-Off

- PPM seems to be a better form for analog pulse modulation from noise performance standpoint. However, its noise performance is very similar to (analog) FM modulation as:
  - Its figure of merit is proportional to the square of transmission bandwidth (i.e., $1/T$) normalized with respect to the message bandwidth ($W$).
  - There exists a threshold effect as SNR is reduced.

- Question: Can we do better than the “square” law in figure-of-merit improvement? Answer: Yes, by means of Digital Communication, we can realize an “exponential” law (with respect to error rates)!

See Slide 2-161: figure-of-merit $\propto D^2 = \left(\frac{1}{2} \frac{B_{T,Carson}}{W} - 1\right)^2 = \left(\frac{1}{2} \frac{B_{n,Carson}}{W} - 1\right)^2$
3.6 Quantization Process

- Transform the continuous-amplitude $m = m(nT_s)$ to discrete approximate amplitude $v = v(nT_s)$

- Such a discrete approximate is adequately good in the sense that any human ear or eye can detect only finite intensity differences.
3.6 Quantization Process

- We may drop the time instance $nT_s$ for convenience, when the quantization process is memoryless and instantaneous (hence, the quantization at time $nT_s$ is not affected by earlier or later samples of the message signal.)

- Types of quantization
  - Uniform
    - Quantization step sizes are of equal length.
  - Non-uniform
    - Quantization step sizes are not of equal length.
An alternative classification of quantization

- **Midtread**
- **Midrise**

![Diagram of quantization classification](image)
3.6 Quantization Noise

Uniform midtread quantizer
3.6 Quantization Noise

- Define the quantization noise to be \( Q = M - V = M - g(M) \), where \( g(\ ) \) is the quantizer.

- Let the message \( M \) be uniformly distributed in \((-m_{\text{max}}, m_{\text{max}})\). So, \( M \) has zero mean.

- Assume \( g(\ ) \) is symmetric and of midrise type; then, \( V = g(M) \) also has zero-mean, and so does \( Q = M - V \).

- Then, the step size of the quantizer is given by:

\[
\Delta = \frac{2m_{\text{max}}}{L}
\]

where \( L \) is the total number of representation levels.
3.6 Quantization Noise

Assume that \( g( ) \) assigns the midpoint of each step interval to be the representation level. Then,

\[
V = M - M \mod \Delta + \frac{\Delta}{2} \\
\Rightarrow Q = M - V = M \mod \Delta - \frac{\Delta}{2}.
\]

\[
\Pr\{Q \leq q\} = \Pr\left\{(M \mod \Delta) - \frac{\Delta}{2} \leq q\right\} = \begin{cases} 
0, & q < -\frac{\Delta}{2} \\
\frac{q}{\Delta} + \frac{1}{2}, & -\frac{\Delta}{2} \leq q < \frac{\Delta}{2} \\
1, & q \geq \frac{\Delta}{2} 
\end{cases}
\]

Example.

\[
m_{\text{max}} = 1 \quad \Delta = \frac{1}{2} \\
L = 4
\]
3.6 Quantization Noise

- So, the output signal-to-noise ratio is equal to:

\[
SNR_O = \frac{P}{\int_{-\Delta/2}^{\Delta/2} q^2 \frac{1}{\Delta} dq} = \frac{P}{\frac{1}{12} \Delta^2} = \frac{P}{\frac{1}{12} \left( \frac{2m_{\text{max}}}{L} \right)^2} = \frac{3P}{m_{\text{max}}^2} L^2
\]

- The transmission bandwidth of a quantization system is conceptually proportional to the number of bits required per sample, i.e., \( R = \log_2(L) \).

- We then conclude that \( SNR_O \propto 4^R \), which increases exponentially with transmission bandwidth.
Example 3.1 Sinusoidal Modulating Signal

Let \( m(t) = A_m \cos(2\pi f_m t) \). Then

\[
P = \frac{A_m^2}{2} \quad \text{and} \quad m_{\text{max}} = A_m
\]

\[
\Rightarrow SNR_O = \frac{3(A_m^2/2)}{A_m^2} L^2 = \frac{3}{2} 4^R = 10 \log_{10}(3/2) + R \cdot 10 \log_{10}(4) \, \text{dB} \approx (1.8 + 6R) \, \text{dB}
\]

<table>
<thead>
<tr>
<th>( L )</th>
<th>( R )</th>
<th>( SNR_O , \text{(dB)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>5</td>
<td>31.8</td>
</tr>
<tr>
<td>64</td>
<td>6</td>
<td>37.8</td>
</tr>
<tr>
<td>128</td>
<td>7</td>
<td>43.8</td>
</tr>
<tr>
<td>256</td>
<td>8</td>
<td>49.8</td>
</tr>
</tbody>
</table>

* Note that in this example, we assume a full-load quantizer, in which no quantization loss is encountered due to saturation.
3.6 Quantization Noise

- In the previous analysis of quantization error, we assume the quantizer assigns the mid-point of each step interval to be the representative level.

- Questions:
  - Can the power of quantization noise be further reduced by adjusting the representative levels?
  - Can the power of quantization noise be further reduced by adopting a non-uniform quantizer?
3.6 Optimality of Scalar Quantizers

Let $d(m, v_k)$ be the distortion by representing $m$ by $v_k$.

**Goal:** To find $\{I_k\}$ and $\{v_k\}$ such that the average distortion $D = E[d(M, g(M))]$ is minimized.

\[ \bigcup_{k=1}^{L} I_k = [-A, A) \]

Notably, interval $I_k$ may not be a “consecutive” single interval.
3.6 Optimality of Scalar Quantizers

Solution:

\[
\min_{\{v_k\}} \min_{\{I_k\}} \min_{\{v_k\}} \min_{\{I_k\}} \sum_{k=1}^{L} \int_{I_k} d(m, v_k) f_M(m) dm
\]

(I) For fixed \(\{v_k\}\), determine the optimal \(\{I_k\}\).

(II) For fixed \(\{I_k\}\), determine the optimal \(\{v_k\}\).

(I) If \(d(m, v_k) \leq d(m, v_j)\), then \(m\) should be assigned to \(I_k\) rather than \(I_j\).

\[
\Rightarrow I_k = \left\{ m \in [-A, A] : d(m, v_k) \leq d(m, v_j) \text{ for all } 1 \leq j \leq L \right\}
\]
(II) For fixed \( \{I_k\} \), determine the optimal \( \{v_k\} \).

\[
\min_{\{v_k\}} \sum_{k=1}^{L} \int_{I_k} d(m, v_k) f_M(m) dm
\]

Since
\[
\frac{\partial}{\partial v_j} \left( \sum_{k=1}^{L} \int_{I_k} d(m, v_k) f_M(m) dm \right) = \frac{\partial}{\partial v_j} \left( \int_{I_j} d(m, v_j) f_M(m) dm \right)
\]
\[
= \int_{I_j} \frac{\partial d(m, v_j)}{\partial v_j} f_M(m) dm
\]

a necessary condition for the optimal \( v_j \) is:
\[
\int_{I_j} \frac{\partial d(m, v_j)}{\partial v_j} f_M(m) dm = 0.
\]

Lloyd-Max algorithm is to repetitively apply (I) and (II) for the search of the optimal quantizer.
Example: Mean-Square Distortion

\[ d(m, v_k) = (m - v_k)^2 \]

\[ (I) \ I_k = \{ m \in [-A, A) : (m - v_k)^2 \leq (m - v_j)^2 \text{ for all } 1 \leq j \leq L \} \]

should be a consecutive interval.

<table>
<thead>
<tr>
<th>Representation level</th>
<th>( v_1 )</th>
<th>( v_2 )</th>
<th>...</th>
<th>( v_{L-1} )</th>
<th>( v_L )</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Partitions</th>
<th>( I_1 )</th>
<th>( I_2 )</th>
<th>...</th>
<th>( I_{L-1} )</th>
<th>( I_L )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_1 = -A )</td>
<td>( m_2 )</td>
<td>( m_3 )</td>
<td>...</td>
<td>( m_{L-1} )</td>
<td>( m_L )</td>
</tr>
<tr>
<td>( m_L + 1 = +A )</td>
<td>( 2A )</td>
<td>( ... )</td>
<td>( ... )</td>
<td>( ... )</td>
<td>( ... )</td>
</tr>
</tbody>
</table>

\( \\)
Example: Mean-Square Distortion

(II) A necessary condition for the optimal $v_j$ is:

$$\int_{m_j}^{m_{j+1}} \frac{\partial (m - v_j)^2}{\partial v_j} f_M(m) dM = -2 \int_{m_j}^{m_{j+1}} (m - v_j) f_M(m) dm = 0.$$ 

$$\Rightarrow v_{j,\text{optimal}} = \frac{\int_{m_j}^{m_{j+1}} mf_M(m) dm}{\int_{m_j}^{m_{j+1}} f_M(m) dm} = E[M \mid m_j \leq M < m_{j+1}]$$

Exercise: What is the best $\{m_k\}$ and $\{v_k\}$ if $M$ is uniformly distributed over $[-A,A)$.

