Chapter 4 Baseband Pulse Transmission

Techniques for the transmission of (originally) digital data over a baseband channel are the main focus of this chapter.

4.1 Introduction

- Transmission of digital data (bit stream) over a noisy baseband channel typically suffers two channel imperfections:
  - Intersymbol interference (ISI)
  - Background noise (e.g., AWGN)
- These two interferences/noises often occur simultaneously. However, for simplicity, they are often separately considered in analysis.
4.1 ISI

ISI channel

Impulse response

4.2 Matched Filter

- Matched filter is a device for the optimal detection of a digital pulse. It is so named because the impulse response of the matched filter matches the pulse shape.
- System model without ISI
4.2 Design Criterion

To find $h(t)$ such that the output signal-to-noise ratio $\text{SNR}_O$ is maximized.

$x(t) = g(t) + w(t)$ for $0 \leq t < T$

$y(t) = [g(t) + w(t)]*h(t)$

$= g(t)*h(t) + w(t)*h(t)$

$= g_o(t) + n(t)$

$\text{SNR}_O = \frac{|g_o(T)|^2}{E[n^2(T)]}$

4.2 Analysis of Matched Filter

$g_o(t) = \int_{-\infty}^{\infty} H(f)G(f)\exp(j2\pi ft)df$

$\Rightarrow |g_o(T)|^2 = \left| \int_{-\infty}^{\infty} H(f)G(f)\exp(j2\pi fT)df \right|^2$

With $w(t)$ being white with PSD $\frac{N_o}{2}$,

$S_y(f) = S_w(f)|H(f)|^2 = \frac{N_o}{2} |H(f)|^2$

$\Rightarrow E[n^2(T)] = \int_{-\infty}^{\infty} S_y(f)df = \frac{N_o}{2} \int_{-\infty}^{\infty} |H(f)|^2 df$
4.2 Analysis of Matched Filter

By Cauchy-Schwarz’s inequality,

\[
\left| \int_{-\infty}^{\infty} H(f)G(f) \exp(j2\pi T) df \right|^2 \leq \int_{-\infty}^{\infty} |H(f)|^2 df \cdot \int_{-\infty}^{\infty} |G(f)|^2 df \
\Rightarrow \eta \leq \frac{\int_{-\infty}^{\infty} |H(f)|^2 df \cdot \int_{-\infty}^{\infty} |G(f)|^2 df}{2N_0 \int_{-\infty}^{\infty} |H(f)|^2 df} = \frac{2}{N_0} \int_{-\infty}^{\infty} |G(f)|^2 df
\]

This is a constant bound, independent of the choice of \( h(t) \). Hence, the optimal \( \eta \) is achieved by:

\[
H(f) = k \cdot G^*(f) \exp(-j2\pi T)
\]
4.2 Analysis of Matched Filter

$$h_{\text{opt}}(t) = \int_{-\infty}^{\infty} k \cdot G^*(f) \exp(-j2\pi fT) \exp(j2\pi ft) df$$

$$= k \left( \int_{-\infty}^{\infty} G(f) \exp(j2\pi f(T-t)) df \right)^*$$

$$= kg^*(T-t).$$

Hence, under additive white noise, the optimal received filter matches the input signal in the sense that it is a time-inversed and delayed version of the complex-conjugated input signal $g(t)$.

4.2 Properties of Matched Filter

- The maximum output signal-to-noise ratio only depends on the energy of the input, and is nothing to do with the pulse shape itself.
  - Namely, whether the pulse shape is sinusoidal, rectangular, triangular, etc is irrelevant to the maximum output signal-to-noise ratio, as long as these pulse shapes have the same energy.

$$\eta_{\text{max}} = \frac{2E_s}{N_0}, \text{ where } E_s = \int_{-\infty}^{\infty} |G(f)|^2 df.$$
Example 4.1 Matched Filter for Rectangular Pulse

- $h_{opt}(t)$ in this example can be implemented as integrate-and-dump circuit

- Rectangular pulse through an integrator, sample at time $t = T$
4.3 Error Rate due to Noise

In what follows, we analyze the error rate of *polar non-return-to-zero* (NRZ) signaling in a system with optimal matched filter receiver over AWGN channel.

\[ s(t) = I \cdot g(t), \text{ where } I \in \{-1, +1\}. \]

\[ y(T) = \left[ I \cdot g(t) \right]^* h(t) \big|_{-T}^{+T} + w(t) \big|_{-T}^{+T} \]

\[ = I \int_{-\infty}^{\infty} h(\tau) g(T-\tau) d\tau + \int_{-\infty}^{\infty} h(\tau) w(T-\tau) d\tau \]

\[ = I \int_{-\infty}^{\infty} k^* g(\tau) (T-\tau) g(T-\tau) d\tau + \int_{-\infty}^{\infty} k^* g(\tau) w(T-\tau) d\tau \]

\[ = I \cdot k \int_{-\infty}^{\infty} |g(\tau)|^2 d\tau + k \int_{-\infty}^{\infty} g^*(\tau) w(\tau) d\tau \]

\[ = I \cdot kE_s + kn, \text{ where } E_s = \int_{-\infty}^{\infty} |g(\tau)|^2 d\tau \text{ and } n = \int_{-\infty}^{\infty} g^*(\tau) w(\tau) d\tau. \]

For notational convenience, brief \( y(T)/k \) by \( y \).

(The integration can be taken over \([0, T]\) since \( g(t) \) is zero outside this range, as does in text. I however use the entire real line as the integration range here for convenience.)
By AWGN assumption of \( w(t) \), and real \( g(t) \) assumption,
\[
n = \int_{-\infty}^{\infty} g^{*}(\tau)w(\tau)d\tau \text{ is Gaussian distributed with}
\]
\[
E[n] = \int_{-\infty}^{\infty} g^{*}(\tau)E[w(\tau)]d\tau = 0.
\]
\[
E[n^{2}] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(s)g(t)E[w(s)w(t)]dsdt
\]
\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(s)g(t)\frac{N_{0}}{2}\delta(s-t)dsdt
\]
\[
= \frac{N_{0}}{2} \int_{-\infty}^{\infty} g^{*}(s)ds = \frac{N_{0}}{2} E_{g}
\]
\[
y = I \cdot E_{g} + n \Rightarrow \begin{cases} 
\phi_{+}(y) = \text{Normal}(E_{g}, E_{g}N_{0}/2), \text{if } I = +1; \\
\phi_{-}(y) = \text{Normal}(-E_{g}, E_{g}N_{0}/2), \text{if } I = -1
\end{cases}
\]

Let \( \Psi \) be the set for which a decision favors +1 is made.
\[
BER = \Pr[I = +1] \Pr\{\text{guess (-1)} | I = +1\} + \Pr[I = -1] \Pr\{\text{guess (+1)} | I = -1\}
\]
\[
= \Pr[I = +1] \Pr\{y \in \Psi | I = +1\} + \Pr[I = -1] \Pr\{y \in \Psi | I = -1\}
\]
\[
= p(1 - \Pr\{y \in \Psi | I = +1\}) + (1 - p) \Pr\{y \in \Psi | I = -1\}
\]
\[
= p + (1 - p) \Pr\{y \in \Psi | I = +1\} - p \Pr\{y \in \Psi | I = -1\}
\]
\[
= p + \int_{\Psi}[(1 - p)\phi_{+}(y) - p\phi_{-}(y)]dy, \text{ where } p = \Pr[I = +1].
\]
To minimize \( BER \), the optimal set \( \Psi_{opt} = \{y \in \Re : (1 - p)\phi_{+}(y) - p\phi_{-}(y) < 0\} \)
Thus, the optimal decision maker is given by:
\[
d(y) = \begin{cases} 
+1, & (1 - p)\phi_{+}(y) < p\phi_{-}(y) \\
-1, & (1 - p)\phi_{+}(y) \geq p\phi_{-}(y)
\end{cases}
\]
\[ \begin{align*}
\phi_{+1}(y) &= \text{Normal}(E_g E, N_0 / 2), \text{if } I = +1; \\
\phi_{-1}(y) &= \text{Normal}(-E_g E, N_0 / 2), \text{if } I = -1
\end{align*} \]

Let \( \mu = E_g \) and \( \sigma^2 = E_g N_0 / 2 \).

\[ \begin{align*}
(1 - p) < \phi_{+1}(y) &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{(y - \mu)^2}{2\sigma^2} \right\} \\
p > \phi_{-1}(y) &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{(y + \mu)^2}{2\sigma^2} \right\} \\
&= \exp\left\{ \frac{2\mu y}{\sigma^2} \right\} = \exp\left\{ \frac{2E_g y}{E_g N_0 / 2} \right\} = \exp\left\{ \frac{4y}{N_0} \right\}
\end{align*} \]

\[ \begin{align*}
y > N_0 \log\left[ \frac{(1 - p)}{p} \right] \text{ This threshold depends on } N_0; \text{ hence, the best} \\
\text{decision relies on the accuracy of } N_0 \text{ estimate.}
\end{align*} \]

\[ \begin{align*}
4.3 \text{ Error Rate due to Noise under Uniform Input}
\end{align*} \]

- In order to free the system dependence on \( N_0 \) estimate, a uniform \( I \) is transmitted in which case, \( p = \frac{1}{2} \).

