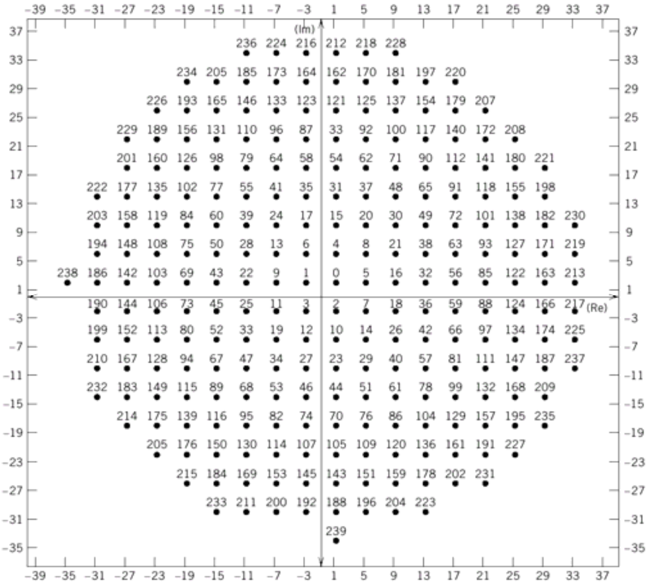


Superconstellation of V.34 modem with 960 signal points.

6.11 Voiceband (PSTN) modems

- Adaptive bandwidth
  - A set of probe tones will be transmitted for measurement of SNR as a function of frequency.
  - Then the appropriate carrier frequency and bandwidth will be appropriately selected based on the measurement results.
- Adaptive bit rates
  - Selection of bit rates subject to bit error rate requirement.

$$10^{-5} \geq \text{BER} \geq 10^{-6}$$

6.11 Voiceband (PSTN) modems

- Trellis coding
  - Compulsory trellis coding provides 3.6 dB coding gain
  - Optional trellis coding provides 4.7 dB coding gain
- Decision feedback equalizer
  - Decision feedback equalizer requires immediate decision that cannot be directly obtained when channel coding technology is introduced.
  - Hence, the DFE is implemented at the transmitter end, rather than the receiver end (Tomlinson-Harashima precoding).

6.12 Multichannel modulation

- Shannon information capacity theorem

$$C = B\log_2(1+\mathrm{SNR})\ \mathrm{b/s} = \tfrac{1}{2}\log_2(1+\mathrm{SNR})\ \mathrm{b/transmission}$$

where B is the baseband bandwidth. One transmission takes $\frac{1}{2B}$ seconds.

- Alternatively, we can write:

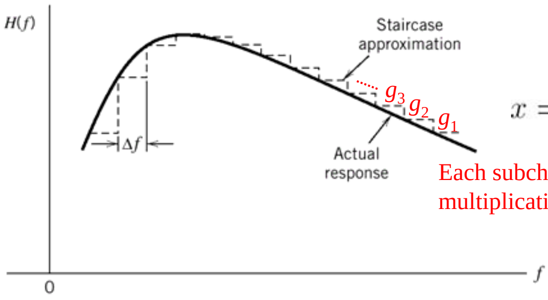
$$\mathrm{SNR} = 2^{2C} - 1$$

- Therefore, we can define the gap between ideal Shannon SNR, and the SNR attainable for a rate R below C as

$$\Gamma = \frac{2^{2C}-1}{2^{2R}-1} = \frac{\mathrm{SNR}}{2^{2R}-1} \quad \text{or} \quad R = \frac{1}{2}\log_2\left(1+\frac{\mathrm{SNR}}{\Gamma}\right)$$

6.12 Multichannel modulation

- ☐ Multichannel modulation
  - Partition a channel (with squared magnitude response |*H*(*f*)| as shown below) into a number of subchannels such that each subchannel becomes AWGN



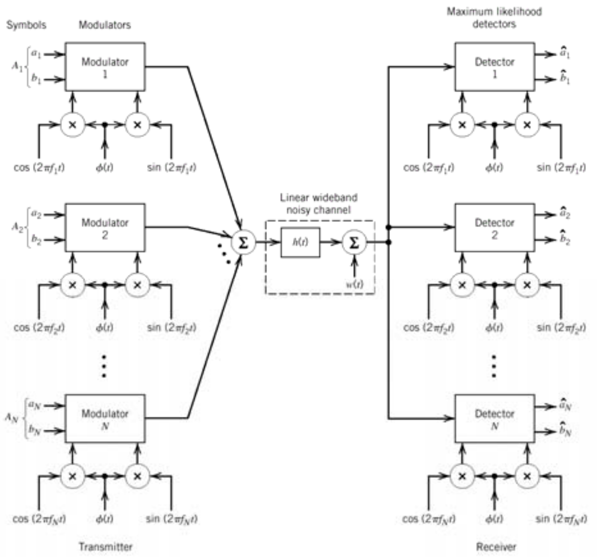
$$x = g \cdot s + w$$

Each subchannel has different multiplicative gain.

- ☐ Block diagram of multichannel data transmission system

$$f_n = \frac{n}{T},\ n = 1, 2, \cdots, N$$

$$\phi(t) = \sqrt{\frac{2}{T}}\text{sinc}\left(\frac{t}{T}\right)$$
for $-\infty < t < \infty$



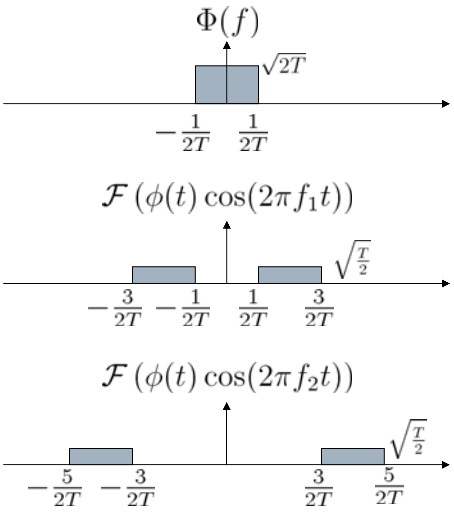
6.12 Multichannel modulation

- ☐ Properties of multichannel modulation
  - Property 1: Orthogonality of the two quadrature-modulated sinc functions (in the sense of integration over the entire real line)

$$\begin{aligned}&\langle \phi(t)\cos(2\pi f_n t), \phi(t)\sin(2\pi f_n t)\rangle\\ =\ &\int_{-\infty}^{\infty} \phi(t)\cos(2\pi f_n t)\phi(t)\sin(2\pi f_n t)dt = 0\end{aligned}$$

$$\int_{-\infty}^{\infty} f(t)g^*(t)dt = \int_{-\infty}^{\infty} F(f)G^*(f)df$$
where $F(f) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi ft}dt$
and $G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt.$

- Property 2: We also have orthogonality among subchannel signals



Introduction of $\phi(t)$ makes no “band overlap” between adjacent subchannels.

6.12 Multichannel modulation

- Property 3: Orthogonality among subchannel signals remains after passing subchannel signals through **linear** channel with arbitrary response *h*.

