Chapter 7 Spread-Spectrum Modulation

Spread Spectrum Technique simply consumes spectrum in excess of the minimum spectrum necessary to send the data.

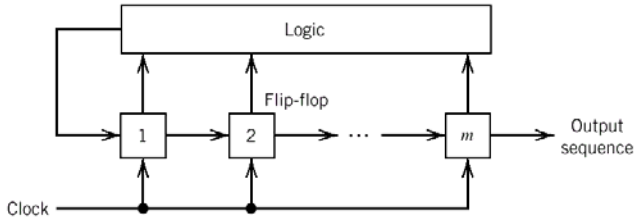
7.1 Introduction

- Definition of spread-spectrum modulation
  - Weakly sense
    - Occupy a bandwidth that is much larger than the minimum bandwidth ($1/2T$) necessary to transmit a data sequence.
  - Strict sense
    - Spectrum is spreading by means of a pseudo-white or pseudo-noise code.

7.2 Pseudo-noise sequences

- A (digital) code sequence that mimics the (second-order) statistical behavior of a white noise.
- For example,
  - balance property
  - run property
  - correlation property
- From implementation standpoint, the most convenient way to generate a pseudo-noise sequence is to employ several shift-registers and a feedback through combinational logic.

7.2 Pseudo-noise sequences

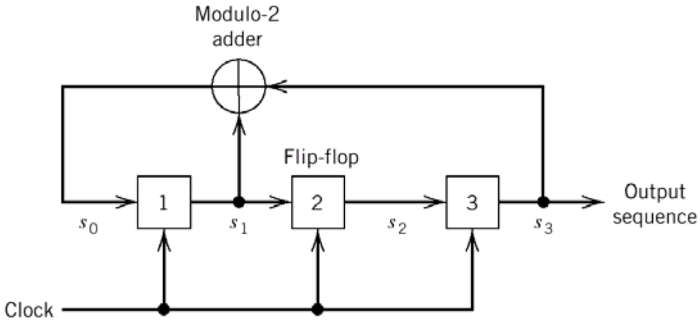


Exemplified block diagram of PN sequence generators

- Feedback shift register becomes "linear" if the feedback logic consists entirely of modulo-2 adders.

7.2 Pseudo-noise sequences

- Example of linear feedback shift register



7.2 Pseudo-noise sequences

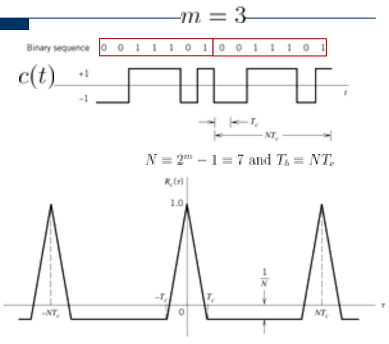
- A PN sequence generated by a feedback shift register must eventually become periodic with period at most 2^m, where *m* is the number of shift registers.
- A PN sequence generated by a linear feedback shift register must eventually become periodic with period at most $2^m - 1$, where *m* is the number of shift registers.
- A PN sequence whose period reaches its maximum value is named the *maximum-length sequence* or simply *m-sequence*.

7.2 Pseudo-noise sequences

- A *maximum-length sequence* generated from a linear shift register satisfies all three properties:
  - Balance property
    - The number of 1s is one more than that of 0s.
  - Run property (total number of runs = 2^{m-1})
    - ½ of the runs is of length 1
    - ¼ of the runs is of length 2
    - …

7.2 Pseudo-noise sequences

- Correlation property
  - Autocorrelation of an ideal discrete white process = $a\cdot\delta[\tau]$, where $\delta[\tau]$ is the Kronecker delta function.



$$R_c(\tau) = \frac{1}{T_b}\int_{-T_b/2}^{T_b/2} c(t)c(t-\tau)\,dt = \begin{cases} 1-\frac{N+1}{NT_c}|\tau|, & |\tau|\le T_c \\ -\frac{1}{N}, & \text{the remainder of the period} \end{cases}$$