- The best decision now becomes \( y \geq 0 \).

\[ \begin{align*}
\text{BER}_{\text{opt}} &= \frac{1}{2} \int_{0}^{+} \phi_{+1}(y)dy + \frac{1}{2} \int_{-}^{-} \phi_{-1}(y)dy \\
&= \frac{1}{2} \int_{0}^{+} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{(y + \mu)^2}{2\sigma^2} \right\}dy \\
&+ \frac{1}{2} \int_{-}^{-} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{(y - \mu)^2}{2\sigma^2} \right\}dy
\end{align*} \]
\[ BER_{\text{opt}} = \frac{1}{2} \int_{\mu}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{y^2}{2\sigma^2}\right\} dy \]
\[ \quad + \frac{1}{2} \int_{-\infty}^{\mu} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{y^2}{2\sigma^2}\right\} dy \]
\[ = \int_{\mu}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{y^2}{2\sigma^2}\right\} dy, \quad z = \frac{y}{\sqrt{2\sigma^2}} \]
\[ = \frac{1}{\sqrt{\pi}} \int_{\mu/\sqrt{2\sigma^2}}^{\infty} \exp\left\{-z^2\right\} dz \]
\[ = \frac{1}{2} \text{erfc}\left(\frac{\mu}{\sqrt{2\sigma^2}}\right) = \frac{1}{2} \text{erfc}\left(\frac{E}{N_0}\right) \]

where \( \text{erfc}(u) = \frac{2}{\sqrt{\pi}} \int_{u}^{\infty} \exp(-z^2) dz \) is the complementary error function.

\[ \text{4.3 Error Function} \]

- Error function \( \text{erf}(u) = \frac{2}{\sqrt{\pi}} \int_{0}^{u} \exp(-z^2) dz \)

- Complementary error function \( \text{erfc}(u) = \frac{2}{\sqrt{\pi}} \int_{u}^{\infty} \exp(-z^2) dz \)

- Q-function \( Q(u) = \frac{1}{\sqrt{2\pi}} \int_{u}^{\infty} \exp\left(-\frac{z^2}{2}\right) dz \)

\[
\begin{align*}
erf(-u) &= -\text{erf}(u) \\
erfc(u) &= 1 - \text{erf}(u) \\
Q(u) &= \frac{1}{2} \text{erfc}\left(\frac{u}{\sqrt{2}}\right)
\end{align*}
\]
4.3 Error Function

- Bounds for error function

\[
\text{erfc}(x) = \frac{1}{x\sqrt{\pi}} e^{-x^2} \left( 1 - \frac{1}{2x^2} + \frac{1\cdot3}{2^2x^4} - \frac{1\cdot3\cdot5}{2^3x^6} + \cdots \right)
\]

For \( x > 0 \),

\[
\frac{1}{x\sqrt{\pi}} e^{-x^2} \left( 1 - \frac{1}{2x^2} \right) < \text{erfc}(x) < \frac{1}{x\sqrt{\pi}} e^{-x^2}
\]

(The bound is good when \( x \) is large.)

4.3 Error rate due to noise

- The optimal BER formula is important in communications:

\[
\text{BER}_{\text{opt}} = \frac{1}{2} \text{erfc}\left( \frac{E_g}{\sqrt{N_0}} \right) = Q\left( \sqrt{\frac{2E_g}{N_0}} \right)
\]

- The best decision is \( y \gtrless 0 \).
4.4 Intersymbol Interference

- The channel is usually dispersive in nature.
- In this section, we only consider discrete pulse-amplitude modulation (PAM). Consideration of PDM and PPM will be out of the scope of this section.

\[ b_k \in \{0, 1\}, \quad a_k = 2b_k - 1 \text{ and } s(t) = \sum_{k=-\infty}^{\infty} a_k g(t - kT_b). \]
4.4 Intersymbol Interference

- Notably, in the previous section, we only consider one interval of input.
\[ s(t) = I \cdot g(t) \]
This is justifiable because of no ISI.
- However, in this section, we have to consider
\[ s(t) = \sum_{k=-\infty}^{\infty} a_k g(t - kT_b). \]
since ISI is involved.
- We also assume perfect synchronization to simplify the analysis.

Information of \( a_k \) is carried at \([kT_b, (k+1)T_b)\).
Without ISI,

\[ H(f) = 1 \Rightarrow p(t) = \int_{-\infty}^{\infty} G(f) C(f) \exp\{-j2\pi ft\} dt. \]

With matched filter \( C(f) = G^*(f) \exp\{-j2\pi ft\} \), or \( c(t) = g^*(T_b - t) \),

\[
p(iT_b) = \int_{-\infty}^{\infty} c(\tau) g(iT_b - \tau) d\tau
= \int_{-\infty}^{\infty} g^*(T_b - \tau) g(iT_b - \tau) d\tau \quad \text{(Let} \ s = T_b - \tau) \\
= \int_{-\infty}^{\infty} g^*(s) g(s + (i - 1)T_b) ds = \begin{cases} 0, & \text{if } i \neq 1 \\
\int_{-\infty}^{\infty} |g(s)|^2 ds, & \text{if } i = 1 \end{cases}
\]
The text sets $p(0) = 1$ for simplicity but this is a little confused (See Slide 4-28)! The text is correct when information of $a_i$ is carried during $[(i-1)T_b, iT_b]$. Information of $a_i$ is actually carried at $[iT_b, (i+1)T_b]$. So in order to recover $a_i$, “correlation” (convolution) operation should start at $iT_b$, and end (i.e., is sampled) at $(i+1)T_b$. Hence, $y((i+1)T_b)$ is used to reconstruct $a_i$.

**4.5 Nyquist’s Criterion for Distortionless Baseband Binary Transmission**

**Is it possible to completely eliminate ISI (in principle) by selecting a proper $g(t)$?**

Choose $g(t)$ and $c(t)$ such that $p(t) = \int_{-\infty}^{\infty} G(f)H(f)C(f) \exp(j2\pi ft)dt$ satisfies $p(iT_b) = \begin{cases} 0, & \text{if } i \neq 0 \\ 1, & \text{if } i = 0. \end{cases}$

(Here, I assume that information of $a_i$ is carried at $[(i-1)T_b, iT_b]$.)
4.5 Nyquist’s Criterion for Distortionless Baseband Binary Transmission

- Let \( P(f) = G(f)H(f)C(f) \).
- Sample \( p(t) \) with sampling period \( T_b \) to produce \( P_\delta(f) \).
- From Slide 3-4, we get:

\[
P_\delta(f) = \frac{1}{T_b} \sum_{n=-\infty}^{\infty} P\left(f - \frac{n}{T_b}\right)
\]

- Also from Slide 3-4, we have:

\[
P_\delta(f) = \sum_{n=-\infty}^{\infty} p(nT_b) \exp(-j2\pi nT_b f) = 1
\]

Because \( p(nT_b) = \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{if } n \neq 0 \end{cases} \)

This concludes that the condition for zero ISI is:

\[
\sum_{n=-\infty}^{\infty} P\left(f - \frac{n}{T_b}\right) = T_b
\]

(Indeed, \( \sum_{n=-\infty}^{\infty} P\left(f - \frac{n}{T_b}\right) = \text{constant.} \))

This is named the Nyquist criterion.