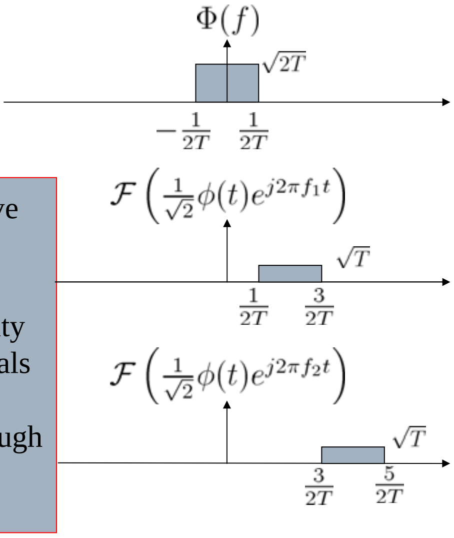
$$\left\{\frac{1}{\sqrt{2}}\phi(t)\exp(j2\pi f_n t)\right\}_{n=1}^{N} \rightarrow \boxed{h(t)} \rightarrow \left\{\frac{1}{\sqrt{2}}\phi(t)\exp(j2\pi f_n t) * h(t)\right\}_{n=1}^{N}$$

Still, no “band overlap” between “convolved” adjacent subchannels.

$$\int_{-\infty}^{\infty} f(t)g^*(t)dt = \int_{-\infty}^{\infty} F(f)G^*(f)df$$

where $F(f) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi ft}dt$

and $G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt.$



- Property 2: We also have orthogonality among subchannel signals
- Property 3: Orthogonality among subchannel signals remains after passing subchannel signals through **linear** channel with arbitrary response *h*.

Introduction of $\phi(t)$ makes no “band overlap” between adjacent subchannels.

6.12 Multichannel modulation

- Geometric SNR for multichannel modulation
  - The average rate (in bits per transmission per subchannel)

$$\begin{aligned} R &= \frac{1}{N}\sum_{n=1}^{N} R_n \\ &= \frac{1}{2N}\sum_{n=1}^{N}\log_2\left(1+\frac{\mathrm{SNR}_n}{\Gamma}\right) \\ &= \frac{1}{2N}\log_2\prod_{n=1}^{N}\left(1+\frac{\mathrm{SNR}_n}{\Gamma}\right) \\ &= \frac{1}{2}\log_2\prod_{n=1}^{N}\left(1+\frac{\mathrm{SNR}_n}{\Gamma}\right)^{1/N} \end{aligned}$$

$$R = \frac{1}{2}\log_2\left(1+\frac{\mathrm{SNR}_{\mathrm{overall}}}{\Gamma}\right)$$

6.12 Multichannel modulation

We then obtain:

$$\mathrm{SNR}_{\mathrm{overall}} = \Gamma\left(\prod_{n=1}^{N}\left(1+\frac{\mathrm{SNR}_n}{\Gamma}\right)^{1/N}-1\right)$$

Assume that

$$\frac{\mathrm{SNR}_n}{\Gamma} \gg 1.$$

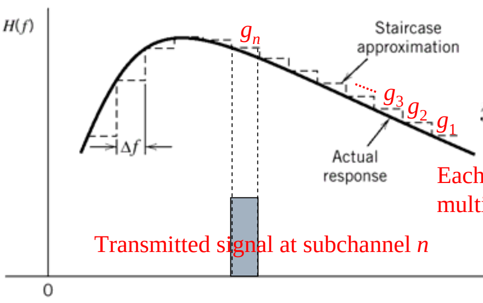
$$\mathrm{SNR}_{\mathrm{overall}} \approx \Gamma\left(\prod_{n=1}^{N}\left(\frac{\mathrm{SNR}_n}{\Gamma}\right)^{1/N}\right) = \left(\prod_{n=1}^{N}\mathrm{SNR}_n\right)^{1/N}$$

Geometric sum of individual **SNR**

$$\boxed{\mathrm{SNR}_{\mathrm{overall}} \ge \left(\prod_{n=1}^{N}\mathrm{SNR}_n\right)^{1/N} \text{ with equality if, and only if, } \mathrm{SNR}_1 = \mathrm{SNR}_2 = \cdots = \mathrm{SNR}_N.}$$

6.12 Multichannel modulation

- ☐ Solution of loading problem – water filling
  - The process of allocating the transmit power *P* to the individual subchannel so as to maximize the system bit rate is called *loading*.



Transmitted signal at subchannel *n*

$$\boxed{\mathrm{SNR}_n = \frac{g_n^2 P_n}{\sigma_n^2}}$$

$$x_n = g_n \cdot s_n + w_n$$

Each subchannel has different multiplicative gain.

6.12 Multichannel modulation

$$\max_{\{(P_1,P_2,\cdots,P_n):\sum_{n=1}^{n} P_n \le P\}} \frac{1}{2N}\sum_{n=1}^{N}\log_2\left(1+\frac{g_n^2 P_n}{\Gamma\sigma_n^2}\right)$$

$$= \max_{\{(P_1,P_2,\cdots,P_n):\sum_{n=1}^{n} P_n = P\}} \frac{1}{2N}\sum_{n=1}^{N}\log_2\left(1+\frac{g_n^2 P_n}{\Gamma\sigma_n^2}\right)$$

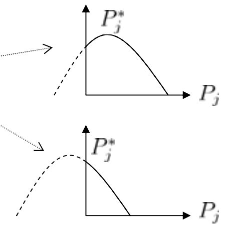
$$= \max_{\{(P_1,P_2,\cdots,P_n):\sum_{n=1}^{n} P_n = P\}} \left[\frac{1}{2N}\sum_{n=1}^{N}\log_2\left(1+\frac{g_n^2 P_n}{\Gamma\sigma_n^2}\right) + \underset{\substack{\uparrow \\ \text{Lagrange multiplier}}}{\lambda}\left(P-\sum_{n=1}^{N}P_n\right)\right]$$

6.12 Multichannel modulation

$$f(P_1,P_2,\cdots,P_n|\lambda)=\frac{1}{2N}\sum_{n=1}^{N}\log_2\left(1+\frac{g_n^2P_n}{\Gamma\sigma_n^2}\right)+\lambda\left(P-\sum_{n=1}^{N}P_n\right)$$ concave with respect to P_j

Hence,

$$\left.\frac{\partial f(P_1,P_2,\cdots,P_n|\lambda)}{\partial P_j}\right|_{P_j=P_j^*}\begin{cases}=0, & \text{if } P_j^*>0\\ \le 0, & \text{if } P_j^*=0\end{cases}$$