Hint: $\min_{\{I_k\}} \min_{\{v_k\}} D = \frac{1}{2A} \min_{\{m_k\}} \sum_{k=1}^{L} \int_{m_k}^{m_{k+1}} \left( m - \frac{m_k + m_{k+1}}{2} \right)^2 dm$. 

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Chapter 3-45
3.7 Pulse-Code Modulation

(a) Transmitter

(b) Receiver

Source of continuous-time message signal → Low-pass filter → Sampler → Quantizer → Encoder → PCM signal applied to channel input

Final channel output → Regeneration circuit → Decoder → Reconstruction filter → Destination
3.7 Pulse-Code Modulation

- Non-uniform quantizers used for telecommunication (ITU-T G.711)
  - ITU-T G.711: Pulse Code Modulation (PCM) of Voice Frequencies (1972)
    - It consists of two laws: A-law (mainly used in Europe) and $\mu$-law (mainly used in US and Japan)
    - This design helps to protect weak signal, which occurs more frequently in, say, human voice.
3.7 Laws

- Quantization Laws
  - A-law
    - 13-bit uniformly quantized
    - Conversion to 8-bit code
  - μ-law
    - 14-bit uniformly quantized
    - Conversion to 8-bit code.

These two are referred to as *compression laws* since they use 8-bit to (lossily) represent 13-(or 14-)bit information.
3.7 A-law in G.711

A-law (A=87.6)

\[ F_{\text{A-law}}(m) = \begin{cases} 
  \frac{A}{1 + \log(A)} m, & |m| \leq \frac{1}{A} \\
  \text{sgn}(m) \left[ \frac{1 + \log(A|m|)}{1 + \log(A)} \right], & \frac{1}{A} \leq |m| \leq 1 
\end{cases} \]

Linear mapping

Logarithmic mapping
$F_{A\text{-law}}(m)$
8 bit PCM code

A piecewise linear approximation to the law.

$F_{A\text{-law}}(m)$

13 bit uniform quantization
Compressor of A-law (assume nonnegative $m$)

<table>
<thead>
<tr>
<th>Input Values</th>
<th>Compressed Code Word</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bits:</td>
<td>Chord</td>
</tr>
<tr>
<td>11 10 9 8 7 6 5 4 3 2 1 0</td>
<td>Bits: 6 5 4 3 2 1 0</td>
</tr>
<tr>
<td>0 0 0 0 0 0 0 a b c d x</td>
<td>0 0 0 a b c d</td>
</tr>
<tr>
<td>0 0 0 0 0 0 1 a b c d x</td>
<td>0 0 1 a b c d</td>
</tr>
<tr>
<td>0 0 0 0 0 1 a b c d x x</td>
<td>0 1 0 a b c d</td>
</tr>
<tr>
<td>0 0 0 0 1 a b c d x x x</td>
<td>0 1 1 a b c d</td>
</tr>
<tr>
<td>0 0 0 1 a b c d x x x x</td>
<td>1 0 0 a b c d</td>
</tr>
<tr>
<td>0 0 1 a b c d x x x x x</td>
<td>1 0 1 a b c d</td>
</tr>
<tr>
<td>0 1 a b c d x x x x x x</td>
<td>1 1 0 a b c d</td>
</tr>
<tr>
<td>1 a b c d x x x x x x x</td>
<td>1 1 1 a b c d</td>
</tr>
</tbody>
</table>

E.g. $(3968)_{10} \rightarrow (1111,1000,0000)_2 \rightarrow (111,1111)_2 \rightarrow (127)_{10}$

E.g. $(2176)_{10} \rightarrow (1000,1000,0000)_2 \rightarrow (111,0001)_2 \rightarrow (113)_{10}$
Expander of A-law (assume nonnegative $m$)

<table>
<thead>
<tr>
<th>Compressed Code Word</th>
<th>Raised Output Values</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Chord</strong></td>
<td><strong>Step</strong></td>
</tr>
<tr>
<td>0 0 0 a b c d</td>
<td>0 0 0 0 0 0 0 a b c d 1</td>
</tr>
<tr>
<td>0 0 1 a b c d</td>
<td>0 0 0 0 0 0 1 a b c d 1</td>
</tr>
<tr>
<td>0 1 0 a b c d</td>
<td>0 0 0 0 0 1 a b c d 1 0</td>
</tr>
<tr>
<td>0 1 1 a b c d</td>
<td>0 0 0 1 a b c d 1 0 0</td>
</tr>
<tr>
<td>1 0 0 a b c d</td>
<td>0 0 1 a b c d 1 0 0 0</td>
</tr>
<tr>
<td>1 0 1 a b c d</td>
<td>0 1 a b c d 1 0 0 0 0</td>
</tr>
<tr>
<td>1 1 0 a b c d</td>
<td>1 a b c d 1 0 0 0 0 0</td>
</tr>
<tr>
<td>1 1 1 a b c d</td>
<td></td>
</tr>
</tbody>
</table>

E.g. $(113)_{10} \rightarrow (111,0001)_2 \rightarrow (1000,1100,0000)_2 \rightarrow (2240)_{10}$

In other words, \[
\frac{(1001,0000,0000)_2 + (1000,1000,0000)_2}{2} = \frac{(2304)_10 + (2176)_{10}}{2} = (2240)_{10}
\]
3.7 $\mu$-law in G.711

$\mu$-law ($\mu = 255$)

$$F_{\mu\text{-law}}(m) = \text{sgn}(m) \frac{\log(1 + \mu|m|)}{1 + \log(\mu)} \quad \text{for } |m| \leq 1.$$ 

- It is approximately linear at low $m$.
- It is approximately logarithmic at large $m$. 
$F_{\mu\text{-law}}(m)$

![Graph showing $F_{\mu\text{-law}}(m)$]
14 bit uniform quantization ($2^{13} = 8192$)
Compressor of $\mu$-law (assume nonnegative $m$)

Raised Input = Input + (33)$_{10}$ = Input + 21H
(For negative $m$, the raised input becomes (input – 33).)
An additional 7th bit is used to indicate whether the input signal is positive (1) or negative (0).

<table>
<thead>
<tr>
<th>Raised Input Values</th>
<th>Compressed Code Word</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Chord</td>
</tr>
<tr>
<td>Bits: 12 11 10 9 8 7 6 5 4 3 2 1 0</td>
<td>Bits: 6 5 4 3 2 1 0</td>
</tr>
<tr>
<td>---------------------</td>
<td>----------------------</td>
</tr>
<tr>
<td>0 0 0 0 0 0 0 1 a b c d x</td>
<td>0 0 0 a b c d</td>
</tr>
<tr>
<td>0 0 0 0 0 0 1 a b c d x x</td>
<td>0 0 1 a b c d</td>
</tr>
<tr>
<td>0 0 0 0 0 1 a b c d x x x</td>
<td>0 1 0 a b c d</td>
</tr>
<tr>
<td>0 0 0 0 1 a b c d x x x x</td>
<td>0 1 1 a b c d</td>
</tr>
<tr>
<td>0 0 0 1 a b c d x x x x x</td>
<td>1 0 0 a b c d</td>
</tr>
<tr>
<td>0 0 1 a b c d x x x x x x</td>
<td>1 0 1 a b c d</td>
</tr>
<tr>
<td>0 1 a b c d x x x x x x x</td>
<td>1 1 0 a b c d</td>
</tr>
<tr>
<td>1 a b c d x x x x x x x x</td>
<td>1 1 1 a b c d</td>
</tr>
</tbody>
</table>
Expander of $\mu$-law (assume nonnegative $m$)

<table>
<thead>
<tr>
<th>Compressed Code Word</th>
<th>Raised Output Values</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Chord</strong></td>
<td><strong>Step</strong></td>
</tr>
<tr>
<td>0 0 0 a b c d</td>
<td></td>
</tr>
<tr>
<td>0 0 1 a b c d</td>
<td></td>
</tr>
<tr>
<td>0 1 0 a b c d</td>
<td></td>
</tr>
<tr>
<td>0 1 1 a b c d</td>
<td></td>
</tr>
<tr>
<td>1 0 0 a b c d</td>
<td></td>
</tr>
<tr>
<td>1 0 1 a b c d</td>
<td></td>
</tr>
<tr>
<td>1 1 0 a b c d</td>
<td></td>
</tr>
<tr>
<td>1 1 1 a b c d</td>
<td></td>
</tr>
</tbody>
</table>

Output = Raised Output – 33

Note that the combination of a compressor and an expander is called a compander.
Comparison of $A$-law and $\mu$-law specified in G.711.
3.7 Coding

- After the quantizer provides a symbol, representing one of 256 possible levels (8 bits of information) at each sampled time, the encoder will transform the symbol (or several symbols) into a code character (or code word) that is suitable for transmission over a noisy channel.