- The overall system frequency function \( P(f) \) suffers no ISI for samples taken at interval \( T_b \) if it satisfies the above equation.
- Notably, \( P(f) \) represents the overall accumulative effect of transmit filter, channel response, and receive filter.
4.5 Ideal Nyquist Channel

The simplest $P(f)$ that satisfies Nyquist criterion is the rectangular function:

$$P(f) = \begin{cases} T_b, & |f| < W = \frac{1}{2T_b} \\
0, & |f| > W = \frac{1}{2T_b} 
\end{cases}$$

and $P(-W) + P(W) = T_b$.

$$\Rightarrow p(t) = \frac{\sin(2\pi W t)}{2\pi W t} = \text{sinc}(2Wt)$$
4.5 Infeasibility of Ideal Nyquist Channel

- Rectangular $P(f)$ is infeasible because:
  - $p(t)$ extends to negative infinity, which means that each $a_k$ have already been transmitted at $t = -\infty$!
  - A system response being flat from $-W$ to $W$, and zero elsewhere is physically unrealizable.
  - The margin of error is quite small, as a slight shift (error) in sampling time (such as, $iT_b+\varepsilon$) would cause very large ISI.
  - Note that $p(t)$ decays to zero at a very slow rate of $1/|t|$.
4.5 Infeasibility of Ideal Nyquist Channel

- Examination of timing error margin
  - Let $\Delta t$ be the sampling time difference between transmitter and receiver.
    \[
    y(iT_b + \Delta t) = \sum_{k=-\infty}^{\infty} a_k p((i - k)T_b + \Delta t)
    \]
  - For simplicity, set $i = 0$.
    \[
    y(\Delta t) = \sum_{k=-\infty}^{\infty} a_k p(\Delta t - kT_b) = \sum_{k=-\infty}^{\infty} a_k \frac{\sin[2\pi W(\Delta t - kT_b)]}{2\pi W(\Delta t - kT_b)}
    \]

There exists $\{a_k\}$ such that $\sum_{k=-\infty}^{\infty} (-1)^k a_k = \infty$ for any fixed small $\Delta t > 0$.

Question: How to make $p(t)$ decays faster?
Answer: Make $P(f)$ smoother.
4.5 Raised Cosine Spectrum

For a nonnegative function \( p(t) \),

\[
\text{if } \int_{-\infty}^{\infty} t^k p(t) dt < \infty, \text{ then } \frac{\delta^k P(f)}{\delta f^k} \text{ exists.}
\]

We extend the bandwidth of \( p(t) \) from \( W \) to \( 2W \), and require that

\[
P(f) + P(f - 2W) + P(f + 2W) = \frac{1}{2W} \quad \text{for } |f| < W.
\]

- So the price to pay here is a larger bandwidth.
- One of the \( P(f) \) that satisfies the above condition is the raised cosine spectrum.

\[
P(f) = \begin{cases} 
  1, & 0 \leq |f| < (1 - \alpha)W \\
  \frac{1}{2W} \left[ 1 + \cos \left( \frac{\pi}{2} \left( \frac{|f|}{2 \alpha W} \right) \right) \right], & (1 - \alpha)W \leq |f| < (1 + \alpha)W \\
  0, & |f| \geq (1 + \alpha)W
\end{cases}
\]
4.5 Raised Cosine Spectrum

- The transmission bandwidth of the raised cosine spectrum is equal to:

\[ B_T = 2W(1 + \alpha) \]

where \( \alpha \) is the rolloff factor, which is the excess bandwidth over the ideal solution.

\[ p(t) = \text{sinc}(2Wt) \left( \frac{\cos(2\pi \alpha Wt)}{1 - 16\alpha^2 W^2 t^2} \right) \]

\[ \sim \frac{1}{|t|^3} \text{ as } |t| \text{ large} \]
4.5 Raised Cosine Spectrum

\( p(t) = \text{sinc}(2Wt) \left( \frac{\cos(2\pi \alpha Wt)}{1-16\alpha^2 W^2 t^2} \right) \) consists of two terms:

- The first term ensures the desired zero crossing of \( p(t) \).
- The second term provides the necessary tail convergence rate of \( p(t) \).

The special case of \( \alpha = 1 \) is known as the full-cosine rolloff characteristic.

\[
p(t) = \frac{\text{sinc}(4Wt)}{1-16W^2 t^2}
\]

4.5 Raised Cosine Spectrum

Useful property of full-cosine spectrum.

\[
p \left( \pm \frac{iT_b}{2} \right) = \begin{cases} 1, & i = 0 \\ \frac{1}{2}, & i = 1 \\ 0, & i \geq 2 \end{cases}
\]

- We have more “zero-crossing” at \( \pm 3T_b/2, \pm 5T_b/2, \pm 7T_b/2, \ldots \) in addition to the desired \( \pm T_b, \pm 2T_b, \pm 3T_b, \ldots \)
- This is useful in synchronization. (Think of when “synchronized,” the quantity should be small both at \( \pm 3T_b/2, \pm 5T_b/2, \pm 7T_b/2, \ldots \) and at \( \pm T_b, \pm 2T_b, \pm 3T_b, \ldots \))
- However, the price to pay for this excessive synchronization information is to “double the bandwidth.”
Example 4.2 Bandwidth Requirement of the T1 System

- For T1 transmission, a frame consists of 24 PCM-encoded voice channels and 1 framing bit.
  - The resultant number of bits in a frame is $24 \times 8 + 1 = 193$.
- The duration of each frame is $125 \, \mu s$.
- Hence,
  
  \[
  T_b = \frac{125 \, \mu s}{193} = 0.647688 \, \mu s
  \]
  
  \[
  \Rightarrow W = \frac{1}{2T_b} = 772 \, kHz.
  \]
  
  \[
  \Rightarrow B_{T1(Band))} = W(1 + \alpha) = 772(1 + \alpha) \, kHz
  \]

4.6 Correlative-Level Coding

- ISI, when generated in an uncontrolled manner, is an undesirable phenomenon.
- However, **ISI may become a friend** if it is added to the transmitted signal in a controlled manner.
  - **Known fact**: A signal of bandwidth $W$ can be distortionlessly transmitted using its samples with sampling rate $\geq 2W$.
  - Conversely, in a channel with bandwidth $W$ Hz, the theoretical maximum signal rate is $2W$ symbols per second.
4.6 Correlative-Level Coding

- Why intentionally adding ISI? Answer: To have better bandwidth efficiency.
  - **Ideal Nyquist pulse shaping** is efficient; it cannot be realized.
  - **Raised cosine pulse shaping** is realizable; it is bandwidth inefficient.
  - By adding ISI to the transmitted symbols in a controlled manner, we can achieve the Nyquist rate $2W$ in a channel bandwidth of $W$ Hertz.
  - Correlative-level coding or Partial-response signaling
4.6 One Example of Correlative-Level Coding

- Duobinary signaling (or class I partial response)

\[
a_k = \begin{cases} 
+1 & \text{if symbol } b_k \text{ is } 1 \\
-1 & \text{if symbol } b_k \text{ is } 0 
\end{cases}
\]

where \( \{b_k\} \) i.i.d.

Let us ignore the effect of \( H_{\text{Nyquist}}(f) \) first in the block diagram in the previous slide. We directly obtain:

\[
c_k = a_k + a_{k-1} \\
\implies H_{\text{DuoB}}(f) = 1 + \exp(-j2\pi T_b)
\]

- Note that \( c_k \) has three levels \((-2,0,2)\).

- The transfer function of the overall system is thus:

\[
H_d(f) = H_{\text{Nyquist}}(f)[1 + \exp(-j2\pi f T_b)] \\
= H_{\text{Nyquist}}(f)[\exp(j\pi f T_b) + \exp(-j\pi f T_b)] \exp(-j\pi f T_b) \\
= 2H_{\text{Nyquist}}(f) \cos(\pi f T_b) \exp(-j\pi f T_b)
\]
4.6 Duobinary Signaling

- $H_{\text{Nyquist}}(f)$:
  - Only for derivation purpose (do not need it finally)

$$H_{\text{Nyquist}}(f) = \begin{cases} 1, & |f| \leq 1/2T_b \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow H(f) = \begin{cases} 2 \cos(\pi f T_b) \exp(-j\pi f T_b), & |f| \leq 1/2T_b \\ 0, & \text{otherwise} \end{cases}$$

- As shown in the next slide, the response $H(f)$ is realizable.
4.6 Duobinary Signaling

$h_f(t)$:

\[ H(f) = H_{Nyquist}(f)[1 + \exp(-j2\pi f T_b)] \]

\[ = H_{Nyquist}(f) + H_{Nyquist}(f)\exp(-j2\pi f T_b) \]

\[ \Rightarrow h_f(t) = h_{Nyquist}(t) + h_{Nyquist}(t - T_b) \]

\[ = \left[ \frac{\sin(\pi t / T_b)}{\pi t / T_b} + \frac{\sin(\pi (t - T_b) / T_b)}{\pi (t - T_b) / T_b} \right] \times T_b \]

\[ = \left[ \frac{\sin(\pi t / T_b)}{\pi t / T_b} - \frac{\sin(\pi t / T_b)}{\pi (t - T_b) / T_b} \right] \times T_b \]

\[ = \frac{T_b}{T_b - t} \frac{\sin(\pi t / T_b)}{\pi t / T_b} \times T_b \]

Text omits this term by saying "except for a scaling factor."
4.6 Duobinary Signaling

- Bandwidth efficiency of duobinary signaling

Example.