As a result,

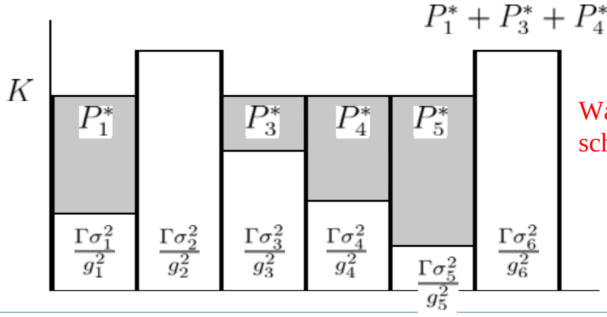
$$\frac{\log_2(e)}{2N}\frac{g_n^2/(\Gamma\sigma_j^2)}{1+g_j^2P_j^*/(\Gamma\sigma_j^2)}-\lambda\begin{cases}=0, & \text{if } P_j^*>0\\ \ge 0, & \text{if } P_j^*=0\end{cases}$$

$$\frac{1}{\Gamma\sigma_j^2/g_j^2+P_j^*}\begin{cases}=\lambda\frac{2N}{\log_2(e)}=\frac{1}{K}, & \text{if } P_j^*>0\\ \le\lambda\frac{2N}{\log_2(e)}=\frac{1}{K}, & \text{if } P_j^*=0\end{cases}$$ and $$\sum_{n=1}^{n}P_j^*=P$$

$$\Downarrow$$

$$\Gamma\sigma_j^2/g_j^2+P_j^*\begin{cases} =K, & \text{if } P_j^*>0\\ =\Gamma\sigma_j^2/g_j^2\ \ge K, & \text{if } P_j^*=0\end{cases}$$ and $$\sum_{n=1}^{n}P_j^*=P$$

$$P_1^*+P_3^*+P_4^*+P_5^*=P$$



Water filling scheme

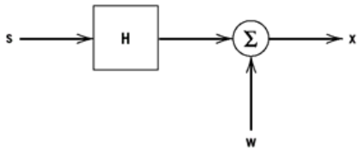
6.13 Discrete multitone

- Impractical in (analog) multichannel modulation
  - Sinc function is time-unlimited (as it is band-limited).
  - The inner product (defined in time-domain) requires to perform integration over the entire real line.
    - Performing integration over a practically finite range will make the (analog) multichannel modulation suboptimal as text has put it.

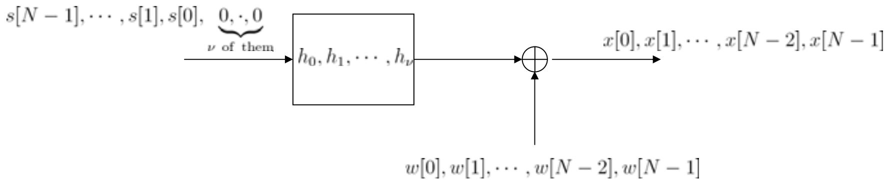
$$\begin{aligned} & \langle \phi(t)\cos(2\pi f_n t), \phi(t)\sin(2\pi f_n t)\rangle \\ = & \int_{-\infty}^{\infty} \phi(t)\cos(2\pi f_n t)\phi(t)\sin(2\pi f_n t)dt = 0 \end{aligned}$$

6.13 Discrete multitone

- Solution
  - Discrete multitone (DMT)
  - Transform *linear discrete convolution* to *circular discrete convolution* by adding *cyclic prefix*.
- Procedure of DMT
  - Sampling the analog signals with sufficiently large sampling rate $1/T_s$.



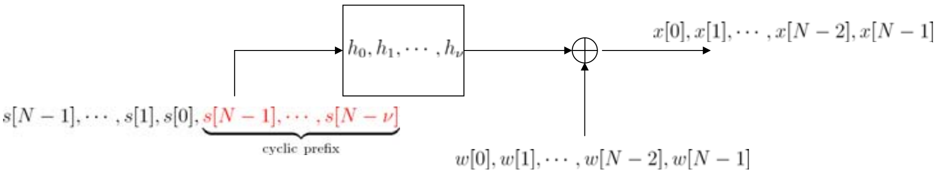
☐ Linear convolution



$$\begin{bmatrix} x[N-1] \\ x[N-2] \\ \vdots \\ x[N-\nu-1] \\ x[N-\nu-2] \\ \vdots \\ x[0] \end{bmatrix} = \begin{bmatrix} h_0 & h_1 & h_2 & \cdots & h_{\nu-1} & h_\nu & 0 & \cdots & 0 \\ 0 & h_0 & h_1 & \cdots & h_{\nu-2} & h_{\nu-1} & h_\nu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & h_0 & h_1 & \cdots & h_\nu \\ 0 & 0 & 0 & \cdots & 0 & 0 & h_0 & \cdots & h_{\nu-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & h_0 \end{bmatrix} \begin{bmatrix} s[N-1] \\ s[N-2] \\ \vdots \\ s[N-\nu-1] \\ s[N-\nu-2] \\ \vdots \\ s[0] \end{bmatrix} + \begin{bmatrix} w[N-1] \\ w[N-2] \\ \vdots \\ w[N-\nu-1] \\ w[N-\nu-2] \\ \vdots \\ w[0] \end{bmatrix}$$

- The above formula is valid under the assumption that
$$s[-\nu]=s[-\nu+1]=\cdots=s[-1]=0$$
- Without the "guard period", ISI occurs.

☐ Circular convolution



$$\begin{bmatrix} x[N-1] \\ X[N-2] \\ \vdots \\ x[N-\nu-1] \\ x[N-\nu-2] \\ \vdots \\ x[0] \end{bmatrix} = \begin{bmatrix} h_0 & h_1 & h_2 & \cdots & h_{\nu-1} & h_\nu & 0 & \cdots & 0 \\ 0 & h_0 & h_1 & \cdots & h_{\nu-2} & h_{\nu-1} & h_\nu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & h_0 & h_1 & \cdots & h_\nu \\ h_\nu & 0 & 0 & \cdots & 0 & 0 & h_0 & \cdots & h_{\nu-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ h_1 & h_2 & h_3 & \cdots & h_\nu & 0 & 0 & \cdots & h_0 \end{bmatrix} \begin{bmatrix} s[N-1] \\ s[N-2] \\ \vdots \\ s[N-\nu-1] \\ s[N-\nu-2] \\ \vdots \\ s[0] \end{bmatrix} + \begin{bmatrix} w[N-1] \\ w[N-2] \\ \vdots \\ w[N-\nu-1] \\ w[N-\nu-2] \\ \vdots \\ w[0] \end{bmatrix}$$

- Instead of zeroing the guard period, how about letting
$$s[-k]=s[N-k] \text{ for } k=1,2,\cdots,\nu$$

Circular convolution to Discrete Fourier Transform

$$\boldsymbol{x} = \mathbb{H}_{\text{circulant}}\boldsymbol{s} + \boldsymbol{w}$$

- Spectral decomposition of a circulant matrix

$$\mathbb{H} = \mathbb{Q}^{\dagger}\boldsymbol{\Lambda}\mathbb{Q}$$

where Λ is a diagonal matrix, and

$$\mathbb{Q} = \frac{1}{\sqrt{N}}\begin{bmatrix} e^{-j\frac{2\pi}{N}(N-1)(N-1)} & \cdots & e^{-j\frac{2\pi}{N}2(N-1)} & e^{-j\frac{2\pi}{N}(N-1)} & 1 \\ e^{-j\frac{2\pi}{N}(N-1)(N-2)} & \cdots & e^{-j\frac{2\pi}{N}2(N-2)} & e^{-j\frac{2\pi}{N}(N-2)} & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ e^{-j\frac{2\pi}{N}(N-1)} & \cdots & e^{-j\frac{2\pi}{N}2} & e^{-j\frac{2\pi}{N}} & 1 \\ 1 & \cdots & 1 & 1 & 1 \end{bmatrix}$$

Circular convolution to Discrete Fourier Transform

$$\boldsymbol{x} = \mathbb{Q}^{\dagger}\boldsymbol{\Lambda}\mathbb{Q}\boldsymbol{s} + \boldsymbol{w}$$

$$\Rightarrow \mathbb{Q}\boldsymbol{x} = \mathbb{Q}\mathbb{Q}^{\dagger}\boldsymbol{\Lambda}\mathbb{Q}\boldsymbol{s} + \mathbb{Q}\boldsymbol{w} \qquad \boxed{\mathbb{Q}\mathbb{Q}^{\dagger} = \mathbb{I}}$$

$$\Rightarrow \boldsymbol{X} = \boldsymbol{\Lambda}\boldsymbol{S} + \boldsymbol{W}$$

$$\Rightarrow X_k = \lambda_k S_k + W_k \text{ for } k = 0, 1, \cdots, N-1$$

(Here, $\{\lambda_k\}_{k=0}^{N-1}$ are assumed "known" or "can-be-accurately estimated".)