- Example. Binary code.

```
11100100
```

0 = change
1 = unchange

Diagram:

```
1 1 1 0 0 1 0 0
```

Diagram:

```
1 1 1 0 0 1 0 0
```

Diagram:

```
1 1 1 0 0 1 0 0
```

Diagram:

```
1 1 1 0 0 1 0 0
```

Diagram:

```
1 1 1 0 0 1 0 0
```

Diagram:

```
1 1 1 0 0 1 0 0
```

Diagram:

```
1 1 1 0 0 1 0 0
```

Diagram:

```
1 1 1 0 0 1 0 0
```

Diagram:

```
1 1 1 0 0 1 0 0
```

Diagram:

```
1 1 1 0 0 1 0 0
```

Diagram:

```
1 1 1 0 0 1 0 0
```

Diagram:

```
1 1 1 0 0 1 0 0
```

Diagram:

```
1 1 1 0 0 1 0 0
```

Diagram:

```
1 1 1 0 0 1 0 0
```

Diagram:

```
1 1 1 0 0 1 0 0
```
3.7 Coding

Example. Ternary code (Pseudo-binary code).

00011011→ACABBCBB

Through the help of coding, the receiver may be able to detect (or even correct) the transmission errors due to noise. For example, it is impossible to receive ABABBABB, since this is not a legitimate code word (character).
3.7 Coding

Example of error correcting code: *Three-times repetition code* (to protect Bluetooth packet header).

\[00011011 \rightarrow 000,000,000,111,111,000,111,111\]

Then, the so-called *majority law* can be applied at the receiver to correct one-bit error.

*Channel* (error correcting) *codes* are designed to compensate the channel noise, while *line codes* are simply used as the electrical representation of a binary data stream over the electrical line.
3.7 Line Codes

(a) Unipolar nonreturn-to-zero (NRZ) signaling
(b) Polar nonreturn-to-zero (NRZ) signaling
(c) Unipolar return-to-zero (RZ) signaling
(d) Bipolar return-to-zero (BRZ) signaling
(e) Split-phase (Manchester code)
3.7 Derivation of PSD

From Slide 1-117, we obtain that the general formula for PSD is:

$$\overline{\text{PSD}} = \lim_{T \to \infty} \frac{1}{2T} E[S(f)S_{2T}^*(f)],$$

where $$s_{2T}(t) = s(t) \cdot 1\{|t| \leq T\}$$.

For a line coded signal, $$s(t) = \sum_{n=-\infty}^{\infty} a_n g(t-nT_b)$$, where $$g(t) = 0$$ outside $$[0, T_b)$$.

Hence, $$S(f) = G(f) \sum_{n=-\infty}^{\infty} a_n e^{-j2\pi fnT_b}$$ and $$S_{2NT_b}(f) = G(f) \sum_{n=-N}^{N-1} a_n e^{-j2\pi fnT_b}$$.

$$\Rightarrow \overline{\text{PSD}} = \lim_{N \to \infty} \frac{1}{2NT_b} |G(f)|^2 \left( \sum_{n=-\infty}^{\infty} \sum_{m=-N}^{N-1} E[a_n a_m^*] e^{-j2\pi f(n-m)T_b} \right).$$
For i.i.d. \( \{a_n\} \),

\[
\frac{1}{T_b} \left( \sum_{k=-\infty}^{\infty} \phi_a(k) e^{-j2\pi fk T_b} \right) = \frac{\sigma_a^2}{T_b} + \frac{\mu_a^2}{T_b} \sum_{k=-\infty}^{\infty} e^{-j2\pi fk T_b}
\]

(See Slide 3-4.)

\[
\frac{1}{T_b} \left( \sum_{k=-\infty}^{\infty} \phi_a(k) e^{-j2\pi fk T_b} \right) = \frac{\sigma_a^2}{T_b} + \frac{\mu_a^2}{T_b} \sum_{k=-\infty}^{\infty} \delta(f - k/T_b)
\]
3.7 Power Spectral of Line Codes

- **Unipolar nonreturn-to-zero (NRZ) signaling**
  - Also named *on-off signaling*.
  - Disadvantage: Waste of power due to the non-zero-mean nature (i.e., PSD does not approach zero at zero frequency).

\[ s(t) = \sum_{n=-\infty}^{\infty} a_n g(t - nT_b), \]
\[ \text{where } \begin{cases} 
\{a_n\}_{n=-\infty}^{\infty} \text{ is zero/one i.i.d.,} \\
\left\{ \begin{array}{ll} A, & 0 \leq t < T_b \\
0, & \text{otherwise} \end{array} \right. 
\end{cases} \]

![Diagram showing binary data and corresponding waveforms](image-url)
3.7 Power Spectral of Line Codes

- PSD of Unipolar NRZ

\[
\text{PSD}_{\text{U-NRZ}} = |G(f)|^2 \left( \frac{\sigma_a^2}{T_b} + \frac{\mu_a^2}{T_b^2} \sum_{k=-\infty}^{\infty} \delta(f - k/T_b) \right)
\]

\[
= A^2 T_b^2 \text{sinc}^2(f T_b) \left( \frac{\sigma_a^2}{T_b} + \frac{\mu_a^2}{T_b^2} \sum_{k=-\infty}^{\infty} \delta(f - k/T_b) \right)
\]

\[
= \frac{A^2 T_b^2}{4} \text{sinc}^2(f T_b) \left( 1 + \frac{1}{T_b} \sum_{k=-\infty}^{\infty} \delta(f - k/T_b) \right)
\]

\[
= \frac{A^2 T_b}{4} \text{sinc}^2(f T_b) \left( 1 + \frac{1}{T_b} \delta(f) \right)
\]
3.7 Power Spectral of Line Codes

- Polar nonreturn-to-zero (NRZ) signaling
  - The previous PSD of Unipolar NRZ suggests that a zero-mean data sequence is preferred.

\[ s(t) = \sum_{n=-\infty}^{\infty} a_n g(t - nT_b), \text{ where } \begin{cases} \{a_n\}_{n=-\infty}^{\infty} \text{ is } \pm 1 \text{ i.i.d.}, \\ g(t) = \begin{cases} A, & 0 \leq t < T_b \\ 0, & \text{otherwise} \end{cases} \end{cases} \]

\[ \text{PSD}_{P-NRZ} = |G(f)|^2 \left( \frac{\sigma_a^2}{T_b} + \frac{\mu_a^2}{T_b^2} \sum_{k=-\infty}^{\infty} \delta(f - k/T_b) \right) \]

\[ = A^2 T_b \text{sinc}^2(fT_b) \]
3.7 Power Spectral of Line Codes

- Unipolar return-to-zero (RZ) signaling
  - An attractive feature of this line code is the presence of delta functions at $f = -1/T_b$, 0, $1/T_b$ in the PSD, which can be used for bit-timing recovery at the receiver.
  - Disadvantage: It requires 3dB more power than polar return-to-zero signaling.

$$s(t) = \sum_{n=-\infty}^{\infty} a_n g(t - nT_b),$$

where

$$\begin{cases} 
\{a_n\}_{n=-\infty}^{\infty} \text{ is zero/one i.i.d.}, \\
g(t) = \begin{cases} 
A, & 0 \leq t < T_b / 2 \\
0, & \text{otherwise}
\end{cases}
\end{cases}$$
3.7 Power Spectral of Line Codes