The transmitted signal is

\[
\sum_{k=-\infty}^{\infty} a_k g(t - kT_b) = \left[ \sum_{k=-\infty}^{\infty} a_k \delta(t - kT_b) \right] * g(t)
\]

The input to this filter may not be WSS! Then we should use the time-average autocorrelation function.

4.6 Time-Average Autocorrelation Function

\[
\begin{align*}
X(t) & \quad H(f) \quad Y(t) \\
\text{Time Average Autocorrelation Function} & \\
\overline{R}_x(\tau) & = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} E[X(t)X(t+\tau)]dt = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} R_x(t,t+\tau)dt \\
\Rightarrow \quad \overline{R}_y(\tau) & = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} R_y(t,t+\tau)dt \\
& = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} h(\tau_1)h(\tau_2)R_x(t-\tau_1,t+\tau_2)dt_1dt_2 \\
& \quad (Assume that limit and integration are interchangeable.)
\end{align*}
\]
\[
\mathcal{R}_X(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1) h(\tau_2) \left( \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} R_X(t - \tau_1, t + \tau - \tau_2) dt \right) d\tau_1 d\tau_2
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1) h(\tau_2) \mathcal{R}_h(\tau - \tau_2 + \tau_1) d\tau_1 d\tau_2
\]

\[
\mathcal{S}_X(f) = \int \mathcal{R}_X(\tau) \exp(-j2\pi\tau f) d\tau
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1) h(\tau_2) \int \mathcal{R}_h(\tau - \tau_2 + \tau_1) \exp(-j2\pi(\tau - \tau_2 + \tau_1)) d\tau_1 d\tau_2
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_2) e^{-j2\pi\tau_2} d\tau_2 \cdot \int_{-\infty}^{\infty} h(\tau_1) e^{-j2\pi\tau_1} d\tau_1 \cdot \int \mathcal{R}_h(\tau') e^{-j2\pi\tau'} d\tau'
\]

\[
= H(f)H^*(f)\mathcal{S}_h(f), \text{ if } h(\tau) \text{ is real}
\]

\[
= |H(f)|^2 \mathcal{S}_h(f)
\]

4.6 Duobinary Signaling

Now back to the example.

\[
X(t) = \sum_{k=-\infty}^{\infty} a_k \delta(t - kT_b) \quad Y(t) = \sum_{k=-\infty}^{\infty} a_k g(t - kT_b) \quad \text{(to channel)}
\]

\[
\mathcal{R}_X(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} E \left( \sum_{k=-\infty}^{\infty} a_k \delta(t - kT_b) \left( \sum_{j=-\infty}^{\infty} a_j \delta(t + \tau - jT_b) \right) \right) dt
\]

\[
= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} E[a_ia_k] \delta(t + \tau - jT_b) \delta(t - kT_b) dt
\]

Assume \( E[a_ia_k] = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases} \)
4.6 Duobinary Signaling

\[ \bar{R}_x(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left( \sum_{k=-\infty}^{\infty} \delta(t-kT_b) \delta(t+\tau-kT_b) \right) dt \]

\[ = \delta(\tau) \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left( \sum_{k=-\infty}^{\infty} \delta(t-kT_b) \right) dt \]

\[ = \frac{1}{T_b} \delta(\tau) \Rightarrow \mathcal{S}_y(f) = \frac{1}{T_b} \left| G(f) \right|^2 \]

Approximately \( \frac{2T}{T_c} \) of them

\[ X(t) = \sum_{k=-\infty}^{\infty} a_k \delta(t-kT_b) \]

\[ \Rightarrow \mathcal{S}_y(f) = \frac{1}{T_b} \left| G(f) \right|^2 \left| H_{Duob}(f) \right|^2 \]
4.6 Duobinary Signaling

\[ H_{\text{Doub}}(f) = 2 \cos(\pi f T_b) \exp(-j\pi f T_b) \]

Assume \( g(t) = \begin{cases} 1, & 0 \leq t < T_b, \\ 0, & \text{otherwise} \end{cases} \Rightarrow |G(f)|^2 = T_b^2 \text{sinc}^2(f T_b) \)

\[ \Rightarrow \frac{S_r(f)}{S_r(0)} = \begin{cases} \text{sinc}^2(f T_b), & \text{No Signal ISI} \\ \cos^2(\pi f T_b) \text{sinc}^2(f T_b), & \text{With Signal ISI} \end{cases} \]

\[ = \begin{cases} \text{sinc}^2(f T_b), & \text{No Signal ISI} \\ \text{sinc}^2(2f T_b), & \text{With Signal ISI} \end{cases} \]
Conclusions

By adding ISI to the transmitted signal in a controlled (and reversible) manner, we can reduce the requirement of bandwidth of the transmitted signal.

Hence, in the previous example, \( \{c_k\} \) can be transmitted in every \( T_b/2 \) seconds!

Doubling the transmission capacity without introducing additional requirement in bandwidth!

*Duobinary signaling*: “Duo” means “doubling the transmission capacity of a straight binary system.”

A larger SNR is required to yield the same error rate because of an increase in the number of signal levels (from \(-1, +1\) to \(-2, 0, 2\)). Detailed discussion on error rate impact is omitted here!

Conclusions (cont.)

The duobinary signaling is also named *class I partial response*.

Full response: The transmission wave at each time instance is fully determined by a single information symbol.

Partial response: The transmission wave at each time instance is only partially determined by one information symbol (i.e., is fully determined by two or more information symbols).
4.6 Decision Feedback for Correlative-Level Coding

- Recovering of \{a_k\} from \{c_k\}
  \[ \hat{a}_k = c_k - \hat{a}_{k-1} \]

- It requires the previous decision to determine the current symbol.
- So the system should feedback the previous decision.
- Error therefore may propagate!

- How to avoid error propagation? Answer: Precoding.

4.6 Precoding of Correlative Coding

Without precoding
\[ \{b_k \in \{0,1\} \text{ i.i.d.}\} \rightarrow a_k = 2b_k - 1 \rightarrow c_k = a_k + a_{k-1} \]

With precoding
\[ \{b_k \in \{0,1\} \text{ i.i.d.}\} \rightarrow \tilde{b}_k = b_k \oplus \tilde{b}_{k-1} \rightarrow a_k = 2\tilde{b}_k - 1 \rightarrow c_k = a_k + a_{k-1} \]

\[
\begin{aligned}
c_k &= a_k + a_{k-1} \\
(2\tilde{b}_k - 1) + (2\tilde{b}_{k-1} - 1) \\
2\tilde{b}_k + 2\tilde{b}_{k-1} - 2 \\
\end{aligned}
\]

<table>
<thead>
<tr>
<th>(\tilde{b}_k)</th>
<th>(\tilde{b}_{k-1})</th>
<th>(b_k)</th>
<th>(c_k)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-2</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>
Final notes

- The precode must not change the “duo" of the transmission capacity of a straight binary system.
- Hence, \{\tilde{b}_k\} must have the same distribution as \{b_k\} and hence must be i.i.d.

4.6 Precoding of Correlative Coding

4.6 Precoding of Correlative Coding

- Uniform i.i.d. of \{\tilde{b}_k\}
- For i.i.d., it suffices to show \( \Pr(\tilde{b}_k \mid \tilde{b}_{k-1}, \tilde{b}_{k-2}, \ldots) = \Pr(\tilde{b}_k) \)

\[ \tilde{b}_k = b_k \oplus \tilde{b}_{k-1} \Rightarrow \Pr(\tilde{b}_k \mid \tilde{b}_{k-1}, \tilde{b}_{k-2}, \ldots) = \Pr(\tilde{b}_k) \]

\[
\begin{align*}
\Pr(\tilde{b}_k = 0 \mid \tilde{b}_{k-1} = 0) &= \Pr(b_k = 0) = 1/2 \\
\Pr(\tilde{b}_k = 0 \mid \tilde{b}_{k-1} = 1) &= \Pr(b_k = 1) = 1/2 \\
\Pr(\tilde{b}_k = 1 \mid \tilde{b}_{k-1} = 0) &= \Pr(b_k = 1) = 1/2 \\
\Pr(\tilde{b}_k = 1 \mid \tilde{b}_{k-1} = 1) &= \Pr(b_k = 0) = 1/2
\end{align*}
\]

\( \Rightarrow \Pr(\tilde{b}_k \mid \tilde{b}_{k-1}, \tilde{b}_{k-2}, \ldots) = \Pr(\tilde{b}_k) \)
For uniformity,
\[
\begin{align*}
\Pr(\tilde{b}_k = 0) &= \Pr(\tilde{b}_{k-1} = 0) \Pr(\tilde{b}_k = 0 | \tilde{b}_{k-1} = 0) \\
&\quad + \Pr(\tilde{b}_{k-1} = 1) \Pr(\tilde{b}_k = 0 | \tilde{b}_{k-1} = 1) \\
&= \Pr(\tilde{b}_{k-1} = 0) \frac{1}{2} + \Pr(\tilde{b}_{k-1} = 1) \frac{1}{2} \\
&= \frac{1}{2} \\
\Pr(\tilde{b}_k = 1) &= \Pr(\tilde{b}_{k-1} = 0) \Pr(\tilde{b}_k = 1 | \tilde{b}_{k-1} = 0) \\
&\quad + \Pr(\tilde{b}_{k-1} = 1) \Pr(\tilde{b}_k = 1 | \tilde{b}_{k-1} = 1) \\
&= \Pr(\tilde{b}_{k-1} = 0) \frac{1}{2} + \Pr(\tilde{b}_{k-1} = 1) \frac{1}{2} \\
&= \frac{1}{2}
\end{align*}
\]
Q.E.D.