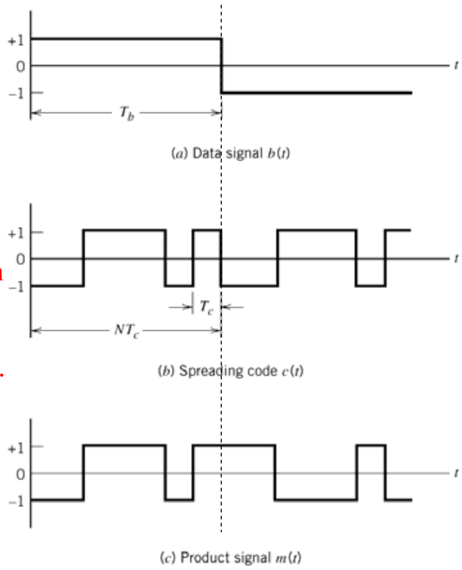
7.2 Pseudo-noise sequences

- Power spectrum view

Suppose $c(t)$ is perfectly white with $c^2(t) = 1$. Then,

$m(t) = b(t)c(t)$ and $b(t) = m(t)c(t)$.

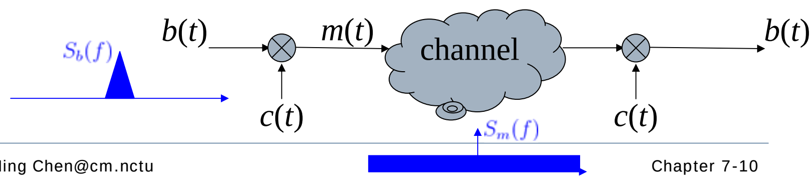
Question: What will be the power spectrum of $m(t)$?



7.2 Pseudo-noise sequences

$$
\begin{aligned}
& E[m(t)m(t+\tau)] \\
&\quad = E[b(t)c(t)b(t+\tau)c(t+\tau)] \\
&\quad = E[b(t)b(t+\tau)]E[c(t)c(t+\tau)] \\
&\qquad\quad \text{(Independence assumption between } b(t) \text{ and } c(t)) \\
&\quad = R_b(\tau)\delta(\tau) \quad (\text{WSS of } b(t) \text{ and white of } c(t))
\end{aligned}
$$

$$\Rightarrow R_m(\tau) = R_b(0)\delta(\tau) \text{ and } S_m(f) = R_b(0)$$



7.2 Pseudo-noise sequences

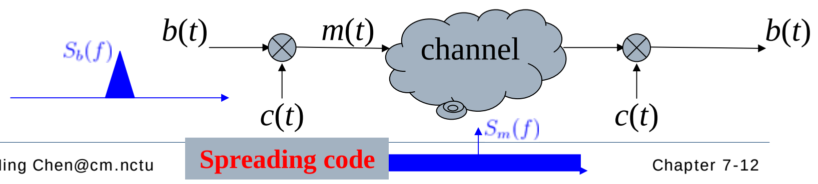
- Please self-study Example 7.2 for examples of the *m*-sequences.
  - Its understanding will be part of the exam.

7.3 A notion of spread spectrum

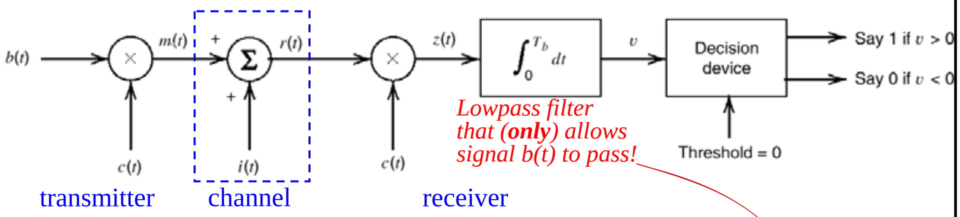
Make the transmitted signal to blend (and hide behind) the background noise.