- PSD of Unipolar RZ

\[
\text{PSD}_{\text{U-RZ}} = |G(f)|^2 \left( \frac{\sigma_a^2}{T_b} + \frac{\mu_a^2}{T_b^2} \sum_{k=-\infty}^{\infty} \delta(f - k/T_b) \right)
\]

\[
= \frac{A^2 T_b^2}{4} \text{sinc}^2 \left( \frac{f T_b}{2} \right) \left( \frac{\sigma_a^2}{T_b} + \frac{\mu_a^2}{T_b^2} \sum_{k=-\infty}^{\infty} \delta(f - k/T_b) \right)
\]

\[
= \frac{A^2 T_b}{16} \text{sinc}^2 \left( \frac{f T_b}{2} \right) \left( 1 + \frac{1}{T_b} \sum_{k=-\infty}^{\infty} \delta(f - k/T_b) \right)
\]

\[
= \frac{A^2 T_b}{16} \text{sinc}^2 \left( \frac{f T_b}{2} \right) \left( 1 + \frac{1}{T_b} \sum_{k=\infty}^{\infty} \delta(f - k/T_b) \right)
\]
3.7 Power Spectral of Line Codes

- Bipolar return-to-zero (BRZ) signaling
  - Also named alternate mark inversion (AMI) signaling
  - No DC component and relatively insignificant low-frequency components in PSD.

\[ s(t) = \sum_{n=-\infty}^{\infty} a_n g(t - nT_b), \text{ where } g(t) = \begin{cases} A, & 0 \leq t < T_b / 2 \\ 0, & \text{otherwise} \end{cases} \]

<table>
<thead>
<tr>
<th>Binary data</th>
<th>0 1 1 0 1 0 0 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1 1 1 0 1 0 0 1</td>
</tr>
<tr>
<td>A</td>
<td>[ \begin{array}{c} \hline \end{array} \text{Pattern of BRZ signaling} ]</td>
</tr>
</tbody>
</table>
3.7 Power Spectral of Line Codes

- PSD of BRZ
  - \{a_n\} is no longer i.i.d.

\[
E[a_n^2] = (0) \frac{1}{2} + (-1)^2 \frac{1}{4} + (+1)^2 \frac{1}{4} = \frac{1}{2}
\]

\[
E[a_n a_{n+1}] = (-1) \frac{1}{4} = -\frac{1}{4}
\]

\[
E[a_n a_{n+2}] = (1)(1) \frac{1}{16} + (1)(-1) \frac{1}{16} + (-1)(1) \frac{1}{16} + (-1)(-1) \frac{1}{16} = 0
\]

\[ \vdots \]

\[
E[a_n a_{n+m}] = 0 \text{ for } m > 1.
\]
3.7 Power Spectral of Line Codes

\[ \text{PSD}_{\text{BRZ}} = |G(f)|^2 \frac{1}{T_b} \left( \sum_{k=-\infty}^{\infty} \phi_a(k) e^{-j2\pi fkT_b} \right) \]

\[ = \frac{A^2 T_b^2}{4} \text{sinc}^2 \left( \frac{fT_b}{2} \right) \cdot \frac{1}{T_b} \left( -\frac{1}{4} e^{j2\pi fT_b} + \frac{1}{2} - \frac{1}{4} e^{-j2\pi fT_b} \right) \]

\[ = \frac{A^2 T_b^2}{4} \text{sinc}^2 \left( \frac{fT_b}{2} \right) \cdot \frac{1}{T_b} \left( \frac{1}{2} - \frac{1}{2} \cos(2\pi fT_b) \right) \]

\[ = \frac{A^2 T_b}{4} \text{sinc}^2 \left( \frac{fT_b}{2} \right) \sin^2(\pi fT_b) \]
3.7 Power Spectral of Line Codes

- Split-phase (Manchester code)
  - This signaling suppressed the DC component, and has relatively insignificant low-frequency components, regardless of the signal statistics.
  - Notably, for P-NRZ and BRZ, the DC component is suppressed only when the signal has the right statistics.

\[
s(t) = \sum_{n=-\infty}^{\infty} a_n g(t - nT_b), \quad \text{where}
\begin{align*}
\{a_n\}_{n=-\infty}^{\infty} & \text{ is } \pm 1 \text{ i.i.d.,} \\
g(t) & = \begin{cases} 
A, & 0 \leq t < T_b / 2 \\
-A, & T_b / 2 \leq t < T_b \\
0, & \text{otherwise}
\end{cases}
\end{align*}
\]
3.7 Power Spectral of Line Codes

- PSD of Manchester code

\[
\text{PSD}_{\text{Manchester}} = |G(f)|^2 \left( \frac{\sigma_a^2}{T_b} + \frac{\mu_a^2}{T_b^2} \sum_{k=-\infty}^{\infty} \delta(f - k/T_b) \right)
\]

\[
= A^2 T_b^2 \text{sinc}^2 \left( \frac{f T_b}{2} \right) \sin^2 \left( \frac{\pi f T_b}{2} \right) \left( \frac{\sigma_a^2}{T_b} + \frac{\mu_a^2}{T_b^2} \sum_{k=-\infty}^{\infty} \delta(f - k/T_b) \right)
\]

\[
= A^2 T_b \text{sinc}^2 \left( \frac{f T_b}{2} \right) \sin^2 \left( \frac{\pi f T_b}{2} \right)
\]
Let $T_b=1$, and adjust $A$ such that the total power of each line code is 1. This gives a fair comparison among line codes.

\[
\text{power}= \frac{1}{2} \quad \text{PSD}_{U-NRZ} = \frac{1}{2} \text{sinc}^2(f) + \frac{1}{2} \delta(f) \quad A = \sqrt{2}
\]

\[
\text{power}= 1 \quad \text{PSD}_{P-NRZ} = \text{sinc}^2(f) \quad A = 1
\]

\[
\text{power}= \frac{1}{2} \quad \text{PSD}_{U-RZ} = \frac{1}{4} \text{sinc}^2\left(\frac{f}{2}\right) + \frac{1}{4} \sum_{k=-\infty}^{\infty} \text{sinc}^2\left(\frac{k}{2}\right) \delta(f - k) \quad A = 2
\]

\[
\text{power}= 1 \quad \text{PSD}_{BRZ} = \text{sinc}^2\left(\frac{f}{2}\right) \sin^2(\pi f) \quad A = 2
\]

\[
\text{power}= 1 \quad \text{PSD}_{Manchester} = \text{sinc}^2\left(\frac{f}{2}\right) \sin^2\left(\frac{\pi f}{2}\right) \quad A = 1
\]
3.7 Differential Encoding with Unipolar NRZ Line Coding

- 1 = no change and 0 = change.

(a) Original binary data

\[ 0 \ 1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ \bar{o}_n \]

(b) Differentially encoded data

\[ 1 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 1 \ 1 \ \bar{d}_n \]

(c) Waveform

\[ d_n = d_{n-1} \oplus \bar{o}_n = d_{n-1} \ominus \bar{o}_n \]
3.7 Regeneration

- Regenerative repeater for PCM system
  - It can completely remove the distortion if the decision making device makes the right decision (on 1 or 0).
3.7 Decoding & Filtering

- After regenerating the received pulse at the last stage, the receiver decodes and regenerates the original message signal (with acceptable quantization error).

- Finally, a lowpass reconstruction filter whose cutoff frequency is equal to the message bandwidth $W$ is applied at the end (to remove the unnecessary high-frequency components due to “quantization”).
3.8 Noise Consideration in PCM Systems

- Two major noise sources in PCM systems
  - (Message-independent) Channel noise
  - (Message-dependent) Quantization noise

- The quantization noise is often under designer’s control, and can be made negligible by taking adequate number of quantization levels.
3.8 Noise Consideration in PCM Systems

- The main effect of channel noise is to introduce bit errors.
  - Notably, the *symbol error rate* is quite different from the *bit error rate*.
  - A symbol error may be caused by one-bit error, or two-bit error, or three-bit error, or …; so, in general, one cannot derive the *symbol error rate* from the *bit error rate* (or vice versa) unless some special assumption is made.
  - Considering the reconstruction of original analog signal, a bit error in the most significant bit is more harmful than a bit error in the least significant bit.
3.8 Error Threshold

- \( E_b/N_0 \)
  - \( E_b \): Transmitted signal energy per information bit
  - E.g., information bit is encoded using three-times repetition code, in which each code bit is transmitted using one BPSK symbol with symbol energy \( E_c \).
  - Then \( E_b = 3 \ E_c \).