Example 4.3 Duobinary Coding with Precoding

Table 4.1 in text

\[
\begin{array}{cccccccc}
{b_k} & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
{\hat{b}_k} & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\
{a_k} & +1 & +1 & +1 & -1 & -1 & +1 & -1 \\
{c_k} & +2 & +2 & 0 & -2 & 0 & 0 & -2 \\
{\hat{b}_k} & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
\end{array}
\]

\[
\{b_k \in \{0,1\} \text{ i.i.d.}\} \rightarrow \tilde{b}_k = b_k \oplus \tilde{b}_{k-1} \rightarrow a_k = 2\tilde{b}_k - 1 \rightarrow c_k = a_k + a_{k-1}
\]
4.6 Modified Duobinary Signaling

- The PSD of the signal is nonzero at the origin.
- This is considered to be an **undesirable feature** in some applications, since many communication channels cannot transmit a DC component.
- Solution: Class IV partial response or modified duobinary technique.

\[
\{b_k \in \{0, 1\} \text{ i.i.d.} \rightarrow \tilde{b}_k = b_k \oplus \tilde{b}_{k-2} \rightarrow a_k = 2\tilde{b}_k - 1 \rightarrow c_i = a_i - a_{i-2}
\]

\[
\Rightarrow H_{MDuoB}(f) = 1 - \exp(-j4\pi f T_b)
\]
4.6 Modified Duobinary Signaling

\[ H_{\text{MDuoB}}(f) = 1 - \exp(-j4\pi f T_b) \]
\[ = [\exp(j2\pi f T_b) - \exp(-j2\pi f T_b)]\exp(-j2\pi f T_b) \]
\[ = 2 \sin(2\pi f T_b) \exp(-j2\pi f T_b) \]

Assume \( g(t) = \begin{cases} 1, & 0 \leq t < T_b \\ 0, & \text{otherwise} \end{cases} \Rightarrow |G(f)|^2 = T_b^2 \sin^2(f T_b) \]

\[ \Rightarrow \begin{cases} \tilde{S}_y(f)/T_b = \sin^2(2f T_b), & \text{Duobinary} \quad \text{(See Slide 4-61)} \\ \tilde{S}_y(f)/(4T_b) = \sin^2(2\pi f T_b)\sin^2(f T_b), & \text{Modified Duobinary} \end{cases} \]
4.6 Modified Duobinary Signaling

Precoding is added to eliminate error propagation in decision system.

\[
\begin{align*}
    c_k &= a_k - a_{k-2} \\
    &= (2\tilde{b}_k - 1) - (2\tilde{b}_{k-2} - 1) \\
    &= 2\tilde{b}_k - 2\tilde{b}_{k-2} \\
    b_k &= \tilde{b}_k \oplus \tilde{b}_{k-2}
\end{align*}
\]

\[
\begin{array}{c|c|c|c}
    b_k & \tilde{b}_{k-2} & b_k & c_k \\
    \hline
    0 & 0 & 0 & 0 \\
    0 & 1 & 1 & -2 \\
    1 & 0 & 1 & 2 \\
    1 & 1 & 0 & 0 \\
\end{array}
\]

\[
\{b_k \in \{0,1\} \text{ i.i.d.} \} \rightarrow \tilde{b}_k = b_k \oplus \tilde{b}_{k-2} \rightarrow a_k = 2\tilde{b}_k - 1 \rightarrow c_k = a_k - a_{k-2}
\]

4.6 Generalized Form of Correlative Level Coding (CLC) or Partial Response Signaling

\[
H_{CLC}(f) = w_0 + w_i z^{-1} + \cdots + w_{N-1} z^{N-1},
\]

where \( z = \exp(-j2\pi f T_s) \).
### 4.6 Generalized Form of Correlative-Level Coding or Partial-Response Signaling

<table>
<thead>
<tr>
<th>Type of Class</th>
<th>$N$</th>
<th>$w_0$</th>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$w_3$</th>
<th>$w_4$</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td>Duobinary coding</td>
</tr>
<tr>
<td>II</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>III</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>IV</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td></td>
<td></td>
<td>Modified duobinary coding</td>
</tr>
<tr>
<td>V</td>
<td>5</td>
<td>-1</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>-1</td>
<td></td>
</tr>
</tbody>
</table>

$$\Rightarrow \mathcal{S}_f(f) = \frac{|G(f)|^2}{T_s} \times \begin{cases} 
4\cos^2(\pi f T_s) & I \\
16\cos^4(\pi f T_s) & II \\
4\cos^2(\pi f T_s) + 8\sin^2(2\pi f T_s) & III \\
4\sin^4(2\pi f T_s) & IV \\
16\sin^4(2\pi f T_s) & V 
\end{cases}$$

Assume $g(t) = \begin{cases} 
1, & 0 \leq t < T_s \\
0, & \text{otherwise}
\end{cases} \Rightarrow |G(f)|^2 = T_s^2 \text{sinc}^2(f T_s)$
4.7 Baseband $M$-ary PAM Transmission

- For $M$-ary PAM transmission, there are $M$ possible symbols with symbol duration $T$.
  - $1/T$ is referred to as the **signaling rate** or **symbol rate** or **symbols per second** or **baud**.

- Some equivalences
  - Each symbol can be equivalently identified with $\log_2 M$ bits.
  - So the baud rate $1/T$ can be equivalently transformed to bps as:
    $$T = T_b \log_2 (M)$$
4.7 Baseband $M$-ary PAM Transmission

Some equivalences

- Virtually fix the symbol error, namely, fix the level distance (to be 2). For example, $(+1, -1)$ for $M = 2$, and $(+3, +1, -1, -3)$ for $M = 4$. Then the transmitted power per unit time for $M$-ary PAM transmission becomes:

$$\frac{E[S^2]}{T} = \frac{\frac{1}{M} \left[(-(M-1))^2 + \cdots + (M-3)^2 + (M-1)^2\right]}{T_b \log_2(M)}$$

$$= \frac{(M^2-1)}{3T_b \log_2(M)} = \left(\frac{1}{T_b}\right) \frac{(M^2-1)}{3 \log_2(M)}$$

For fixed $R_b = 1/T_b$ (bps) and level distance = 2, the transmitted power of an $M$-ary PAM transmission signal is increased by a factor $M^2/\log_2 M$. 

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4.8 Digital Subscriber Lines (DSL)

- A DSL operates over a local loop (often less than 1.5km) that provides a direct connection between a user terminal (e.g., computer) and a telephone company’s central office (CO).
  - Since it is a direct connection, no dialup is necessary.
  - The information-bearing signal is kept in the digital domain all the way from the user terminal to an Internet service provider.

![Diagram of DSL setup]

SONET: Synchronous Optical Networking

---

4.8 Digital Subscriber Lines

- DSL is intended to provide high data-rate, full-duplex, digital transmission capability using local cost configuration (such as twisted pairs for ordinary telephonic communications).