$$
\begin{aligned}
&E[m(t)m(t+\tau)] \\
&\quad = E[b(t)c(t)b(t+\tau)c(t+\tau)] \\
&\quad = E[b(t)b(t+\tau)]E[c(t)c(t+\tau)] \\
&\qquad\quad \text{(Independence assumption between } b(t) \text{ and } c(t)) \\
&\quad = R_b(\tau)\delta(\tau) \quad \text{(WSS of } b(t) \text{ and white of } c(t))
\end{aligned}
$$

$$\Rightarrow R_m(\tau) = R_b(0)\delta(\tau) \text{ and } S_m(f) = R_b(0)$$



7.3 A notion of spread spectrum



$$r(t) = m(t) + i(t) = c(t)b(t) + i(t)$$

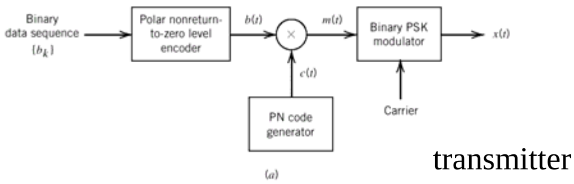
$$z(t) = c(t)r(t) = c^2(t)b(t) + c(t)i(t) = b(t) + c(t)i(t)$$

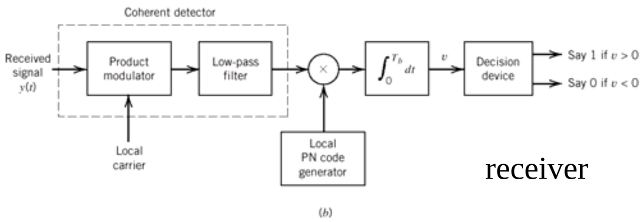
$c^2(t) = 1$

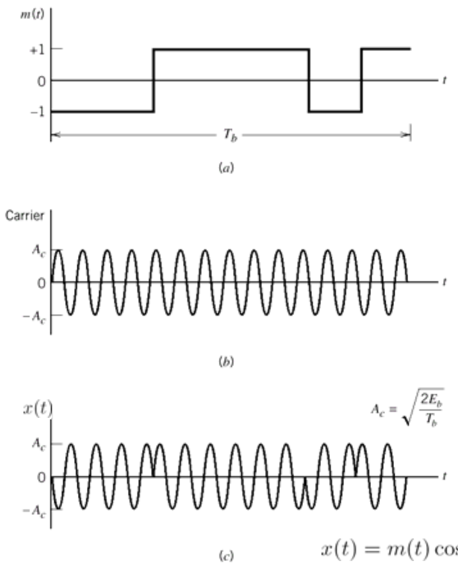
Multiplication of $c(t)$ = Spreading the spectrum
Hence, most of the power in the wideband $c(t)i(t)$ is filtered out.

7.4 Direct-sequence spread spectrum with coherent binary phase-shift keying

- DSSS system







$x(t) = m(t)\cos(2\pi f_c t) = b(t)c(t)\cos(2\pi f_c t)$

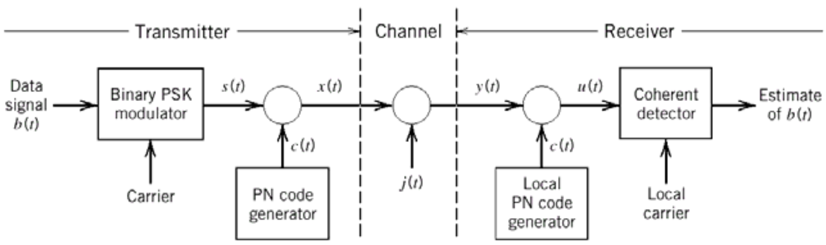
Let $s(t) = b(t)\cos(2\pi f_c t)$.

Then, $x(t) = c(t)s(t)$. (As the conceptual system below.)

$\Rightarrow y(t) = x(t) + i(t) = c(t)s(t) + j(t)$.

In text, $i(t)$ is the baseband interference, while $j(t)$ is the passband interference.

$\Rightarrow u(t) = c(t)y(t) = c^2(t)s(t) + c(t)j(t) = s(t) + c(t)j(t)$.



7.5 Signal-space dimensionality and processing gain

- SNR before spreading
  $$y(t) = c(t)s(t) + j(t).$$
- SNR after spreading
  $$u(t) = s(t) + c(t)j(t).$$
- Orthonormal basis used at the receiver end

$$\phi_k(t) = \begin{cases} \sqrt{\frac{2}{T_c}}\cos(2\pi f_c t), & kT_c \le t < (k+1)T_c \\ 0, & \text{otherwise} \end{cases}$$

where $k = 0, 1, \cdots, N-1$.

