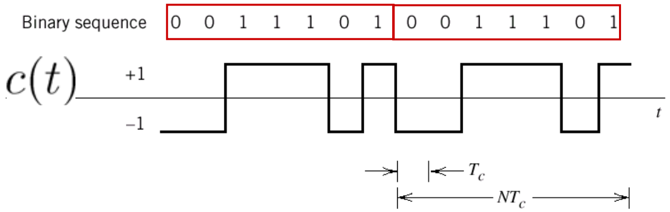
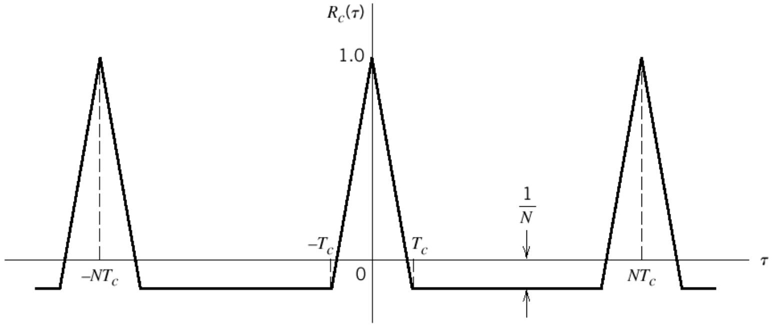
7.2 Pseudo-noise sequences

- Correlation property
  - Autocorrelation of an ideal discrete white process = $a\cdot\delta[\tau]$, where $\delta[\tau]$ is the Kronecker delta function.

$m = 3$



$N = 2^m - 1 = 7$ and $T_b = NT_c$



$$R_c(\tau) = \frac{1}{T_b}\int_{-T_b/2}^{T_b/2} c(t)c(t-\tau)dt$$

$$= \begin{cases} 1 - \frac{N+1}{NT_c}|\tau|, & |\tau| \le T_c \\ -\frac{1}{N}, & \text{the remainder of the period} \end{cases}$$

$$c(t) = \sum_{k=-\infty}^{\infty} c_k \cdot g(t - kT_c),$$

where sequence $\{c_k\}_{k=-\infty}^{\infty}$ is periodic with period N, $c_k \in \{-1, +1\}$,

$$\sum_{k=0}^{N-1} c_k c_{k+\ell} = \begin{cases} N, & \text{if } \ell = 0 \\ -1, & \text{if } \ell \neq 0 \end{cases} \quad \text{and } g(t) = \begin{cases} 1, & 0 \leq t < T_c \\ 0, & \text{otherwise} \end{cases}$$

Hence,

$$\begin{aligned} R_c(\tau) &= \frac{1}{T_b} \int_{-T_b/2}^{T_b/2} c(t)c(t-\tau)dt \\ &= \frac{1}{T_b} \int_0^{T_b} c(t)c(t-\tau)dt \end{aligned}$$

since $c(t)$ and $c(t-\tau)$ are both periodic functions in t with period T_b.

$$
\begin{aligned}
R_c(\tau) &= \frac{1}{T_b}\int_0^{T_b} c(t)c(t-\tau)dt \\
&= \frac{1}{T_b}\int_0^{T_b}\left(\sum_{k=-\infty}^{\infty} c_k \cdot g(t-kT_c)\right)\left(\sum_{\ell=-\infty}^{\infty} c_\ell \cdot g(t-\tau-\ell T_c)\right)dt \\
&= \frac{1}{T_b}\int_0^{T_b}\left(\sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} c_k c_\ell \cdot g(t-kT_c)g(t-\tau-\ell T_c)\right)dt \\
&= \begin{cases} \frac{1}{T_b}\int_0^{T_b}\left(\sum_{k=-\infty}^{\infty}\sum_{\ell=\lceil k-u-1\rceil}^{\lfloor k-u+1\rfloor} c_k c_\ell \cdot g(t-kT_c)g(t-\tau-\ell T_c)\right)dt & \text{if } u \neq \text{integer} \\ \frac{1}{T_b}\int_0^{T_b}\left(\sum_{k=-\infty}^{\infty} c_k c_{k-u} \cdot g(t-kT_c)\right)dt & \text{if } u = \text{integer} \end{cases}
\end{aligned}
$$

where $\tau = u \cdot T_c$. Note that:

$g(t-kT_c)g(t-\tau-\ell T_c) = 0$ iff $\tau + \ell T_c \geq (k+1)T_c$ or $\tau + (\ell+1)T_c \leq kT_c$.
Hence, $g(t-kT_c)g(t-\tau-\ell T_c) \neq 0$ iff $k-1-u < \ell < k+1-u$