6.13 Discrete multitone

- ☐ With “cyclic prefix”, the discrete Fourier transform (DTF) technique can then be used straightforwardly.
  - DFT transform pair
    - ☐ Analysis equation versus synthesis equation

$$\begin{cases} X[k] = \dfrac{1}{\sqrt{N}} \displaystyle\sum_{n=0}^{N-1} x[n] \exp\left(-j\frac{2\pi}{N}kn\right) & \text{for } k = 0, 1, \cdots, N-1 \\ x[n] = \dfrac{1}{\sqrt{N}} \displaystyle\sum_{k=0}^{N-1} X[k] \exp\left(j\frac{2\pi}{N}kn\right) & \text{for } n = 0, 1, \cdots, N-1 \end{cases}$$

$$\boxed{\begin{cases} \boldsymbol{X} = \mathbb{Q}\, \boldsymbol{x} \\ \boldsymbol{x} = \mathbb{Q}^\dagger \boldsymbol{X} \end{cases}}$$

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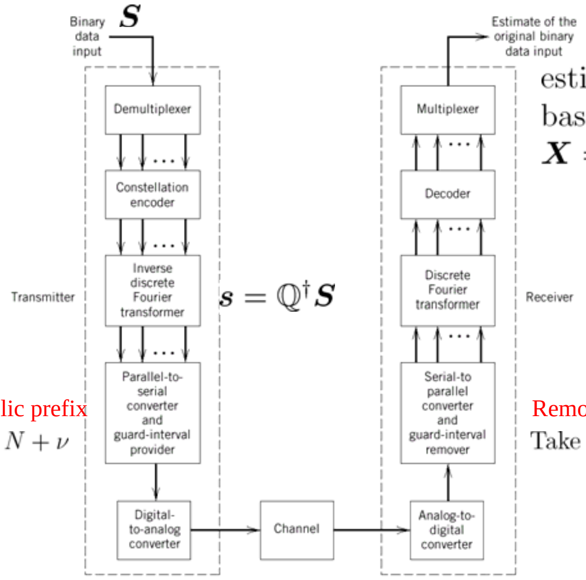
Chapter 6-223



$\boldsymbol{S}$

estimate $\hat{\boldsymbol{S}}$ based on $\boldsymbol{X} = \boldsymbol{\Lambda}\boldsymbol{S} + \boldsymbol{W}$

$\boldsymbol{s} = \mathbb{Q}^\dagger \boldsymbol{S}$

$\boldsymbol{X} = \mathbb{Q}\boldsymbol{x}$

Adding cyclic prefix
Of length $N + \nu$

Removing cyclic prefix
Take only N symbols

$$\boldsymbol{x} = \mathbb{Q}^\dagger \boldsymbol{\Lambda} \mathbb{Q} \boldsymbol{s} + \boldsymbol{w}$$

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Chapter 6-224

6.14 Synchronization

- ☐ Modes of synchronization
  - ■ Carrier synchronization (carrier recovery)
    - ☐ Including the estimate of carrier phase and frequency
  - ■ Symbol synchronization (clock recovery)
    - ☐ So as to know the timing for sampling and product-integrator

---

6.14 Synchronization

- ☐ Example: Decision-directed synchronization for *M*-ary PSK

$$s_k(t) = \sqrt{\frac{2E}{T}}\cos(2\pi f_c t + \alpha_k) \text{ for } 0 \le t < T$$

where E symbol energy, T symbol period, and $\alpha_k = 0, \frac{2\pi}{M}, \cdots, (M-1)\frac{2\pi}{M}$.

  - ■ Due to channel effect, we receive

$$x_k(t) = \sqrt{\frac{2E}{T}}\cos(2\pi f_c t + \alpha_k + \theta) + w(t) \text{ for } \tau \le t \le T + \tau$$

Note that for $-T+\tau \le t < \tau$, the received signal will be $x_{k-1}(t)$, a function of α_{k-1}. Hence, knowing τ is essential.

6.14 Synchronization

- Given that τ is accurately estimate, we shall estimate θ through likelihood ratio function.

$$\begin{cases} \phi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t), & \tau \le t < T+\tau \\ \phi_2(t) = \sqrt{\frac{2}{T}}\sin(2\pi f_c t), & \tau \le t < T+\tau \end{cases}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \text{ where} \begin{cases} x_i = \int_{\tau}^{T+\tau} x_k(t)\phi_i(t)dt \\ s_i = \int_{\tau}^{T+\tau} \sqrt{\frac{2E}{T}}\cos(2\pi f_c t + \alpha_k + \theta)\phi_i(t)d \\ w_i = \int_{\tau}^{T+\tau} w(t)\phi_i(t)dt \end{cases}$$

6.14 Synchronization

- For an observation window of size L_0, i.e., $k = 0, 1, \ldots, L_0-1$,

$$\begin{aligned} \hat{\theta} &= \arg\max_{\theta} f(\boldsymbol{x}_0, \boldsymbol{x}_1, \cdots, \boldsymbol{x}_{L_0-1}|\theta) \\ &= \arg\max_{\theta} \frac{1}{(\pi N_0)^{L_0}} \exp\left(-\frac{1}{N_0}\sum_{k=0}^{L_0-1}\|\boldsymbol{x}_k - \boldsymbol{s}_k(\theta)\|^2\right) \\ &= \arg\min_{\theta} \sum_{k=0}^{L_0-1}\|\boldsymbol{x}_k - \boldsymbol{s}_k(\theta)\|^2 \\ &= \arg\max_{\theta} \sum_{k=0}^{L_0-1} \boldsymbol{x}_k^T \boldsymbol{s}_k(\theta) \end{aligned}$$

$$\boxed{\boldsymbol{s}_k(\theta) = \begin{bmatrix} s_{1,k} \\ s_{2,k} \end{bmatrix} = \begin{bmatrix} \sqrt{E}\cos(\alpha_k + \theta) \\ -\sqrt{E}\sin(\alpha_k + \theta) \end{bmatrix}}$$

$$\hat{\theta} = \arg\max_{\theta} \sum_{k=0}^{L_0-1}[x_{1,k}\cos(\alpha_k+\theta) - x_{2,k}\sin(\alpha_k+\theta)] = \arg\max_{\theta} \ell(\theta)$$

which implies

$$\begin{aligned} 0 &= \partial\ell(\theta)/\partial\theta = \partial\left(\sum_{k=0}^{L_0-1}[x_{1,k}\cos(\alpha_k+\theta) - x_{2,k}\sin(\alpha_k+\theta)]\right)\Big/\partial\theta \\ &= \sum_{k=0}^{L_0-1}[-x_{1,k}\sin(\alpha_k+\theta) - x_{2,k}\cos(\alpha_k+\theta)] \\ &= \sum_{k=0}^{L_0-1}\mathrm{Im}\left\{(x_{1,k}-jx_{2,k})e^{-j(\alpha_k+\theta)}\right\} \\ \Bigg(&= \sum_{k=0}^{L_0-1}\mathrm{Im}\left\{a_k^*\tilde{x}_k e^{-j\theta}\right\}\Bigg) \end{aligned}$$