Assume "coherent detection". In other words, perfect synchronization and no phase mismatch.

- SNR before spreading (SNR_I)

$$\begin{aligned} y(t) &= c(t)s(t) + j(t) \\ &= \pm\sqrt{\frac{2E_b}{T_b}}c(t)\cos(2\pi f_c t) + j(t) \\ &= \pm\sqrt{\frac{2E_b}{T_b}}\sqrt{\frac{T_c}{2}}\sum_{k=0}^{N-1} c_k\phi_k(t) + j(t) \end{aligned}$$

$c_k \in \{\pm 1\}$ and $j(t)$ white

$$\Rightarrow \boldsymbol{y} = \pm\sqrt{\frac{E_b}{N}}\begin{bmatrix} c_0 \\ \vdots \\ c_{N-1} \end{bmatrix}_{N\times 1} + \begin{bmatrix} j_0 \\ \vdots \\ j_{N-1} \end{bmatrix}_{N\times 1}$$

$$j_k = \int_0^{T_b} j(t)\phi_k(t)dt$$

$$\Rightarrow \text{SNR}_I = \frac{(E_b/N)\cdot E[c_0^2 + \cdots + c_{N-1}^2]}{E\left[j_0^2 + \cdots + j_{N-1}^2\right]} = \frac{(E_b/N)N}{NE[j_0^2]} = \frac{E_b}{NE[j_0^2]}$$

- SNR after spreading (SNR_O)

$$\begin{aligned} u(t) &= s(t) + c(t)j(t) \\ &= \pm\sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_c t) + c(t)j(t) \end{aligned}$$

$$\phi(t) = \begin{cases} \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t), & 0 \le t < T_b \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow u = \pm\sqrt{E_b} + j, \text{ where } j = \int_0^{T_b} c(t)j(t)\phi(t)dt$$

$$\begin{aligned} j &= \int_0^{T_b} c(t)j(t)\phi(t)dt \\ &= \int_0^{T_b} \left(\sum_{m=0}^{N-1} c_m g(t - mT_c)\right) j(t)\phi(t)dt \\ &= \sum_{m=0}^{N-1} c_m \int_0^{T_b} g(t - mT_c)j(t)\phi(t)dt \\ &= \sqrt{\frac{T_c}{T_b}} \sum_{m=0}^{N-1} c_m \int_{mT_c}^{(m+1)T_c} j(t)\sqrt{\frac{2}{T_c}}\cos(2\pi f_c t)dt \\ &= \sqrt{\frac{T_c}{T_b}} \sum_{m=0}^{N-1} c_m \int_0^{T_b} j(t)\phi_m(t)dt \\ &= \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} c_m j_m \end{aligned}$$

$$g(t) = \begin{cases} 1, & 0 \le t < T_c \\ 0, & \text{otherwise} \end{cases}$$

$$
\begin{aligned}
E[j^2] &= \frac{1}{N}E\left[\left(\sum_{m=0}^{N-1} c_m j_m\right)\left(\sum_{k=0}^{N-1} c_k j_k\right)\right] \\
&= \frac{1}{N}\sum_{m=0}^{N-1}\sum_{k=0}^{N-1} E[c_m c_k]E[j_m j_k] \\
&= \frac{1}{N}\sum_{k=0}^{N-1} E[j_k^2] \\
&= E[j_0^2]
\end{aligned}
$$

$$\Rightarrow \text{SNR}_O = \frac{E_b}{E[j_0^2]} = N \cdot \text{SNR}_I$$

$N = \dfrac{T_b}{T_c}$ is hence named the processing gain of spread spectrum technique.

Mismatch with the text in SNR_I

- SNR before spreading

  $y(t) = c(t)s(t) + j(t)$.

- SNR after spreading

  $u(t) = s(t) + c(t)j(t)$.