- One of two possible modes can be used to achieve the full-duplex goal.
  - Time compression multiplexing (TCM) mode
  - Echo-cancellation (EC) mode
4.8 Digital Subscriber Lines

- Time-compression multiplexing (TCM) mode
  - A guard time is often inserted between bursts in the two opposite directions of data.
  - So the required line rate is slightly greater than twice the data rate.

```
Transmitter  =>  Receiver
              \  /
               \|
  Transmitter  =>  Receiver
```

- Echo-cancellation (EC) mode
  - Support the simultaneous flow of data along the common line in both directions.
  - In this mode, the line rate is the same as the data rate.

```
Transmitter  =>  Echo canceller
              \  /
               \|
  Echo-free signal

Receiver  =>  Hybrid
              \  /
               \|
  Common line

Hybrid  =>  Hybrid
              \  /
               \|
  Echo canceller

Transmitter  =>  Receiver
```

4.8 Digital Subscriber Lines

- Hybrid transformer for DSL
  - Two-to-four-wire conversion

Comparison between TCM mode and EC mode
- EC offers a much better data transmission performance at the expense of increase complexity.
- However, with the recent advance in VLSI, complexity is no longer a main system concern. So in North America, the EC mode has been adopted as the basis for designing the transceiver.
4.8 Digital Subscriber Lines

- Other impairments to DSL
  - ISI and Crosstalk

- The transfer function of a twisted pair line can be approximated by

\[
|H_{\text{twist pair}}(f)|^2 = \exp(-\alpha \sqrt{f})
\]

where \(\alpha = k \frac{l}{l_0}\), \(k\) is a physical constant of the twisted pair, and \(l_0\) and \(l\) are respectively the reference length and actual length of the twisted pair.

\[
H_{\text{twist pair}}(f) = \exp\left(-\frac{1}{2} \sqrt{|f|}\right) \exp(-j2\pi f \tau_0)
\]

\[
h_{\text{twist pair}}(\tau) = \frac{4}{1 + 16\pi^2(\tau - \tau_0)^2}
\]

\(\tau_0 = 0.1\)
4.8 Digital Subscriber Lines

- Crosstalk
  - Capacitive coupling that exists between adjacent twisted pairs in a cable
  - Near-end crosstalk (NEXT) and Far-end crosstalk (FEXT)

Near-end crosstalk

Cable containing a bundle of twisted pairs

- Transmitting end

- Receiver

- Disturbed end

Far-end crosstalk

Cable containing a bundle of twisted pairs

- Transmitting end

- Receiver

- Disturbed end

4.8 Digital Subscriber Lines

- Crosstalk (cont.)
  - FEXT suffers the *same line loss* as the signal, whereas NEXT does not.
    - This is close to the phenomenon of *near-far effect* of wireless channel.
  - Accordingly, NEXT will be a more serious problem than FEXT. So we can ignore the effect of FEXT, and add NEXT filter to the twisted pair channel model (as shown in the figure in the next slide).
4.8 Digital Subscriber Lines

Interference (input of $H_{\text{NEXT}}(f)$) often assumes to have the same PSD as the transmitted signal, but is Gaussian distributed.

$H_{\text{NEXT}}(f) = \beta f^{3/2}$

Interfering signal responsible for generating near-end crosstalk (NEXT)

Other features of DSL channel

- The PSD of the transmitted signal should be zero at zero frequency because no DC transmission through a hybrid transformer is possible.
- The PSD of the transmitted signal should be low at high frequencies because
  - transmission attenuation in a twisted pair is most severe at high frequency;
  - crosstalk due to capacitive coupling between adjacent twisted pairs increases dramatically at high frequency (recall that the impedance of a capacitor is inversely proportional to frequency).
4.8 Digital Subscriber Lines

- Possible candidates for line codes that are suitable for DSL
  - Manchester code
    - Zero DC component but large spectrum at high frequency so it is vulnerable to NEXT and ISI.
  - Bipolar return to zero (BRZ) or Alternate mark inversion (AMI) code
    - Successive 1’s are represented alternately by positive and negative but equal levels, and 0 is represented by a zero level.
    - Zero DC component. Its NEXT and ISI performance is slightly inferior to the modified duobinary code on all digital subscriber loops.

- Modified duobinary code
  - Of no DC component and moderately spectrally efficient. However, its robustness against NEXT and ISI is about 2 to 3 dB poorer than that of (2B1Q) block codes on worst-case subscriber lines.

- 2B1Q code
  - Two binary bits encoded into one quaternary symbol (four-level PAM signal).
  - Zero DC component, and offers the best performance among all the codes introduced. So it is adopted as the standard as the North American standard for DSL.
4.8 Digital Subscriber Lines

Possible candidates for line codes that are suitable for DSL

2B1Q code (cont.)

With 2B1Q line coding, adaptive equalizer and echo cancellation, it is possible to achieve BER = 10^{-7} operating full duplex at 160 kb/s.
4.8 Asymmetric Digital Subscriber Lines

- ADSL is targeted to simultaneously support three services at a single twisted-wire pair
  - Data transmission downstream at 9 Mbps
  - Data transmission upstream at 1Mbps
  - Plain old telephone service (POTS)

- Some notes
  - It is named *asymmetric* because the downstream bit rate is much higher than the upstream bit rate.
  - The actually achievable bit rates depend on the length of the twisted pair used to do the transmission.

- Frequency-division multiplexing (FDM) technique is used to combine analog voice and DSL data.
  - Upstream and downstream data transmission are placed in different frequency band to avoid crosstalk.
4.8 Asymmetric Digital Subscriber Lines

- Various applications can be applied to asymmetric transmissions, such as video-on-demand (VoD).
  - For example
    - Downstream = 1.544 Mbps (DS1) for video data
    - Upstream = 160 kbps for real-time control commands.

4.9 Optimum Linear Receiver

- Zero-forcing equalizer
  - A receiver design is to use a zero-forcing equalizer followed by a decision-making device.
  - The design objective of a zero-forcing equalizer is to force the ISI to zero at all sampling instances \( t = kT_b \) for \( k \neq 0 \), provided that “the channel noise \( w(t) \) is zero.”
4.9 Optimum Linear Receiver

Zero-forcing equalizer (cont.)

- This reduces to the Nyquist criterion.

\[
\sum_{n=-\infty}^{\infty} P\left(f - \frac{n}{T_b}\right) = T_b \quad \text{or} \quad p(nT_b) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}
\]

where \( P(f) = G(f)H(f)C(f) \).

A serious consequence of the ignorance of \( w(t) \) in the design of a zero-forcing equalizer is the performance degradation due to noise enhancement.
4.9 Optimum Linear Receiver

Example of noise enhancement.

- Suppose that the receiver filter is a tapped-delay-line equalizer, which is of the form

\[ c(t) = \sum_{k=0}^{\infty} c_k \delta(t - kT_b) \]

- Assume ideally that \( G(f) = 1 \).
- Hence, the Nyquist criterion becomes:

\[
p(nT_b) = \begin{cases} 
1, & n = 0 \\ 
0, & n \neq 0 
\end{cases}
\]

where \( P(f) = H(f)C(f) \).

\[
p(t) = \int_{-\infty}^{\infty} h(\tau)c(t-\tau)d\tau \\
= \int_{-\infty}^{\infty} \left( \sum_{k=0}^{\infty} c_k \delta(t - \tau - kT_b) \right) d\tau \\
= \sum_{k=0}^{\infty} c_k \int_{-\infty}^{\infty} h(\tau)\delta(t - \tau - kT_b) d\tau \\
= \sum_{k=0}^{\infty} c_k h(t - kT_b) \\
p_n = p(nT_b) = \sum_{k=0}^{\infty} c_k h((n-k)T_b) = \sum_{k=0}^{\infty} c_k h_{n-k} = \begin{cases} 
1, & n = 0 \\ 
0, & n \neq 0 
\end{cases}
\]
It is reasonable to assume that $h_n = 0$ for $n < 0$, and $h_0 = 1$.

\[
\begin{bmatrix}
1 & 1 & 0 & 0 & \cdots & 0 \\
0 & h_1 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & h_N & h_{N-1} & h_{N-2} & \cdots & 1 \\
\end{bmatrix}
= \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{N-1} \\ c_N \end{bmatrix}
\]

⇒ $h_n = 0$ for arbitrary $N > 0$.

Suppose

\[
h(\tau) = \begin{cases} 1 & |\tau|/(2T_b), \quad 0 \leq \tau < 2T_b \\ 0, & \text{otherwise} \end{cases}
\]

⇒ $h_0 = 1, h_1 = \frac{1}{2},$ and $h_n = 0$ for $n \neq 0,1$.

⇒ $c_n = (-1)^n 2^{-n}$ for $(N \geq) n \geq 0$, and zero, otherwise.

The above $c(t)$ can successfully remove ISI, provided $w(t) = 0$.

Now add the additive white Gaussian noise $w(t)$, which also passes the filter $c(t)$.

At any time instance $nT_b$, the sampled noise becomes

\[
\int_{-\infty}^{\infty} w(\tau)c(nT_b - \tau)d\tau = \int_{-\infty}^{\infty} w(\tau) \sum_{k=0}^{\infty} c_k \delta(nT_b - kT_b - \tau)d\tau
\]

\[
= \sum_{k=0}^{\infty} c_k \int_{-\infty}^{\infty} w(\tau) \delta(nT_b - kT_b - \tau)d\tau = \sum_{k=0}^{\infty} c_k w(nT_b - kT_b) = \sum_{k=1}^{\infty} c_k w_{n-k}
\]

The sampled noise variance then becomes:

\[
\text{Var}\left[\sum_{k=0}^{\infty} c_k w_{n-k}\right] = \sum_{k=0}^{\infty} c_k^2 \text{Var}[w_{n-k}] = \sigma_w^2 \sum_{k=0}^{\infty} 2^{-2k} = \frac{4}{3} \sigma_w^2 > \sigma_v^2
\]
An easier way to interpret the noise enhancement phenomenon.