- \( N_0 \): One-sided noise spectral density
  - The bit error rate (BER) is a function of \( E_b/N_0 \) and transmission speed (and implicitly bandwidth, etc).
3.8 Error Threshold

- Influence of $E_b/N_0$ on BER at $10^5$ bit per second (bps)

<table>
<thead>
<tr>
<th>$E_b/N_0$ (dB)</th>
<th>BER</th>
<th>About one bit error in every ...</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.3</td>
<td>$10^{-2}$</td>
<td>$10^{-3}$ second</td>
</tr>
<tr>
<td>8.4</td>
<td>$10^{-4}$</td>
<td>$10^{-1}$ second</td>
</tr>
<tr>
<td>10.6</td>
<td>$10^{-6}$</td>
<td>10 seconds</td>
</tr>
<tr>
<td>12.0</td>
<td>$10^{-8}$</td>
<td>20 minutes</td>
</tr>
<tr>
<td>13.0</td>
<td>$10^{-10}$</td>
<td>1 day</td>
</tr>
<tr>
<td>14.0</td>
<td>$10^{-12}$</td>
<td>3 months</td>
</tr>
</tbody>
</table>

- The usual requirement of BER in practice is $10^{-5}$. 
3.8 Error Threshold

- Error threshold
  - The minimum $E_b/N_0$ to achieves the required BER.

- By knowing the error threshold, one can always add a regenerative repeater when $E_b/N_0$ is about to drop below the threshold; hence, long-distance transmission becomes feasible.
  - Unlike the analog transmission, of which the distortion will accumulate for long-distance transmission.
3.9 Time-Division Multiplexing

- An important feature of sampling process is a *conservation-of-time*.
  - In principle, the communication link is used only at the sampling time instances.
- Hence, it may be feasible to put other message’s samples between adjacent samples of this message on a time-shared basis.
- This forms the time-division multiplex (TDM) system.
  - A joint utilization of a common communication link by a plurality of independent message sources.
3.9 Time-Division Multiplexing

The commutator (1) takes a narrow sample of each of the $N$ input messages at a rate $f_s$ slightly higher than $2W$, where $W$ is the cutoff frequency of the anti-aliasing filter, and (2) interleaves these $N$ samples inside the sampling interval $T_s$. 
The price we pay for TDM is that $N$ samples be squeezed in a time slot of duration $T_s$. 
3.9 Time-Division Multiplexing

- Synchronization is essential for a satisfactory operation of the TDM system.

- One possible procedure to synchronize the transmitter clock and the receiver clock is to set aside a code element or pulse at the end of a frame, and to transmit this pulse every other frame only.
Example 3.2 The T1 System

- T1 system
  - Carries 24 64kbps voice channels with regenerative repeaters spaced at approximately 2-km intervals.
  - Each voice signal is essentially limited to a band from 300 to 3100 Hz.
    - Anti-aliasing filter with $W = 3.1$ KHz
    - Sampling rate = 8 KHz ($> 2W = 6.2$ KHz)
  - ITU G.711 μ-law is used with $\mu = 255$.
  - Each frame consists of $24 \times 8 + 1 = 193$ bits, where a single bit is added at the end of the frame for the purpose of synchronization.
Example 3.2 The T1 System

- In addition to the 193 bits per frame (i.e., 1.544 Megabits per second), a telephone system must also pass signaling information such as “dial pulses” and “on/off-hook.”

- The least significant bit of each voice channel is deleted in every sixth frame, and a signaling bit is inserted in its place.

DSI Frame (24 DS0) (DS=Digital Signal)
3.10 Digital Multiplexers

The introduction of digital multiplexer enables us to combine digital signals of various natures, such as computer data, digitized voice signals, digitized facsimile and television signals.
3.10 Digital Multiplexers

The multiplexing of digital signals is accomplished by using a *bit-by-bit interleaving* procedure with a selector switch that sequentially takes a (or more) bit from each incoming line and then applies it to the high-speed common line.

![Diagram of Pulse Code Modulation (PCM) and Time Division Muxing (TDM)](image)

- 4 kHz ANALOG Voice IN
- 64 kbps DIGITAL Voice OUT
- Pulse Code Modulation (PCM)
- Time Division Muxing (TDM)
- (3 channels/frame)
3.10 Digital Multiplexers

Digital multiplexers are categorized into two major groups.

1. 1st Group: Multiplex digital computer data for TDM transmission over public switched telephone network.
   - Require the use of modem technology.

2. 2nd Group: Multiplex low-bit-rate digital voice data into high-bit-rate voice stream.
   - Accommodate in the hierarchy that is varying from one country to another.
   - Usually, the hierarchy starts at 64 Kbps, named a *digital signal zero* (DS0).
3.10 North American Digital TDM Hierarchy

- **The first level hierarchy**
  - Combine 24 DS0 to obtain a primary rate DS1 at 1.544 Mb/s (T1 transmission)

- **The second-level multiplexer**
  - Combine 4 DS1 to obtain a DS2 with rate 6.312 Mb/s

- **The third-level multiplexer**
  - Combine 7 DS2 to obtain a DS3 at 44.736 Mb/s

- **The fourth-level multiplexer**
  - Combine 6 DS3 to obtain a DS4 at 274.176 Mb/s

- **The fifth-level multiplexer**
  - Combine 2 DS4 to obtain a DS5 at 560.160 Mb/s
3.10 North American Digital TDM Hierarchy

- The combined bit rate is higher than the multiple of the incoming bit rates because of the addition of *bit stuffing* and *control signals*.
3.10 North American Digital TDM Hierarchy

- Basic problems involved in the design of multiplexing system
  - Synchronization should be maintained to properly recover the interleaved digital signals.
  - Framing should be designed so that individual can be identified at the receiver.
  - Variation in the bit rates of incoming signals should be considered in the design.
  - A 0.01% variation in the propagation delay produced by a 1°F decrease in temperature will result in 100 fewer pulses in the cable of length 1000-km with each pulse occupying about 1 meter of the cable.
3.10 Digital Multiplexers

- Synchronization and rate variation problems are resolved by *bit stuffing*.

- Example 3.3. AT&T M12 (second-level multiplexer)
  - 24 control bits are stuffed, and separated by sequences of 48 data bits (12 from each DS1 input).

```
\[ \begin{align*}
\quad M_0 & \quad [48] & \quad C_1 & \quad [48] & \quad F_0 & \quad [48] & \quad C_1 & \quad [48] & \quad C_1 & \quad [48] & \quad F_1 & \quad [48] \\
M_1 & \quad [48] & \quad C_{11} & \quad [48] & \quad F_0 & \quad [48] & \quad C_{11} & \quad [48] & \quad C_{11} & \quad [48] & \quad F_1 & \quad [48] \\
M_1 & \quad [48] & \quad C_{111} & \quad [48] & \quad F_0 & \quad [48] & \quad C_{111} & \quad [48] & \quad C_{111} & \quad [48] & \quad F_1 & \quad [48] \\
M_1 & \quad [48] & \quad C_{1V} & \quad [48] & \quad F_0 & \quad [48] & \quad C_{1V} & \quad [48] & \quad C_{1V} & \quad [48] & \quad F_1 & \quad [48]
\end{align*} \]
```

| \( M_0 \) | \[48\] | \( C_1 \) | \[48\] | \( F_0 \) | \[48\] | \( C_1 \) | \[48\] | \( C_1 \) | \[48\] | \( F_1 \) | \[48\] |
| \( M_1 \) | \[48\] | \( C_{11} \) | \[48\] | \( F_0 \) | \[48\] | \( C_{11} \) | \[48\] | \( C_{11} \) | \[48\] | \( F_1 \) | \[48\] |
| \( M_1 \) | \[48\] | \( C_{111} \) | \[48\] | \( F_0 \) | \[48\] | \( C_{111} \) | \[48\] | \( C_{111} \) | \[48\] | \( F_1 \) | \[48\] |
| \( M_1 \) | \[48\] | \( C_{1V} \) | \[48\] | \( F_0 \) | \[48\] | \( C_{1V} \) | \[48\] | \( C_{1V} \) | \[48\] | \( F_1 \) | \[48\] |