- Orthonormal basis used at the receiver end

$$
\begin{cases}
\phi_k(t) = \begin{cases} \sqrt{\frac{2}{T_c}}\cos(2\pi f_c t), & kT_c \le t < (k+1)T_c \\ 0, & \text{otherwise} \end{cases} \\
\hat{\phi}_k(t) = \begin{cases} \sqrt{\frac{2}{T_c}}\sin(2\pi f_c t), & kT_c \le t < (k+1)T_c \\ 0, & \text{otherwise} \end{cases}
\end{cases}
$$

where $k = 0, 1, \cdots, N-1$.

Assume "coherent detection". In other words, perfect synchronization but with phase mismatch.

- SNR before spreading (SNR_I)

$$\begin{aligned} y(t) &= c(t)s(t) + j(t) \\ &= \pm\sqrt{\frac{2E_b}{T_b}} c(t)\cos(2\pi f_c t) + j(t) \\ &= \pm\sqrt{\frac{2E_b}{T_b}}\sqrt{\frac{T_c}{2}}\sum_{k=0}^{N-1} c_k\phi_k(t) + j(t) \end{aligned}$$

$c_k \in \{\pm 1\}$ and $j(t)$ white

$$\Rightarrow \boldsymbol{y} = \pm\sqrt{\frac{E_b}{N}} \begin{bmatrix} c_0 \\ \vdots \\ c_{N-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{2N\times 1} + \begin{bmatrix} j_0 \\ \vdots \\ j_{N-1} \\ \hat{j}_0 \\ \vdots \\ \hat{j}_{N-1} \end{bmatrix}_{2N\times 1}$$

$$\begin{cases} j_k = \int_0^{T_b} j(t)\phi_k(t)dt \\ \hat{j}_k = \int_0^{T_b} j(t)\hat{\phi}_k(t)dt \end{cases}$$

$$\begin{aligned} \Rightarrow \mathrm{SNR}_I &= \frac{(E_b/N)\cdot E[c_0^2 + \cdots + c_{N-1}^2]}{E\left[j_0^2 + \cdots + j_{N-1}^2 + \hat{j}_0^2 + \cdots + \hat{j}_{N-1}^2\right]} \\ &= \frac{(E_b/N)N}{2NE[j_0^2]} = \frac{E_b}{2NE[j_0^2]} \end{aligned}$$

$$\begin{aligned} \Rightarrow \mathrm{SNR}_O &= \frac{E_b}{E[j_0^2]} \\ &= 2\cdot N\cdot \mathrm{SNR}_I \end{aligned}$$

The factor **2** enters when the "phase" is assumed unknown to the before-spreading receiver; hence, both cosine and sine domains must be "inner-producted".
If the "phase" is also assumed unknown to the after-spreading receiver, then the fact **2** in $\mathrm{SNR}_I/\mathrm{SNR}_O$ formula will (again) disappear.

7.6 Probability of error

$$u = \pm\sqrt{E_b} + j, \text{ where } j = \frac{1}{\sqrt{N}}\sum_{m=0}^{N-1} c_m j_m$$

- Gaussian assumption
  - j is Gaussian distributed due to central limit theorem

$$E[j] = 0 \text{ and } E[j^2] = E[j_0^2]$$

$$\text{where } j_0 = \int_0^{T_c} j(t)\phi_0(t)dt$$

7.6 Probability of error

- Similar to what the textbook has assumed, let J denote the interference power experiencing at the channel with "phase mismatch".

$$\text{In other words, } J = \frac{1}{T_b}\left(\sum_{k=0}^{N-1} E[j_k^2] + \sum_{k=0}^{N-1} E[\hat{j}_k^2]\right)$$

Correction to Eq. (7.20): Both $\{j_k\}_{k=0}^{N-1}$ and $\{\hat{j}_k\}_{k=0}^{N-1}$ are random variables. Hence, "expectation" should be added to make the "average power" J a deterministic number.