The Nyquist criterion requires that:

\[ \sum_{n=-\infty}^{\infty} P\left( f - \frac{n}{T_b} \right) = \sum_{n=-\infty}^{\infty} H\left( f - \frac{n}{T_b} \right) C\left( f - \frac{n}{T_b} \right) = T_b \]

A sufficient condition for the Nyquist criterion is that:

\[ H(f)C(f) = \text{Raised Cosine Spectrum} \]

When \( H(f) \) is very small (or zero) at some frequency range, \( C(f) \) has to be very large (or infinity) at the same frequency range in order to “equalize” the spectrum.

Thus, the noise spectrum \( S_W(f)C(f)^2 \) after passing through \( C(f) \) will be “enhanced.”

4.9 Optimum Linear Receiver

To alleviate noise enhancement phenomenon, it is better to simultaneously consider the ISI and channel noise.

An approach of this kind is to use the mean-square error criterion, and find a balanced solution to the problem of reducing the effects of both channel noise and intersymbol interference.
\[
\begin{align*}
y(t) &= c(t) \ast x(t) = \int_{-\infty}^{\infty} c(\tau) x(t-\tau) d\tau \\
x(t) &= \sum a_k q(t-kT_b) + w(t) \\
q(t) &= g(t) \ast h(t)
\end{align*}
\]

\[\Rightarrow y(iT_b) = \sum a_k \int_{-\infty}^{\infty} c(\tau) q(iT_b - \tau - kT_b) d\tau + \int_{-\infty}^{\infty} c(\tau) w(iT_b - \tau) d\tau = \xi_i + n_i\]

For perfect receiver, \(y(iT_b) = a_i\).

So, the error \(e_i = (\xi_i + n_i) - a_i\).

The mean squared error criterion then wishes to minimize:

\[
J_i = E[e_i^2] = E[(\xi_i + n_i - a_i)^2].
\]

\[
= E[\xi_i^2] + E[n_i^2] + E[a_i^2] + 2E[\xi_i n_i] - 2E[n_i a_i] - 2E[\xi_i a_i]
\]

For i.i.d. \(\{a_k\}\) where \(a_k = \pm 1\),

\[
E[\xi_i^2] = \sum a_k^2 \sum \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c(\tau_1) c(\tau_2) q(iT_b - \tau_1 - kT_b) q(iT_b - \tau_2 - lT_b) d\tau_1 d\tau_2
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c(\tau_1) c(\tau_2) R_q(\tau_1, \tau_2; i) d\tau_1 d\tau_2
\]

where \(R_q(\tau_1, \tau_2; i) = \sum q(iT_b - \tau_1 - kT_b) q(iT_b - \tau_2 - lT_b)\)
Observe that $R_q(\tau_1, \tau_2;i) = \sum_k q(iT_b - kT_b - \tau_1)q(iT_b - kT_b - \tau_2)$ only depends on the difference between $\tau_1$ and $\tau_2$, and is invariant with respect to $i$. We can then re-express it as $R_q(\tau_1 - \tau_2)$.

\[
E[\xi_1^2] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c(\tau_1)c(\tau_2)R_q(\tau_1 - \tau_2)d\tau_1 \tau_2
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c(\tau_1)c(\tau_2)\left(\int_{-\infty}^{\infty} S_q(f)e^{i2\pi f(\tau_1 - \tau_2)} df\right) d\tau_1 \tau_2
= \int_{-\infty}^{\infty} S_q(f) \left(\int_{-\infty}^{\infty} c(\tau_1)e^{-i2\pi (-f)\tau_1} d\tau_1\right) \left(\int_{-\infty}^{\infty} c(\tau_2)e^{-i2\pi f\tau_2} d\tau_2\right) df
= \int_{-\infty}^{\infty} S_q(f)C(-f)C(f)df
= \int_{-\infty}^{\infty} S_q(f)|C(f)|^2 df.
\]

\[
E[n_1^2] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c(\tau_1)c(\tau_2)E[w(iT_b - \tau_1)w(iT_b - \tau_2)]d\tau_1 d\tau_2
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c(\tau_1)c(\tau_2) \frac{N_0}{2} \delta(\tau_1 - \tau_2) d\tau_1 d\tau_2
= \frac{N_0}{2} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} C(f_1)e^{i2\pi f_1\tau_1} df_1\right) \left(\int_{-\infty}^{\infty} C(f_2)e^{i2\pi f_2\tau_2} df_2\right) d\tau
= \frac{N_0}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C(f_1)C(f_2) \left(\int_{-\infty}^{\infty} e^{-i2\pi [-f_1+f_2]\tau} d\tau\right) df_1 df_2
= \frac{N_0}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C(f_1)C(-f_1)\delta(-f_1 - f_2) df_1 df_2
= \frac{N_0}{2} \int_{-\infty}^{\infty} C(f_1)|C(-f_1)|^2 df_1
= \frac{N_0}{2} \int_{-\infty}^{\infty} |C(f)|^2 df.
\]
3rd term

For i.i.d. $\{a_k\}$ where $a_k = \pm 1$, $E[a_k^2] = 1$.

4th and 5th term

By independence of $\{a_k\}$ and $w(t)$, and zero mean of $n$, $E[\xi n_i] = E[\xi] E[n_i] = 0$ and $E[n_i a_i] = E[n_i] E[a_i] = 0$.

6th term

$$E[\xi a_i] = \sum E[a_i a_j] \int_{-\infty}^{\infty} c(\tau) q(iT_h - kT_b - \tau) d\tau = \int_{-\infty}^{\infty} c(\tau) q(-\tau) d\tau$$

$$= \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} C(f_1) e^{i2\pi f_1 \tau} df_1 \right) \left( \int_{-\infty}^{\infty} Q(f_2) e^{i2\pi f_2 (-\tau)} df_2 \right) d\tau$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C(f_1) Q(f_2) \left( \int_{-\infty}^{\infty} e^{-i2\pi (f_2 - f_1) \tau} df_1 \right) df_1 df_2$$

where the last step follows from the observation that $E[\xi a_i]$ must be a real number, and $C_r(f)$ and $C_i(f)$ are respectively the real and imaginary parts of $C(f)$, i.e., $C(f) = C_r(f) + iC_i(f)$, and similarly $Q(f) = Q_r(f) + iQ_i(f)$.

Substitute all six terms into $J_i$.

$$J_i = \int_{-\infty}^{\infty} \left( S_q(f) + \frac{N_0}{2} \right) \frac{|C(f)|^2 - 2Q_r(f)C_r(f) + 2Q_i(f)C_i(f)}{A(f)} df + 1$$
\[ A(f) = \left( S_q(f) + \frac{N_0}{2} \right) |C(f)|^2 - 2Q_t(f)C_t(f) + 2Q_t(f)C_t(f) \]
\[ = \left( S_q(f) + \frac{N_0}{2} \right) C_t^2(f) - 2Q_t(f)C_t(f) \]
\[ + \left( S_q(f) + \frac{N_0}{2} \right) C_t^2(f) + 2Q_t(f)C_t(f) \]
\[ = \left( S_q(f) + \frac{N_0}{2} \right) \left[ C_t(f) - \frac{Q_t(f)}{(S_q(f) + N_0/2)} \right]^2 - \frac{Q_t^2(f)}{(S_q(f) + N_0/2)} \]
\[ + \left( S_q(f) + \frac{N_0}{2} \right) \left[ C_t(f) + \frac{Q_t(f)}{(S_q(f) + N_0/2)} \right]^2 - \frac{Q_t^2(f)}{(S_q(f) + N_0/2)} \]
\[ \Rightarrow C(f) = \frac{Q^*(f)}{S_q(f) + N_0/2} \] for MMSE equalizer.

An equalizer that is so designed is referred to as the minimum-mean square error (MMSE) equalizer.