Case 1: If u is not an integer, then

$$\begin{aligned} R_c(\tau) &= \frac{1}{T_b}\int_0^{T_b}\left(\sum_{k=-\infty}^{\infty}\sum_{\ell=\lceil k-u-1\rceil}^{\lfloor k-u+1\rfloor} c_k c_\ell \cdot g(t-kT_c)g(t-\tau-\ell T_c)\right)dt \\ &= \frac{1}{T_b}\int_0^{T_b}\left(\sum_{k=-\infty}^{\infty}\sum_{\ell=k-\lfloor u\rfloor-1}^{k-\lfloor u\rfloor} c_k c_\ell \cdot g(t-kT_c)g(t-\tau-\ell T_c)\right)dt \\ &= \frac{1}{T_b}\int_0^{T_b}\left(\sum_{k=0}^{N-1}\sum_{\ell=k-\lfloor u\rfloor-1}^{k-\lfloor u\rfloor} c_k c_\ell \cdot g(t-kT_c)g(t-\tau-\ell T_c)\right)dt \end{aligned}$$

because $g(t-kT_c)=0$ for the integration range $0<t<T_b$ if $kT_c \ge T_b$ or $(k+1)T_c \le 0$, namely, $k \ge N$ or $k \le -1$.

$$= \sum_{k=0}^{N-1}\sum_{\ell=k-\lfloor u\rfloor-1}^{k-\lfloor u\rfloor} c_k c_\ell \cdot \frac{1}{T_b}\int_0^{T_b} g(t-kT_c)g(t-uT_c-\ell T_c)dt$$

$$
\begin{aligned}
&= \sum_{k=0}^{N-1} c_k c_{k-\lfloor u \rfloor -1} \cdot \frac{1}{T_b} \int_0^{T_b} g(t-kT_c) g(t-uT_c-(k-\lfloor u \rfloor -1)T_c) dt \\
&\quad + \sum_{k=0}^{N-1} c_k c_{k-\lfloor u \rfloor} \cdot \frac{1}{T_b} \int_0^{T_b} g(t-kT_c) g(t-uT_c-(k-\lfloor u \rfloor)T_c) dt \\
&= \sum_{k=0}^{N-1} c_k c_{k-\lfloor u \rfloor -1} \cdot \frac{(u-\lfloor u \rfloor)T_c}{T_b} + \sum_{k=0}^{N-1} c_k c_{k-\lfloor u \rfloor} \cdot \frac{[1-(u-\lfloor u \rfloor)]T_c}{T_b} \\
&= \sum_{k=0}^{N-1} c_k c_{k-\lfloor u \rfloor -1} \cdot \frac{(u-\lfloor u \rfloor)}{N} + \sum_{k=0}^{N-1} c_k c_{k-\lfloor u \rfloor} \cdot \frac{1-(u-\lfloor u \rfloor)}{N} \\
&= \frac{(u-\lfloor u \rfloor)}{N} \sum_{k=0}^{N-1} c_k c_{k-\lfloor u \rfloor -1} + \frac{1-(u-\lfloor u \rfloor)}{N} \sum_{k=0}^{N-1} c_k c_{k-\lfloor u \rfloor} \\
&= \begin{cases} -\dfrac{u}{N} + 1 - u, & \lfloor u \rfloor = 0 \\ u + 1 - \dfrac{1-(u+1)}{N}, & \lfloor u \rfloor = -1 \\ -\dfrac{1}{N}, & \text{otherwise} \end{cases}
\end{aligned}
$$

$$
= \begin{cases} 1 - u - \dfrac{u}{N}, & 0 < u < 1 \\ 1 - (-u) - \dfrac{(-u)}{N}, & -1 < u < 0 \\ -\dfrac{1}{N}, & \text{otherwise} \end{cases}
$$

$$
= \begin{cases} 1 - |u| - \dfrac{|u|}{N}, & |u| < 1, u \text{ noninteger} \\ -\dfrac{1}{N}, & |u| > 1, u \text{ noninteger} \end{cases}
$$

Case 2: If u is an integer, then

$$
\begin{aligned}
R_c(\tau) &= \frac{1}{T_b}\int_0^{T_b}\left(\sum_{k=-\infty}^{\infty} c_k c_{k-u}\cdot g(t-kT_c)\right)dt \\
&= \frac{1}{T_b}\int_0^{T_b}\left(\sum_{k=0}^{N-1} c_k c_{k-u}\cdot g(t-kT_c)\right)dt \\
&= \sum_{k=0}^{N-1} c_k c_{k-u}\frac{1}{T_b}\int_0^{T_b} g(t-kT_c)dt \\
&= \sum_{k=0}^{N-1} c_k c_{k-u}\frac{T_c}{T_b} \\
&= \sum_{k=0}^{N-1} c_k c_{k-u}\frac{1}{N} \\
&= \frac{1}{N}\sum_{k=0}^{N-1} c_k c_{k-u} = \begin{cases} 1, & \text{if } u=0 \\ -\dfrac{1}{N}, & \text{if } u\neq 0, \text{ integer} \end{cases}
\end{aligned}
$$

Case 1&2:

$$
\begin{aligned}
R(\tau) &= \begin{cases} 1 - |u| - \dfrac{|u|}{N}, & |u| < 1 \\ -\dfrac{1}{N}, & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 - \dfrac{N+1}{N}\left|\dfrac{\tau}{T_c}\right|, & |\tau| < T_c \\ -\dfrac{1}{N}, & \text{otherwise} \end{cases}
\end{aligned}
$$