7.6 Probability of error

As $J = \frac{1}{T_b}\left(\sum_{k=0}^{N-1} E[j_k^2] + \sum_{k=0}^{N-1} E[\hat{j}_k^2]\right) = \frac{2N E[j_0^2]}{T_b} = 2\frac{E[j_0^2]}{T_c}$

$\Rightarrow E[j_0^2] = \frac{1}{2}T_c J \quad \left(= \sigma^2\right)$

$\Rightarrow$ Slide 6-32 said that

$$P(\text{Error}) = \Phi\left(\frac{0-\sqrt{E_b}}{\sigma}\right) = \Phi\left(\frac{0-\sqrt{E_b}}{\sqrt{T_c J/2}}\right) = \Phi\left(-\sqrt{2\frac{E_b}{T_c J}}\right)$$

6.3 Coherent phase-shift keying – Error probability

- Error probability of Binary PSK
  - Based on the decision rule $x \underset{+\sqrt{E_b}}{\overset{-\sqrt{E_b}}{\lessgtr}} 0$

$$\begin{aligned} P(\text{Error}) &= P\left(-\sqrt{E_b}\text{ transmitted}\right) P\left(x > 0 \middle| -\sqrt{E_b}\text{ transmitted}\right) \\ &\quad + P\left(+\sqrt{E_b}\text{ transmitted}\right) P\left(x < 0 \middle| +\sqrt{E_b}\text{ transmitted}\right) \\ &= \frac{1}{2}\int_0^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x+\sqrt{E_b})^2/2\sigma^2} dx + \frac{1}{2}\int_{-\infty}^{0} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\sqrt{E_b})^2/2\sigma^2} dx \\ &= \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\sqrt{E_b})^2/2\sigma^2} dx = \Phi\left(\frac{0-\sqrt{E_b}}{\sigma}\right) = \Phi\left(-\sqrt{2\frac{E_b}{N_0}}\right) \end{aligned}$$

$$\Phi(-x) = \frac{1}{2}\text{erfc}\left(\frac{x}{\sqrt{2}}\right)$$

7.6 Probability of error

- Comparing system performances with/without spreading, we obtain:

$$N_0 = T_c J$$

- With $P = E_b/T_b$, where P is the average signal power,

$$\frac{E_b}{N_0} = \left(\frac{T_b}{T_c}\right)\left(\frac{P}{J}\right) \text{ or equivalently, } \frac{J}{P} = \frac{\text{PG}}{E_b/N_0}$$

  - J/P is termed *the jamming margin (required for a specific error rate).*

$$\begin{aligned}(\text{Jamming margin})_{\text{dB}} =& (\text{Processing gain})_{\text{dB}} \\ & -(\text{required } (E_b/N_0)_{\text{dB}} \text{ for a given } P_e)\end{aligned}$$

- Example 7.3
  - Without spreading, (E_b/N_0) required for $P_e = 10^{-5}$ is around 10 dB.
  - PG = 4095
  - Then, Jamming margin for $P_e = 10^{-5}$ is

$$\begin{aligned}(\text{Jamming margin})_{\text{dB}} &= 10\log_{10}(4095) - 10 = 36.1 - 10 \\ &= 26.1 \text{ dB } = 10\log_{10}(409.5)\end{aligned}$$

  - Information bits can be detected subject to the required error rate, even if the interference level is 409.5 times larger than the received signal power (in the price of the transmission speed is 4095 times slower).

7.7 Frequency-hop spread spectrum

- Basic characterization of frequency hopping
  - Slow-frequency hopping

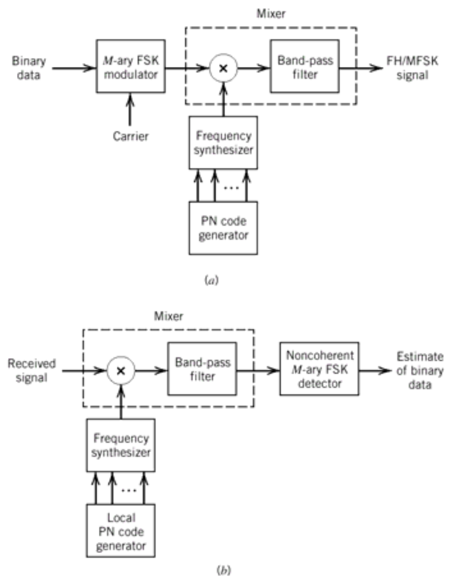
    Symbol rate $R_s >$ hop rate R_h
    (Usually, an interger multiple of)
  - Fast-frequency hopping