4.9 MMSE Equalizer

- Summary
  - The MMSE equalizer can be viewed as the concatenation of two filters:
    - a matched filter \( Q^*(f) \) to \( Q(f) = G(f)H(f) \)
    - an equalizer whose frequency response is the inverse of \( S_q(f) + N_0/2 \).
4.9 MMSE Equalizer

Property of \( S_q(f) \)

The text wrote that 
\[
S_q(f) = \frac{1}{T_b} \sum_{k} Q \left( f + \frac{k}{T_b} \right),
\]
which is periodic with period \( 1/T_b \). This implies that \( R_q(\tau) \) consists of a series of pulse train with width \( T_b \), which is not entirely true.

\[
R_q(\tau_1 - \tau_2) = \sum q(kT_b - \tau_1)q(kT_b - \tau_2)
\]

\[
S_q(f) = \int_{-\infty}^{\infty} R_q(\tau) \exp(-j2\pi f \tau) d\tau = \int_{-\infty}^{\infty} \left( \sum q(kT_b - \tau)q(kT_b) \right) \exp(-j2\pi f \tau) d\tau
\]

\[
= \sum q(kT_b) \int_{-\infty}^{\infty} q(kT_b - \tau) \exp(-j2\pi f \tau) d\tau
\]

\[
= \sum q(kT_b) \int_{-\infty}^{\infty} q(v) \exp(-j2\pi (kT_b - v)) dv
\]

\[
= \sum q(kT_b) \exp(-j2\pi kT_b) \int_{-\infty}^{\infty} q(v) \exp(j2\pi v) dv
\]

\[
= Q' \left( f \right) \sum q(kT_b) \exp(-j2\pi kT_b)
\]

\[
= Q' \left( f \right) \int_{-\infty}^{\infty} \left( \sum q(t) \delta(t - kT_b) \right) \exp(-j2\pi f t) dt
\]

\[
= Q' \left( f \right) \frac{1}{T_b} \sum Q \left( f + \frac{k}{T_b} \right)
\]
4.9 Implementation of MMSE Equalizer

- One can approximate $\frac{1}{S_q(f) + N_0/2}$ by a periodic function with:

$$S_q(f) = Q^*(f) \cdot \frac{1}{T_b} \sum_k Q(f + \frac{k}{T_b}) = \frac{1}{T_b} \sum_k Q(f + \frac{k}{T_b}) = \tilde{S}_q(f)$$

- Since $\Theta_q(f) = \frac{1}{\tilde{S}_q(f) + N_0/2}$ is now periodic with period $1/T_b$, we obtain by Fourier series that

$$\Theta_q(f) = \sum_{k=-\infty}^{\infty} c_k \exp(j2\pi k T_b)$$

where $c_k = T_b \int_{-1/(2T_b)}^{1/(2T_b)} \Theta_q(f) \exp(-j2\pi k T_b) df$. 

---

4.9 Implementation of MMSE Equalizer

- We can approximate $\Theta_q(f)$ by its main $2N+1$ terms as:

$$\Theta_q(f) = \sum_{k=1}^{N} c_k \exp(j2\pi k T_b) \Rightarrow \theta_q(\tau) = \sum_{k=1}^{N} c_k \delta(\tau + k T_b)$$

One can therefore approximate $\frac{1}{[S_q(f) + N_0/2]}$ by a transversal tapped-delay-line equalizer.
4.9 Implementation of MMSE Equalizer

- Final notes
  - In a real-life telecommunication environment, the channel is usually time-varying.
  - Therefore, an adaptive receiver that provides for the adaptive implementation of both the matched filter and the equalizer in a combined manner is usually necessary.

4.10 Adaptive Equalization

- The equalizer is adjusted under the guidance of a training sequence transmitted through the channel.
4.10 Adaptive Equalization

- Least-mean-square (LMS) algorithm
  \[ e[n] = d[n] - y[n] = d[n] - \sum_{i=0}^{N} w_i x[n - k] \]

- Design objective
  - To find the filter coefficients \( w_0, w_1, \ldots, w_N \) so as to minimize index of performance \( J \):
    \[ J = e^2[n] \]

  So when \( g_i > 0 \), \( w_i \) should be decreased.
  - On the contrary, \( w_i \) should be increased if \( g_i < 0 \).
  - Hence, we may define the update rule as:
    \[ \hat{w}_{i,\text{next}} = \hat{w}_{i,\text{current}} - \frac{1}{2} \mu \cdot g_i \]
    where \( \mu \) is a chosen constant step size, and \( \frac{1}{2} \) is included only for convenience of analysis.
4.10 Adaptive Equalization

\[ J = \left( d[n] - \sum_{k=0}^{N} w_k n[k] \right)^2 \]
\[ = d^2[n] - 2 \sum_{k=0}^{N} w_k d[n] n[k] + \sum_{k=0}^{N} \sum_{j=0}^{N} w_k w_j n[k] n[j] \]
\[ g_i = \frac{\partial J}{\partial w_i} = -2 d[n] n[i] + 2 \sum_{j=0}^{N} w_j n[k] n[i] \]
\[ = -2 n[i] \left( d[n] - \sum_{k=0}^{N} w_k n[k] \right) \]
\[ = -2 n[i] e[n] \]

Some notes on LMS algorithm

- There is no guarantee that the algorithm converges to a local minimum (could converge to a saddle point).
- There is even no guarantee that the algorithm converges.
4.10 Adaptive Equalization

- Some notes on LMS algorithm (cont.)
  - If $\mu$ is too large, high excess mean-square error may occur.
  - If $\mu$ is too small, a slow rate of convergence may arise.

4.10 Operation of the Equalizer

- Two modes of operations for adaptive equalizer
  - Training mode (position 1)
  - Decision-directed mode (position 2)
4.10 Decision-Directed Mode

- In normal operation, the decisions made by the receiver are correct with high probability.
- Under such premise, we can use the previous decisions to calibrate or track the tap coefficients.
- In this mode,
  - if $\mu$ is too large, high excess mean-square error may occur.
  - if $\mu$ is too small, a too-slow tracking may arise.
- We can further extend the idea of decision-directed or decision-feedback to the decision-feedback equalizer (DFE).

4.10 Decision-Feedback Equalizer

\[
\begin{align*}
\Sigma &= a_n - e_n^T v_n, \\
\end{align*}
\]

Let $e_n = \begin{bmatrix} \hat{w}_n^{(1)} \\ \hat{w}_n^{(2)} \end{bmatrix}$ and $v_n = \begin{bmatrix} x_n \\ \hat{a}_n \end{bmatrix}$, where $n$ = sample time at $nT$.

Denote $e_n = a_n - e_n^T v_n$, where $a_n$ is the $n$th transmitted symbol.
4.10 Decision-Feedback Equalizer

- Then DFE gives:

$$
\begin{bmatrix}
\mathbf{w}_{n+1}^{(1)} \\
\mathbf{w}_{n+1}^{(2)}
\end{bmatrix} = 
\begin{bmatrix}
\mathbf{w}_{n}^{(1)} \\
\mathbf{w}_{n}^{(2)}
\end{bmatrix} +
\begin{bmatrix}
\mu_{e} x_{e} \\
\mu_{e} \hat{a}_{e}
\end{bmatrix}.
$$

- As anticipated, DFE suffers from error propagation due to incorrect decisions.
- However, error propagation will not persist indefinitely; rather, it tends to occur in bursts.
  - E.g., if the number of taps in the feedback section is $L$, then the influence of one decision error will be flushed out after subsequent $L$ correct decisions.

4.11 Computer Experiments: Eye Patterns

- Eye pattern: The synchronized superposition of all possible realizations of the signal of interest viewed within a particular signaling interval.
4.11 Computer Experiments: Eye Patterns

Eye pattern for pulse shaping function $p(t)$ is half-cycle sine wave with duration $T_b$, and error-free BPSK transmission.

4.11 Computer Experiments: Eye Patterns

Eye pattern for pulse shaping function $p(t)$ is half-cycle sine wave with duration $2T_b$, and error-free BPSK transmission.
Interpretation of Eye Pattern

- Best sampling time
- Distortion at sampling time
- Margin over noise
- Distortion of zero-crossings
- Slope = sensitivity to timing error
- Time interval over which the received signal can be sampled
Experiment 1: Effect of channel noise
(Raise-cosine pulse-shaping with roll-off factor $\alpha = 0.5, W = 0.5$ Hz, $M = 4$)
(a) Eye diagram for noiseless quaternary system.
(b) Eye diagram for quaternary system with SNR = 20 dB.
(c) Eye diagram for quaternary system with SNR = 10 dB.

Experiment 2: Effect of bandwidth limitation
(Raise-cosine pulse-shaping with roll-off factor $\alpha = 0.5, W = 0.5$ Hz, $M = 4$)
(a) Eye diagram for noiseless band-limited quaternary system: cutoff frequency $f_o = 0.975$ Hz
(b) Eye diagram for noiseless band-limited quaternary system: cutoff frequency $f_o = 0.5$ Hz
(The channel is now modeled by a low-pass Butterworth filter with
\[ |H(f)|^2 = \frac{1}{1 + (f/f_o)^{50}} \]