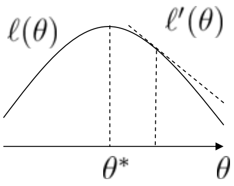
$\boxed{\text{As the text puts } \begin{cases} \tilde{x}_k = x_{1,k} - jx_{2,k} \\ a_k = e^{j\alpha_k} \end{cases}}$.

Question: Is the solution of the above equation unique? Is it certain that when derivative equals zero, we obtain the global maximum?

6.14 Synchronization

☐ Adaptive or recursive algorithm for ML estimation

$$\begin{cases} \text{decrease the current } \theta, & \text{if } \ell'(\theta) < 0 \\ \text{increase the current } \theta, & \text{if } \ell'(\theta) > 0 \end{cases}$$

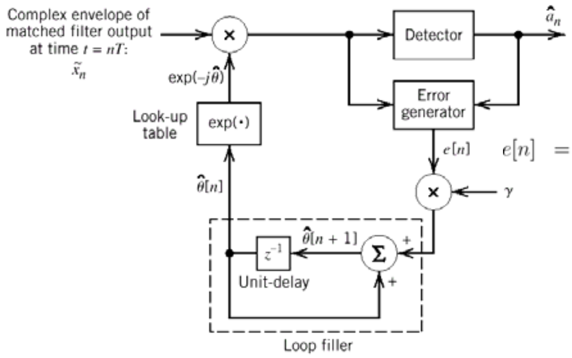


$$\begin{aligned} \hat{\theta}[n+1] &= \hat{\theta}[n] + \gamma \cdot \ell'(\hat{\theta}[n]) \\ &= \hat{\theta}[n] + \gamma \cdot \sum_{k=0}^{L_0-1}\mathrm{Im}\left\{a_k^*\tilde{x}_k e^{-j\hat{\theta}[n]}\right\} \end{aligned}$$

$\boxed{\gamma > 0 \text{ step size}}$

6.14 Synchronization

☐ First-order digital (loop) filter



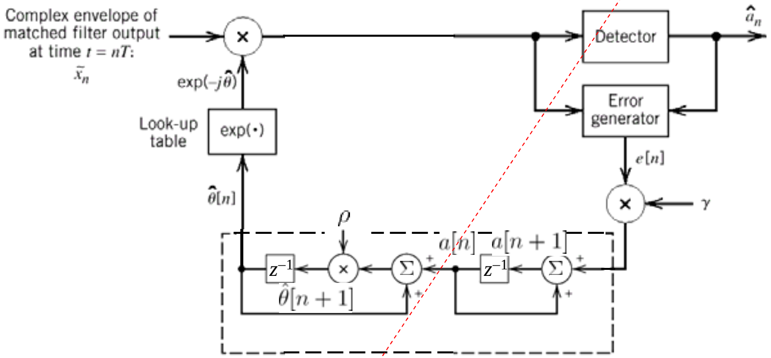
$$e[n] = \text{Im}\left\{a_n^* \tilde{x}_n e^{-j\hat{\theta}[n]}\right\}$$

with $L_0 = 1$

$$\begin{cases}\hat{\theta}[n+1] = \rho\hat{\theta}[n] + \rho a[n] \\ a[n+1] = a[n] + \gamma e[n]\end{cases} \Rightarrow \begin{cases}\hat{\theta}[n+2] = \rho\hat{\theta}[n+1] + \rho a[n+1] & (1) \\ \hat{\theta}[n+1] = \rho\hat{\theta}[n] + \rho a[n] & (2)\end{cases} \Rightarrow (1) - (2)$$

6.14 Synchronization

☐ An example of second-order digital (loop) filter



$$\hat{\theta}[n+2] = (1+\rho)\hat{\theta}[n+1] - \rho\hat{\theta}[n] + \rho\gamma e[n]$$

6.14 Synchronization

- ☐ The previous discussion estimates θ subject to known (or already-accurately-estimated) τ.
- ☐ We now need to estimate τ without the knowledge of θ.

$$\Rightarrow \begin{bmatrix} x_1(\tau) \\ x_2(\tau) \end{bmatrix} = \begin{bmatrix} s_1(\alpha_k, \theta, \tau|\tau_0) \\ s_2(\alpha_k, \theta, \tau|\tau_0) \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \text{ where}$$

$$\begin{cases} x_i(\tau) = \displaystyle\int_{\tau}^{T+\tau} x_k(t)\phi_i(t)dt \\ s_i(\alpha_k, \theta, \tau|\tau_0) = \displaystyle\int_{\tau}^{T+\tau} \sqrt{\frac{2E}{T}} \cos(2\pi f_c t + \alpha_k + \theta)\mathbf{1}\{\tau_0 \le t < \tau_0 + T\}\phi_i(t)dt \\ w_i = \displaystyle\int_{\tau}^{T+\tau} w(t)\phi_i(t)dt \end{cases}$$

$\boxed{\tau_0 \text{ is the true timing delay}}$

$\longrightarrow$ Its statistics is independent of τ.

6.14 Synchronization

- For an observation window of size L_0, i.e., $k = 0, 1, \ldots, L_0-1$,

$$\begin{aligned} \hat{\tau} &= \arg\max_{\tau} f(\boldsymbol{x}_0, \boldsymbol{x}_1, \cdots, \boldsymbol{x}_{L_0-1}|\{\alpha_k\}, \theta, \tau, \tau_0) \\ &= \arg\max_{\tau} \frac{1}{(\pi N_0)^{L_0}} \exp\left(-\frac{1}{N_0}\sum_{k=0}^{L_0-1} \|\boldsymbol{x}_k(\tau) - \boldsymbol{s}_k(\alpha_k, \theta, \tau|\tau_0)\|^2\right) \\ &= \arg\max_{\tau} \prod_{k=0}^{L_0-1} \exp\left(\frac{2}{N_0}\boldsymbol{x}_k(\tau)^T \boldsymbol{s}_k(\alpha_k, \theta, \tau|\tau_0)\right) \end{aligned}$$

$$\boxed{\boldsymbol{s}_k(\alpha_k, \theta, \tau|\tau_0) = \begin{bmatrix} \sqrt{\tilde{E}}\cos(\alpha_k + \theta) \\ -\sqrt{\tilde{E}}\sin(\alpha_k + \theta) \end{bmatrix}}$$

(See the next slide.)