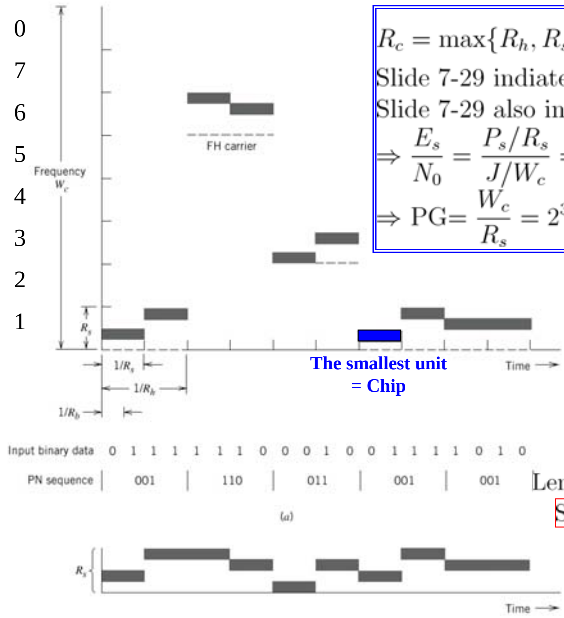
    Symbol rate $R_s <$ hop rate R_h
    (Usually, an interger multiple of)
  - Chip rate (The smallest unit = Chip)

    $$R_c = \max\{R_h, R_s\}$$

7.7 Frequency-hop spread spectrum

- A common modulation scheme for FH systems is the *M-ary frequency-shift keying*





$R_c = \max\{R_h, R_s\} = R_s = \frac{R_b}{2}$

Slide 7-29 indiates again $N_0 = T_c J = J/W_c$

Slide 7-29 also indiates $P_s = E_s/T_s = R_s E_s$

$\Rightarrow \frac{E_s}{N_0} = \frac{P_s/R_s}{J/W_c} = \left(\frac{W_c}{R_s}\right)\left(\frac{P_s}{J}\right) = \text{PG}\left(\frac{P_s}{J}\right)$

$\Rightarrow$ PG$= \frac{W_c}{R_s} = 2^3 = 8.$

4-ary FSK

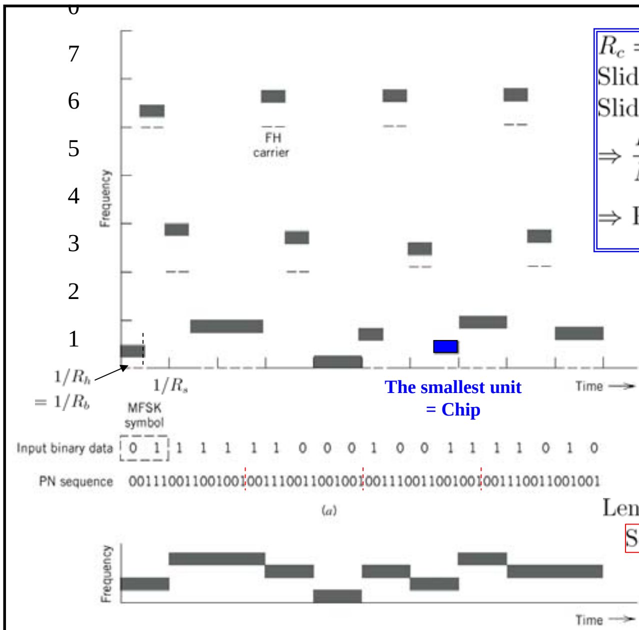
Period of PN sequence=15

Length of PN segment per hop=3

So there are 2^3 hop frequencies.

4-ary FSK

Slow-frequency hopping



$R_c = \max\{R_h, R_s\} = R_h = R_b$

Slide 7-29: $N_0 = T_c J = J/W_c$

Slide 7-29: $P_s = E_s/T_s = R_s E_s$

$\Rightarrow \frac{E_s}{N_0} = \frac{P_s/R_s}{J/W_c} = \text{PG}\left(\frac{P_s}{J}\right)$

$\Rightarrow$ PG$= \frac{W_c}{R_s} = 2^3 = 8.$

4-ary FSK

Period of PN sequence=15

Length of PN segment per hop=3

So there are 2^3 hop frequencies.