- Subframe markers
- First stuffing indicators
- Frame markers
- Second stuffing indicators
- Third stuffing indicators
- Frame markers
- Stuffed bits
DS2 M-Subframe

One DS2 M-Subframe; 294 bits (6 Blocks)
Example 3.3 AT&T M12 Multiplexer

- The control bits are labeled $F$, $M$, and $C$.
  - Frame markers: In sequence of $F_0F_1F_0F_1F_0F_1F_0F_1$, where $F_0 = 0$ and $F_1 = 1$.
  - Subframe markers: In sequence of $M_0M_1M_1M_1$, where $M_0 = 0$ and $M_1 = 1$.
  - Stuffing indicators: In sequences of $C_I C_I C_I C_{II} C_{II} C_{III} C_{III} C_{III} C_{IV} C_{IV} C_{IV} C_{IV}$, where all three bits of $C_j$ equal 1’s indicate that a stuffing bit is added in the position of the first information bit associated with the first DS1 bit stream that follows the $F_1$-control bit in the same subframe, and three 0’s in $C_jC_jC_j$ imply no stuffing.
- The receiver should use majority law to check whether a stuffing bit is added.
Example 3.3 AT&T M12 Multiplexer

These stuffed bits can be used to balance (or maintain) the nominal input bit rates and nominal output bit rates.

- $S = \text{nominal bit stuffing rate}$
  - The rate at which stuffing bits are inserted when both the input and output bit rates are at their nominal values.
- $f_{in} = \text{nominal input bit rate}$
- $f_{out} = \text{nominal output bit rate}$
- $M = \text{number of bits in a frame}$
- $L = \text{number of information bits (input bits) for one input stream in a frame}$
Example 3.3 AT&T M12 multiplexer

For M12 framing,

\[ f_{\text{in}} = 1.544 \text{ Mbps} \]
\[ f_{\text{out}} = 6.312 \text{ Mbps} \]
\[ M = 288 \times 4 + 24 = 1176 \text{ bits} \]
\[ L = 288 \text{ bits} \]

Duration of a frame is

\[
\frac{M}{f_{\text{out}}} = S \frac{L-1}{f_{\text{in}}} + (1-S) \frac{L}{f_{\text{in}}}
\]

One bit is replaced by a stuffed bit.

\[ S = L - \frac{f_{\text{in}}}{f_{\text{out}}} \]
\[ M = 288 - \frac{1.544}{6.312} \times 1176 = 0.334601 \]
Example 3.3 AT&T M12 Multiplexer

- Allowable tolerances to maintain nominal output bit rates

- A sufficient condition for the existence of $S$ such that the nominal output bit rate can be matched.

$$\max_{S \in [0,1]} \left[ S \frac{L-1}{f_{in}} + (1-S) \frac{L}{f_{in}} \right] \geq \frac{M}{f_{out}} \geq \min_{S \in [0,1]} \left[ S \frac{L-1}{f_{in}} + (1-S) \frac{L}{f_{in}} \right]$$

$$\iff \frac{L}{f_{in}} \geq \frac{M}{f_{out}} \geq \frac{L-1}{f_{in}} \iff \frac{L}{M} f_{out} \geq f_{in} \geq \frac{L-1}{M} f_{out}$$

$$1.5458 = \frac{288}{1176} 6.312 \geq f_{in} \geq \frac{287}{1176} 6.312 = 1.54043$$
Example 3.3 AT&T M12 Multiplexer

This results in an allowable tolerance range:

\[ 1.5458 - 1.54043 = 6.312 / 1176 = 5.36735 \text{ kbps} \]

In terms of ppm (pulse per million pulses),

\[
\frac{10^6 - b_{ppm}}{1.54043} = \frac{10^6}{1.544} = \frac{10^6 + a_{ppm}}{1.5458}
\]

\[ \Rightarrow a_{ppm} = 1164.8 \text{ and } b_{ppm} = 2312.18 \]

This tolerance is already much larger than the expected change in the bit rate of the incoming DS1 bit stream.
3.11 Virtues, Limitations, and Modifications of PCM

- Virtues of PCM systems
  - Robustness to channel noise and interference
  - Efficient regeneration of coded signal along the transmission path
  - Efficient exchange of increased channel bandwidth for improved signal-to-noise ratio, obeying an exponential law.
  - Uniform format for different kinds of baseband signal transmission; hence, facilitate their integration in a common network.
  - Message sources are easily dropped or reinserted in a TDM system.
  - Secure communication through the use of encryption/decryption.
3.11 Virtues, Limitations, and Modifications of PCM

- Two limitations of PCM system (in the past)
  - Complexity
  - Bandwidth

- Nowadays, with the advance of VLSI technology, and with the availability of wideband communication channels (such as fiber) and compression technique (to reduce the bandwidth demand), the above two limitations are greatly released.
3.12 Delta Modulation

- Delta Modulation (DM)
  - The message is oversampled (at a rate much higher than the Nyquist rate) to purposely increase the correlation between adjacent samples.
  - Then, the difference between adjacent samples is encoded instead of the sample value itself.
(a) Staircase approximation $m_q(t)$

Binary sequence at modulator output

0 0 1 0 1 1 1 1 1 1 0 1 0 0 0 0 0 0 0 0

(b)
3.12 Math Analysis of Delta Modulation

Let \( m[n] = m(nT_s) \).

Let \( m_q[n] \) be the DM approximation of \( m(t) \) at time \( nT_s \). Then

\[
m_q[n] = m_q[n-1] + e_q[n] = \sum_{j=-\infty}^{n} e_q[j],
\]

where \( e_q[n] = \Delta \cdot \text{sgn}(m[n] - m_q[n-1]) \).

The transmitted code word is \( \{(e_q[n]/\Delta + 1)/2\}_{n=-\infty}^{\infty} \).
3.12 Delta Modulation

- The principle virtue of delta modulation is its simplicity.
- It only requires the use of comparator, quantizer, and accumulator.

\[ m_q[n] = m_q[n - 1] + e_q[n] = \sum_{j=-\infty}^{n} e_q[j], \]

where \( e_q[n] = \Delta \cdot \text{sgn}(m[n] - m_q[n - 1]). \)
3.12 Delta Modulation

- Distortions due to delta modulation
  - Slope overload distortion
  - Granular noise

![Diagram showing slope overload distortion and granular noise in delta modulation]
3.12 Delta Modulation

- Slope overload distortion
  - To eliminate the slope overload distortion, it requires
    
    \[ \frac{\Delta}{T_s} \geq \max \left| \frac{dm(t)}{dt} \right| \]  
    (slope overload condition)
  - So, increasing step size \( \Delta \) can reduce the slope-overload distortion.
  - An alternative solution is to use dynamic \( \Delta \). (Often, a delta modulation with fixed step size is referred to as a linear delta modulator due to its fixed slope, a basic function of linearity.)
3.12 Delta Modulation

- Granular noise
  - $m_q[n]$ will hunt around a relatively flat segment of $m(t)$.
  - A remedy is to reduce the step size.

- A tradeoff in step size is therefore resulted for slope overload distortion and granular noise.
3.12 Delta-Sigma Modulation

- Delta-sigma modulation
  - In fact, the delta modulation distortion can be reduced by increasing the *correlation* between samples.
  - This can be achieved by integrating the message signal \( m(t) \) prior to delta modulation.
  - The “integration” process is equivalent to a pre-emphasis of the low-frequency content of the input signal.
3.12 Delta-Sigma Modulation

A side benefit of “integration-before-delta-modulation,” which is named delta-sigma modulation, is that the receiver design is further simplified (at the expense of a more complex transmitter). Move the accumulator to the transmitter.
3.12 Delta-Sigma Modulation

A straightforward structure

Since integration is a linear operation, the two integrators before comparator can be combined into one after comparator.
3.12 Math Analysis of Delta-Sigma Modulation

Let $i[n] = \int_{-\infty}^{nT_s} m(t)dt$.

Let $i_q[n]$ be the DM approximation of $i(t) = \int_{-\infty}^{t} m(\tau)d\tau$ at time $nT_s$.