As an example, assume $\tau_0 > \tau$ (up to the first three steps).

$$\begin{aligned}
s_1(\alpha_k, \theta, \tau|\tau_0) &= \frac{2\sqrt{E}}{T}\int_{\tau_0}^{T+\tau} \cos(2\pi f_c t + \alpha_k + \theta)\cos(2\pi f_c t)dt \\
&= \frac{\sqrt{E}}{T}\int_{\tau_0}^{T+\tau} [\cos(4\pi f_c t + \alpha_k + \theta) + \cos(\alpha_k + \theta)]\,dt \\
&= \underbrace{\cancel{\frac{\sqrt{E}}{T}\int_{\tau_0}^{T+\tau} \cos(4\pi f_c t + \alpha_k + \theta)dt}}_{\text{approximately zero}} + \frac{\sqrt{E}}{T}\int_{\tau_0}^{T+\tau} \cos(\alpha_k + \theta)dt \\
&\approx \sqrt{E}\left(\frac{T - |\tau_0 - \tau|}{T}\right)\cos(\alpha_k + \theta) \\
&= \sqrt{\tilde{E}}\cos(\alpha_k + \theta).
\end{aligned}$$

This is valid for both $\tau_0 > \tau$ and $\tau_0 \le \tau$.

6.14 Synchronization

- One way to solve the unknown is to average out all possible θ in the maximization operation.

$$\begin{aligned}
\hat{\tau} &= \arg\max_{\tau} \prod_{k=0}^{L_0-1} \frac{1}{2\pi}\int_0^{2\pi} \exp\left(\frac{2}{N_0}\boldsymbol{x}_k(\tau)^T \boldsymbol{s}_k(\alpha_k, \theta, \tau|\tau_0)\right) d\theta \\
&= \arg\max_{\tau} \prod_{k=0}^{L_0-1} \frac{1}{2\pi}\int_0^{2\pi} \exp\left\{\frac{2\sqrt{\tilde{E}}}{N_0}(|a_k\tilde{x}_k(\tau)|\cos(\arg[\tilde{x}_k(\tau)] - \arg[a_k] - \theta))\right\} d\theta \\
&= \arg\max_{\tau} \prod_{k=0}^{L_0-1} \frac{1}{2\pi}\int_0^{2\pi} \exp\left\{\frac{2\sqrt{\tilde{E}}}{N_0}|\tilde{x}_k(\tau)|\cos(\theta)\right\} d\theta \\
&= \arg\max_{\tau} \prod_{k=0}^{L_0-1} I_0\left(\frac{2\sqrt{\tilde{E}}}{N_0}|\tilde{x}_k(\tau)|\right)
\end{aligned}$$

(See the next slide.)

Since $|a_k| = 1$, and the integrand is periodic with period 2π.

Since $I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{x\cos(\varphi)}d\varphi$.

$$\begin{aligned}
&\frac{1}{\sqrt{\tilde{E}}}\boldsymbol{x}_k(\tau)^T\boldsymbol{s}_k(\alpha_k,\theta,\tau)\\
&= x_{1,k}(\tau)\cos(\alpha_k+\theta) - x_{2,k}(\tau)\sin(\alpha_k+\theta)\\
&= \mathrm{Re}\left\{[x_{1,k}(\tau) - jx_{2,k}(\tau)]e^{-j(\alpha_k+\theta)}\right\}\\
&= \mathrm{Re}\left\{a_k^*\tilde{x}_k(\tau)e^{-j\theta}\right\}\\
&= \mathrm{Re}\left\{|a_k\tilde{x}_k(\tau)|\exp\left(j\left[\arg(\tilde{x}_k(\tau)) - \arg(a_k) - \theta\right]\right)\right\}\\
&= |a_k\tilde{x}_k(\tau)|\cos\left(\left[\arg(\tilde{x}_k(\tau)) - \arg(a_k) - \theta\right]\right).
\end{aligned}$$

As the text puts $\begin{cases}\tilde{x}_k &= x_{1,k} - jx_{2,k}\\ a_k &= e^{j\alpha_k}\end{cases}.$

6.14 Synchronization

- Square approximation of logarithm of the modified Bessel function of zero order

$$\begin{aligned}
\log I_0(x) &= \log\left(\sum_{m=0}^{\infty}\frac{(x/2)^{2m}}{(m!)^2}\right)\\
&\approx \log\left(1+\frac{x^2}{4}\right) \quad \text{for } x \text{ small}\\
&\approx \frac{x^2}{4} \quad \text{for } x \text{ small}
\end{aligned}$$

(Take the first two terms, *m*=0 and *m*=1)

$$\hat{\tau} = \arg\max_{\tau}\sum_{k=0}^{L_0-1}\log\left\{I_0\left(\frac{2\sqrt{\tilde{E}}}{N_0}|\tilde{x}_k(\tau)|\right)\right\} \approx \arg\max_{\tau}\sum_{k=0}^{L_0-1}|\tilde{x}_k(\tau)|^2$$

provided that $\frac{2\sqrt{\tilde{E}}}{N_0}|\tilde{x}_k(\tau)|$ small

6.14 Synchronization

- ☐ Realization of square approximation

$$\hat{\tau} = \arg\max_{\tau} \sum_{k=0}^{L_0-1} |\tilde{x}_k(\tau)|^2$$

$$\begin{aligned}
\Rightarrow \sum_{k=0}^{L_0-1} \frac{\partial |\tilde{x}_k(\tau)|^2}{\partial \tau} &= \sum_{k=0}^{L_0-1} \frac{\partial \left(\mathrm{Re}^2\{\tilde{x}_k(\tau)\} + \mathrm{Im}^2\{\tilde{x}_k(\tau)\}\right)}{\partial \tau} \\
&= \sum_{k=0}^{L_0-1} 2 \cdot \mathrm{Re}\{\tilde{x}_k(\tau)\} \cdot \mathrm{Re}'\{\tilde{x}_k(\tau)\} + 2 \cdot \mathrm{Im}\{\tilde{x}_k(\tau)\} \cdot \mathrm{Im}'\{\tilde{x}_k(\tau)\} \\
&= 2 \sum_{k=0}^{L_0-1} \mathrm{Re}\left\{\tilde{x}_k^*(\tau)\tilde{x}_k'(\tau)\right\} = 0
\end{aligned}$$

6.14 Synchronization

- Early-late approximation of the derivative

$$\begin{aligned}
\tilde{x}_n'(\tau) &= \tilde{x}_n'(nT + \hat{\tau}_n) \\
&\approx \frac{\overbrace{\tilde{x}\left(nT + \frac{T}{2} + \hat{\tau}[n+1/2]\right)}^{\text{late}} - \overbrace{\tilde{x}\left(nT - \frac{T}{2} + \hat{\tau}[n-1/2]\right)}^{\text{early}}}{T} \\
&\approx \frac{\tilde{x}\left(nT + \frac{T}{2} + \hat{\tau}[n]\right) - \tilde{x}\left(nT - \frac{T}{2} + \hat{\tau}[n-1]\right)}{T}
\end{aligned}$$

since no estimations for $\hat{\tau}[n+1/2]$ and $\hat{\tau}[n-1/2]$ are performed

6.14 Synchronization

$$\hat{\tau}[n+1] = \hat{\tau}[n] + \gamma \cdot e[n]$$

where $e[n] = \mathrm{Re}\left\{\tilde{x}(nT+\hat{\tau}[n])\left[\tilde{x}\left(nT+\frac{T}{2}+\hat{\tau}[n]\right) - \tilde{x}\left(nT-\frac{T}{2}+\hat{\tau}[n-1]\right)\right]\right\}$.

Final note: For every $\hat{\tau}[n]$, the realization requires the sample values of $\tilde{x}(nT+\hat{\tau}[n])$ and $\tilde{x}(nT+T/2+\hat{\tau}[n])$.

6.15 Computer experiments: Carrier recovery and symbol timing

- Experiment 1: Carrier phase θ recovery subject to known timing information τ
  - QPSK and error-free
  - L_0 = 1 and symbol energy E = 1 for simplicity

$$s_k(t) = \sqrt{\frac{2E}{T}}\cos(2\pi f_c t + \alpha_k) \text{ for } 0 \le t < T$$

where T symbol period, and $\alpha_k = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.

$$\Rightarrow x_k(t) = \sqrt{\frac{2E}{T}}\cos(2\pi f_c t + \alpha_k + \theta) \text{ for } \tau \le t \le T + \tau$$

$$\begin{cases} \phi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t), & \tau \le t < T+\tau \\ \phi_2(t) = \sqrt{\frac{2}{T}}\sin(2\pi f_c t), & \tau \le t < T+\tau \end{cases}$$

$$\Rightarrow \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} = \begin{bmatrix} s_{1,k} \\ s_{2,k} \end{bmatrix}, \text{ where } \begin{cases} x_{i,k} = \int_\tau^{T+\tau} x_k(t)\phi_i(t)dt \\ s_{i,k} = \int_\tau^{T+\tau} \sqrt{\frac{2}{T}}\cos(2\pi f_c t + \alpha_k + \theta)\phi_i(t)dt \end{cases}$$

$$\boxed{\boldsymbol{s}_k(\theta) = \begin{bmatrix} s_{1,k} \\ s_{2,k} \end{bmatrix} = \begin{bmatrix} \cos(\alpha_k + \theta) \\ -\sin(\alpha_k + \theta) \end{bmatrix}}$$

$$\Rightarrow \hat{\theta} = \arg\max_{\tilde{\theta}} \boldsymbol{x}_k^T \boldsymbol{s}_k(\tilde{\theta}) = \arg\max_{\tilde{\theta}} \boldsymbol{s}_k^T(\theta)\boldsymbol{s}_k(\tilde{\theta}) = \arg\max_{\tilde{\theta}} \ell(\tilde{\theta})$$

$$\Rightarrow 0 = \frac{\partial \ell(\tilde{\theta})}{\partial \tilde{\theta}} = \mathrm{Im}\left\{\hat{a}_k^* \tilde{x}_k e^{-j\tilde{\theta}}\right\} = \mathrm{Im}\left\{e^{-j\hat{\alpha}_k} e^{j(\alpha_k+\theta)} e^{-j\tilde{\theta}}\right\} = \mathrm{Im}\left\{e^{j(\alpha_k-\hat{\alpha}_k)} e^{j(\theta-\tilde{\theta})}\right\}$$

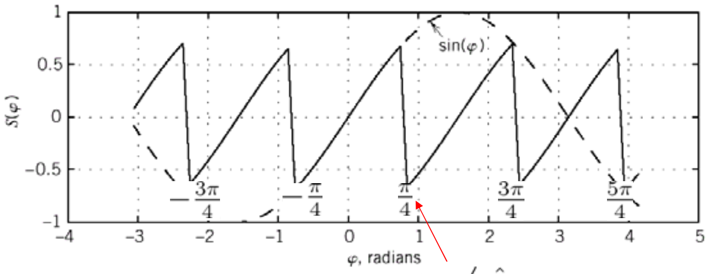
$\hat{\alpha}_k$ is the feedback directed decision.
α_k is the true transmitted signal.

$$\begin{cases} \tilde{x}_k &= x_{1,k} - jx_{2,k} \\ \hat{a}_k &= e^{j\hat{\alpha}_k} \end{cases}$$

$$\varphi = \theta - \hat{\theta}[n]$$

$$S(\varphi) = E\left[e[n] \mid \varphi\right] = \sin(\varphi) \quad \text{(S-curve)}$$

$$e[n] = \mathrm{Im}\left\{e^{j(\alpha_n - \hat{\alpha}_n)} e^{j(\theta - \hat{\theta}[n])}\right\} = \sin(\varphi) \text{ if ideally } \alpha_n = \hat{\alpha}_n$$



$\alpha_n \neq \hat{\alpha}_n$

$$\boxed{\ell(\tilde{\theta}) = \cos(\theta - \hat{\theta}) \quad \text{and} \quad \frac{\partial^2 \ell(\tilde{\theta})}{\partial^2 \tilde{\theta}} = -\cos(\varphi)}$$

6.15 Computer experiments: Carrier recovery and symbol timing

☐ Experiment 1: Carrier phase θ recovery subject to known timing information τ

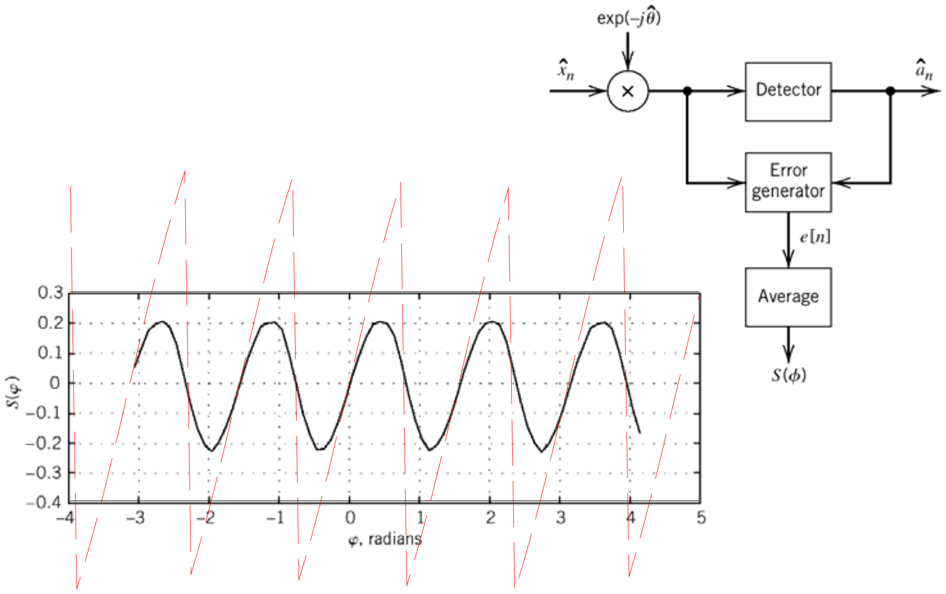
- QPSK and non-error-free
- L_0 = 1 and symbol energy E = 1 for simplicity

$$s_k(t) = \sqrt{\frac{2E}{T}}\cos(2\pi f_c t + \alpha_k) g(t-\tau)$$

where T symbol period, $\alpha_k = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and