Then $i_q[n] = i_q[n-1] + \sigma_q[n]$, where $\sigma_q[n] = \Delta \cdot \text{sgn}(i[n] - i_q[n-1])$.

The transmitted code word is $\{(\sigma_q[n]/\Delta + 1)/2\}_{n=-\infty}^{\infty}$.

Since

$$\sigma_q[n] = i_q[n] - i_q[n-1] \approx i[n] - i[n-1] = \int_{(n-1)T_s}^{nT_s} m(t)dt \approx m(t)T_s,$$

we only need a lowpass filter to smooth out the received signal at the receiver end. (See the previous slide.)
3.12 Delta Modulation

Final notes

- Delta(-sigma) modulation trades channel bandwidth (e.g., much higher sampling rate) for reduced system complexity (e.g., the receiver only demands a lowpass filter).
- Can we trade increased system complexity for a reduced channel bandwidth? Yes, by means of prediction technique.
- In Section 3.13, we will introduce the basics of prediction technique. Its application will be addressed in subsequent sections.
3.13 Linear Prediction

Consider a finite-duration impulse response (FIR) discrete-time filter, where $p$ is the prediction order, with linear prediction

$$\hat{x}[n] = \sum_{k=1}^{p} w_k x[n - k]$$
3.13 Linear Prediction

**Design objective**

- To find the filter coefficient $w_1, w_2, \ldots, w_p$ so as to minimize index of performance $J$:

$$J = E[e^2[n]], \text{ where } e[n] = x[n] - \hat{x}[n].$$
Let \( \{x[n]\} \) be stationary with autocorrelation function \( R_X[k] \).

\[
J = E\left[ \left( x[n] - \sum_{k=1}^{p} w_k x[n-k] \right)^2 \right]
\]

\[
= E[x^2[n]] - 2 \sum_{k=1}^{p} w_k E[x[n]x[n-k]] + \sum_{k=1}^{p} \sum_{j=1}^{p} w_k w_j E[x[n-k]x[n-j]]
\]

\[
= R_X[0] - 2 \sum_{k=1}^{p} w_k R_X[k] + \left( 2 \sum_{k=1}^{p} \sum_{j=k+1}^{p} w_k w_j R_X[k-j] + \sum_{k=1}^{p} w_k^2 R_X[0] \right)
\]

\[
\frac{\partial}{\partial w_i} J = -2 R_X[i] + \left( 2 \sum_{j=i+1}^{p} w_j R_X[i-j] + 2 \sum_{k=1}^{i-1} w_k R_X[k-i] + 2 w_i R_X[0] \right)
\]

\[
= -2 R_X[i] + 2 \sum_{j=1}^{p} w_j R_X[i-j] = 0
\]

\( R_X[k-i] = R_X[i-k] \)
\[
\sum_{j=1}^{p} w_j R_x[i-j] = R_x[i] \quad \text{for} \quad 1 \leq i \leq p.
\]

The above optimality equations are called the \emph{Wiener-Hopf equations} for linear prediction.

It can be rewritten in matrix form as:

\[
\begin{bmatrix}
R_x[0] & R_x[1] & \ldots & R_x[p-1] \\
R_x[1] & R_x[0] & \ldots & R_x[p-2] \\
\vdots & \vdots & \ddots & \vdots \\
R_x[p-1] & R_x[p-2] & \ldots & R_x[0]
\end{bmatrix}
\begin{bmatrix}
w_1 \\
w_2 \\
\vdots \\
w_p
\end{bmatrix}
=
\begin{bmatrix}
R_x[1] \\
R_x[2] \\
\vdots \\
R_x[p]
\end{bmatrix}
\]

or \( R_Xw = r_X \Rightarrow \text{Optimal solution} \ w_o = R_X^{-1}r_X \)
3.13 Toeplitz (Square) Matrix

- Any square matrix of the form

$$\begin{bmatrix}
a_0 & a_1 & \cdots & a_{p-1} \\
a_1 & a_0 & \cdots & a_{p-2} \\
\vdots & \vdots & \ddots & \vdots \\
a_{p-1} & a_{p-2} & \cdots & a_0
\end{bmatrix}_{p \times p}$$

is said to be Toeplitz.

- A Toeplitz matrix (such as $R_X$) can be uniquely determined by $p$ elements, $[a_0, a_1, \ldots, a_{p-1}]$. 

3.13 Linear Adaptive Predictor

- The optimal $w_0$ can only be obtained with the knowledge of autocorrelation function.

- Question: What if the autocorrelation function is unknown?

- Answer: Use \textit{linear adaptive predictor}.
3.13 Idea Behind Linear Adaptive Predictor

- To minimize $J$, we should update $w_i$ toward the bottom of the $J$-bowel.

\[ g_i \equiv \frac{\partial J}{\partial w_i} \]

- So, when $g_i > 0$, $w_i$ should be decreased.
- On the contrary, $w_i$ should be increased if $g_i < 0$.
- Hence, we may define the update rule as:

\[ \hat{w}_i[n + 1] = \hat{w}_i[n] - \frac{1}{2} \mu \cdot g_i[n] \]

where $\mu$ is a chosen constant step size, and $\frac{1}{2}$ is included only for convenience of analysis.
\[ g_i[n] \text{ can be approximated by:} \]

\[
g_i[n] \equiv \frac{\partial J}{\partial w_i} = -2R_X(i) + 2\sum_{j=1}^{p} w_j R_X(i - j) \\
\approx -2x[n]x[n-i] + 2\sum_{j=1}^{p} \hat{w}_j[n]x[n-j]x[n-i] \\
= 2x[n-i]\left(-x[n] + \sum_{j=1}^{p} \hat{w}_j[n]x[n-j]\right) \\
\Rightarrow \hat{w}_i[n + 1] = \hat{w}_i[n] + \mu \cdot x[n-i] \left( x[n] - \sum_{j=1}^{p} \hat{w}_j[n]x[n-j] \right) \\
= \hat{w}_i[n] + \mu \cdot x[n-i]e[n]
\]
3.13 Structure of Linear Adaptive Predictor

\[ x[n] \xrightarrow{z^{-1}} \text{Linear predictor: } \{\hat{w}_k[n]\} \xrightarrow{\Sigma} \hat{x}[n] \]

Input \( x[n] \) to \( z^{-1} \) to linear predictor \( \{\hat{w}_k[n]\} \) to \( \hat{x}[n] \) via 
- Error \( e[n] \) from \( \hat{x}[n] \) to \( \Sigma \) and back to \( x[n] \) via \( \Sigma \) and 
- Prediction \( \hat{x}[n] \) output from linear predictor.
3.13 Least Mean Square

- The pairs in the form of the popular least-mean-square (LMS) algorithm for linear adaptive prediction.

\[
\begin{align*}
\hat{w}_j[n + 1] &= \hat{w}_j[n] + \mu \cdot x[n - j]e[n] \\
\hat{e}[n] &= x[n] - \sum_{j=1}^{p} \hat{w}_j[n]x[n - j]
\end{align*}
\]
Basic idea behind *differential pulse-code modulation*

- Adjacent samples are often found to exhibit a high degree of correlation.
- If we can remove this adjacent redundancy before encoding, a more efficient coded signal can be resulted.
- A way to remove the redundancy is to use *linear prediction*. 
3.14 DPCM

- For DPCM, the quantization error is on $e[n]$, rather on $m[n]$ as for PCM.
- So, the quantization error $q[n]$ is supposed to be smaller.
3.14 DPCM

Derive:

\[ e_q[n] = e[n] + q[n] \]

\[ \Rightarrow m_q[n] = \hat{m}[n] + e_q[n] \]

\[ = \hat{m}[n] + e[n] + q[n] \]

\[ = m[n] + q[n] \]

So, we have the same relation between \( m_q[n] \) and \( m[n] \) (as in Slide 3-110) but with smaller \( q[n] \).
3.14 DPCM

- **Notes**
  - DM system can be treated as a special case of DPCM.