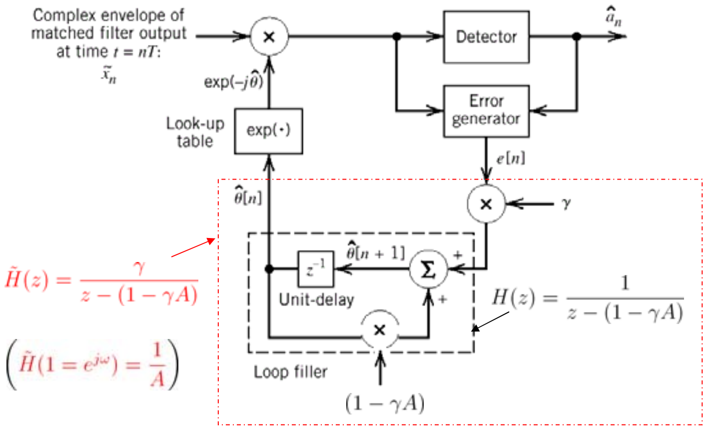
$$g(t) = \frac{\sin(\pi t/T)}{\pi t/T} \cdot \frac{\cos(\pi\alpha t/T)}{1-4\alpha^2 t^2/T^2} \text{ for } 0 \le t < T$$

Raise-cosine with rolloff factor α = 0.5.



$E/N_0 = 10$ dB

- ☐ Further experiment on phase error $\varphi = \theta - \hat{\theta}[n]$



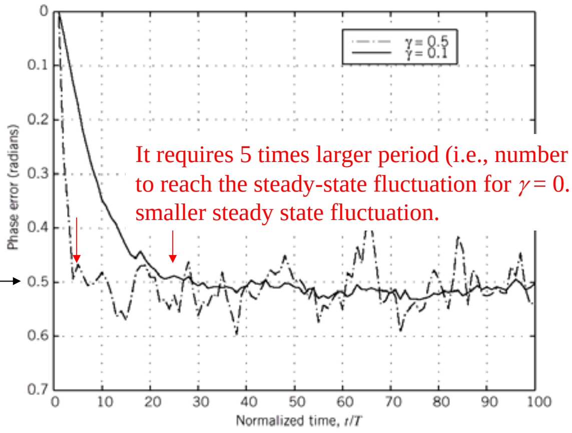
$$\tilde{H}(z) = \frac{\gamma}{z-(1-\gamma A)}$$

$$\left(\tilde{H}(1 = e^{j\omega}) = \frac{1}{A}\right)$$

$$H(z) = \frac{1}{z-(1-\gamma A)}$$

$$\varphi = \theta - \hat{\theta}[n]$$

True phase error →

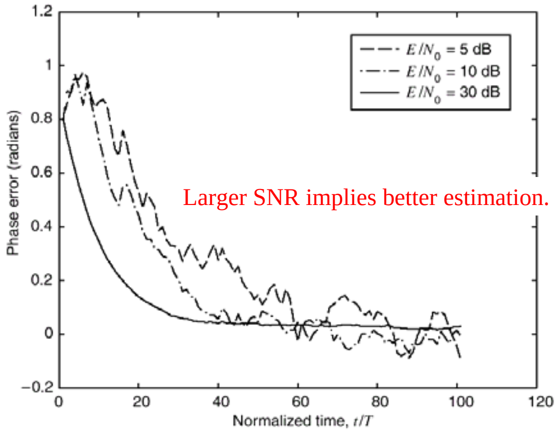


-·-·-	$\gamma = 0.5$
——	$\gamma = 0.1$

It requires 5 times larger period (i.e., number of iterations) to reach the steady-state fluctuation for $\gamma = 0.1$ but with smaller steady state fluctuation.

$$E/N_0 = 20 \text{ dB}$$

$$A = \left.\frac{\partial S(\varphi)}{\partial \varphi}\right|_{\varphi=0}$$

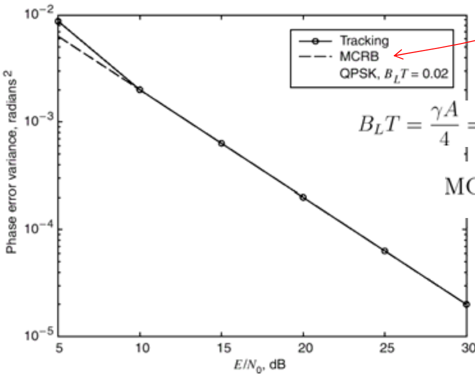


Larger SNR implies better estimation.

$\gamma = 0.08$

MCRB = modified Cramer-Rao (lower) bound

(Phase error variance calculated based on 100 trials)



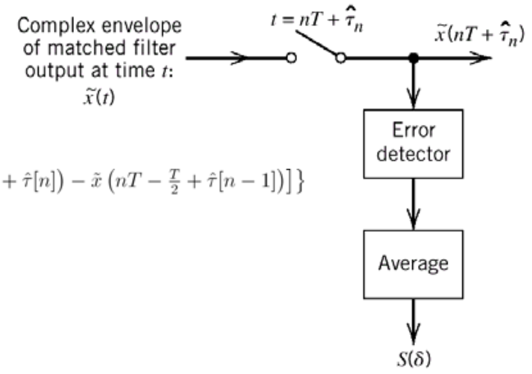
$$B_L T = \frac{\gamma A}{4} = 0.02 \Rightarrow A = 1$$

$$\text{MCRB} = \frac{B_L T}{E/N_0}$$

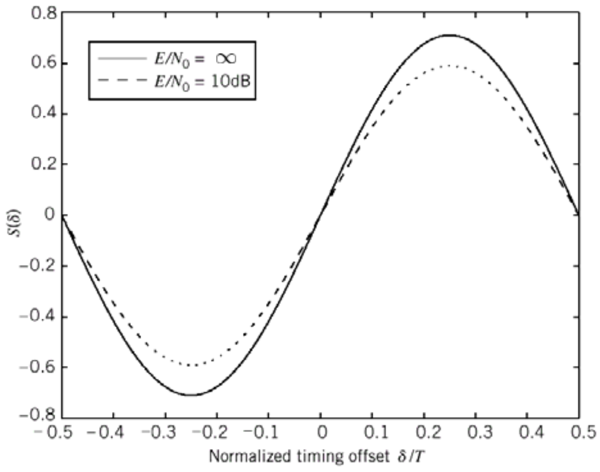
$\gamma = 0.08$

6.15 Computer experiments: Carrier recovery and symbol timing

- ☐ Experiment 2: Symbol timing recovery under error-free situation



$$e[n] = \mathrm{Re}\left\{\tilde{x}(nT+\hat{\tau}[n])\left[\tilde{x}\left(nT+\tfrac{T}{2}+\hat{\tau}[n]\right)-\tilde{x}\left(nT-\tfrac{T}{2}+\hat{\tau}[n-1]\right)\right]\right\}$$
$$\delta = \tau_0 - \hat{\tau}[n]$$
$$S(\delta) = E\left[e[n]|\delta\right]$$



The larger the SNR, the more accurate the estimate.

$$\hat{\tau}[n+1] = \hat{\tau}[n] + \gamma \cdot e[n]$$