  Prediction filter => single delay
  Quantizer => single-bit
3.14 DPCM

- Distortions due to DPCM
  - Slope overload distortion
    - The input signal changes too rapidly for the prediction filter to track it.
  - Granular noise
3.14 Processing Gain

- The DPCM system can be described by:
  \[ m_q[n] = m[n] + q[n] \]
- So, the output signal-to-noise ratio is:
  \[ SNR_O = \frac{E[m^2[n]]}{E[q^2[n]]} \]
- We can re-write \( SNR_O \) as:
  \[ SNR_O = \frac{E[m^2[n]]}{E[e^2[n]]} \cdot \frac{E[e^2[n]]}{E[q^2[n]]} = G_p \cdot SNR_Q \]
  where \( e[n] = m[n] - \hat{m}[n] \) is the prediction error.
3.14 Processing Gain

In terminologies,

\[
G_p = \frac{E[m^2[n]]}{E[e^2[n]]} \quad \text{processing gain}
\]

\[
SNR_Q = \frac{E[e^2[n]]}{E[q^2[n]]} \quad \text{signal to quantization noise ratio}
\]

Notably, \(SNR_Q\) can be treated as the \(SNR\) for system of \(e_q[n] = e[n] + q[n]\).
3.14 Processing Gain

- Usually, the contribution of $SNR_Q$ to $SNR_O$ is fixed.
  - One additional bit in quantization results in 6 dB improvement.
- $G_p$ is the processing gain due to a nice “prediction.”
  - The better the prediction is, the larger $G_p$ is.
3.14 DPCM

- Final notes on DPCM
  - Comparing DPCM with PCM in the case of voice signals, the improvement is around 4-11 dB, depending on the prediction order.
  - The greatest improvement occurs in going from no prediction to first-order prediction, with some additional gain, resulting from increasing the prediction order up to 4 or 5, after which little additional gain is obtained.
  - For the same sampling rate (8KHz) and signal quality, DPCM may provide a saving of about 8~16 Kbps compared to standard PCM (64 Kpbs).
3.14 DPCM


Speech Quality (Mean Opinion Scores)

Bit Rate (kb/s)

Unacceptable
Poor
Fair
Good
Excellent

IS = Interim Standard  GSM = Global System for Mobile Communications  JDC = Japanese Digital Cellular
FS = Federal Standard  MELP = Mixed-Excitation Linear Prediction
3.15 Adaptive Differential Pulse-Code Modulation

- *Adaptive prediction* is used in DPCM.
- Can we also combine *adaptive quantization* into DPCM to yield a comparably voice quality to PCM with 32 Kbps bit rate? The answer is YES from the previous figure.
  - 32 Kbps: 4 bits for one sample, and 8 KHz sampling rate
  - 64 Kbps: 8 bits for one sample, and 8 KHz sampling rate
- So, “adaptive” in ADPCM means being responsive to changing level and spectrum of the input speech signal.
3.15 Adaptive Quantization

- Adaptive quantization refers to a quantizer that operates with a time-varying step size $\Delta[n]$.
- $\Delta[n]$ is adjusted according to the power of input sample $m[n]$.
  - Power = variance, if $m[n]$ is zero-mean.
  - $\Delta[n] = \phi \cdot \sqrt{E[m^2[n]]}$
- In practice, we can only obtain an estimate of $E[m^2[n]]$. 
3.15 Adaptive Quantization

- The estimate of $E[m^2[n]]$ can be done in two ways:
  - Adaptive quantization with **forward** estimation (AQF)
    - Estimate based on *unquantized* samples of the input signals.
  - Adaptive quantization with **backward** estimation (AQB)
    - Estimate based on *quantized* samples of the input signals.
3.15 AQF

- AQF is in principle a more accurate estimator. However, it requires:
  - an additional buffer to store unquantized samples for the learning period.
  - explicit transmission of level information to the receiver (the receiver, even without noise, only has the quantized samples).
  - a processing delay (around 16 ms for speech) due to buffering and other operations for AQF.
- The above requirements can be relaxed by using AQB.
A possible drawback for a feedback system is its potential unstability. However, stability in this system can be guaranteed if $m_q[n]$ is bounded.
3.15 APF and APB

- Likewise, the prediction approach used in ADPCM can be classified into:
  - Adaptive prediction with forward estimation (APF)
    - Prediction based on unquantized samples of the input signals.
  - Adaptive prediction with backward estimation (APB)
    - Prediction based on quantized samples of the input signals.

- The pro and con of APF/APB is the same as AQF/AQB.
- APB/AQB are a preferred combination in practical applications.
3.15 ADPCM

Adaptive prediction with backward estimation (APB).
3.16 Computer Experiment: Adaptive Delta Modulation

- In this section, the simplest form of ADM with AQB is simulated, namely, ADM with AQB.

- Comparison with LDM (i.e., linear DM), where step size is fixed, will also be performed.

This figure is incorrect.
3.16 Computer Experiment: Adaptive Delta Modulation

- In this section, the simplest form of ADM modulation with AQB is simulated, namely, ADM with AQB.
- Comparison with LDM (i.e., linear DM) where step size is fixed will also be performed.
3.16 Computer Experiment: Adaptive Delta Modulation

\[ \Delta[n] = \begin{cases} \Delta[n-1] \times \left(1 + \frac{1}{2} \frac{e_q[n-1]}{e_q[n]} \right), & \text{if } \Delta[n-1] \geq \Delta_{\text{min}} \\ \Delta_{\text{min}}, & \text{if } \Delta[n-1] < \Delta_{\text{min}} \end{cases} \]

where \( \Delta[n] \) is the step size at iteration \( n \), \( e_q[n] \) is the 1 bit quantizer output that equals ±1.

Setting: \( m(t) = 10 \sin \left( 2\pi \frac{f_s}{100} t \right) \), \( \Delta_{\text{LDM}} = 1 \) and \( \Delta_{\text{min}} = \frac{1}{8} \)
Observation: ADM can achieve a comparable performance of LDM with a much lower bit rate.
3.17 MPEG Audio Coding Standard

- The ADPCM and various voice coding techniques introduced above did not consider the human auditory perception.
- In practice, a consideration on human auditory perception can further improve the system performance (from the human standpoint).
- The MPEG-1 standard is capable of achieving transparent, “perceptually lossless” compression of stereophonic audio signals at high sampling rate.
  - A human subjective test shows that a 6-to-1 compression ratio are “perceptually indistinguishable” to human.
3.17 Characteristics of Human Auditory System

- Psychoacoustic characteristic of human auditory system
  - Critical band
    - The inner ear will scale the power spectra of incoming signals nonlinearly in the form of limited frequency bands called the critical bands.
    - Roughly, the inner ear can be modeled as 25 selective overlapping band-pass filters with bandwidth $< 100\text{Hz}$ for the lowest audible frequencies, and up to $5\text{kHz}$ for the highest audible frequencies.
3.17 Characteristics of Human Auditory System

- Auditory masking
  - When a low-(power-)level signal (i.e., the maskee) and a high-(power-)level signal (i.e., the masker) occur simultaneously in the same critical band, and are close to each other in frequency, the low-(power-)level signal will be made inaudible (i.e., masked) by the high-(power-)level signal, if the low-(power-)level one lies below a masking threshold.
3.17 Characteristic of Human Auditory System

- **Masking threshold** is frequency-dependent.

Within a critical band, the quantization noise is inaudible as long as the NMR for the pertinent quantizer is negative.
3.17 MPEG Audio Coding Standard

(a) Diagram showing the process:
- Digital (PCM) audio signal
- Time-to-frequency mapping network
- Quantizer and coder
- Frame-packing unit
- Psychoacoustic model
- Encoded bit stream

(b) Diagram showing the inverse process:
- Encoded bit stream
- Frame-unpacking unit
- Frequency-sample reconstruction network
- Frequency-to-time mapping network
- Digital (PCM) audio signal
3.17 MPEG Audio Coding Standard

- Time-to-frequency mapping network
  - Divide the audio signal into a proper number of subbands, which is a compromise design for computational efficiency and perceptual performance.

- Psychoacoustic model
  - Analyze the spectral content of the input audio signal and thereby compute the signal-to-mask ratio (SMR).

- Quantizer-coder
  - Decide how to apportion the available number of bits for the quantization of the subband signals.

- Frame packing unit
  - Assemble the quantized audio samples into a decodable bit stream.
3.18 Summary and Discussion

- Sampling – transform analog waveform to discrete-time continuous wave
  - Nyquist rate

- Quantization – transform discrete-time continuous wave to discrete data.
  - Human can only detect finite intensity difference.

- PAM, PDM and PPM
- TDM (Time-Division Multiplexing)
- PCM, DM, DPCM, ADPCM
- Additional consideration in MPEG audio coding