4-ary FSK

Fast-frequency hopping

7.7 Frequency-hop spread spectrum

- Fast-frequency hopping is popular in military use because the transmitted signal hops to a new frequency before the jammer is able to sense and jam it.
- Two detection rules are generally used in fast-frequency hopping
  - Make decision separately for each chip, and do majority vote based on these chip-based decisions (Simple)
  - Make maximum-likelihood decision based on all chip receptions (Optimal)

7.8 Computer experiments: Maximum-length and gold codes

- Code-division multiplexing (CDM)
  - Each user is assigned a different spreading code.

For simplicity, assume $s_1(t), s_2(t) \in \{\pm\sqrt{E_b}\}$ (i.e., $f_c = 0$)

Let $c_i(t) = \sum_{k=0}^{N-1} c_{k,i} g(t - kT_c)$, where $c_{k,i} \in \{\pm 1\}$, and $g(t) = 1$ for $0 \le t < T_c$, and zero, otherwise.

Multiplexing $x_1(t) + x_2(t) = c_1(t)s_1(t) + c_2(t)s_2(t)$

$\Rightarrow c_1(t)[x_1(t) + x_2(t)] = s_1(t) + c_1(t)c_2(t)s_2(t)$

$\Rightarrow$ Decision is based on $s_1 \int_0^{T_b} dt + s_2 \int_0^{T_b} c_1(t)c_2(t)dt$

7.8 Computer experiments: Maximum-length and gold codes

- So, if

$$\int_0^{T_b} c_1(t)c_2(t)dt = \sum_{k=0}^{N-1} c_{k,1}c_{k,2} = 0 \qquad \text{cross-correlation}$$

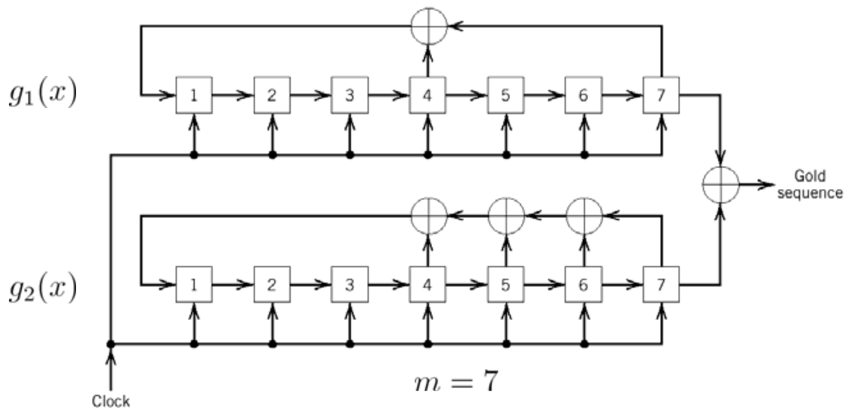
then *signal one* (i.e., s_1) can be exactly reconstructed.

- In practice, it may not be easy to have a big number of PN sequences satisfying the above equality. Instead, we desire

$$\left|\sum_{k=0}^{N-1} c_{k,1}c_{k,2}\right| \text{ small}$$

7.8 Computer experiments: Maximum-length and gold codes

- Gold sequences



7.8 Computer experiments: Maximum-length and gold codes

- Gold sequences
  - $g_1(x)$ and $g_2(x)$ are two maximum-length shift-register sequences of period $2^m - 1$, whose "cross-correlation" lies in:

$$\{-1, -t(m), t(m)-2\},$$
$$\text{where } t(m) = \begin{cases} 2^{(m+1)/2}+1, & m \text{ odd} \\ 2^{(m+2)/2}+1, & m \text{ even} \end{cases}$$

  - Then, the structure in previous slide can gives us $2^m - 1$ sequences (by setting different initial value in the shift registers).
  - Together with the two original *m*-sequences, we have $2^m + 1$ sequences.

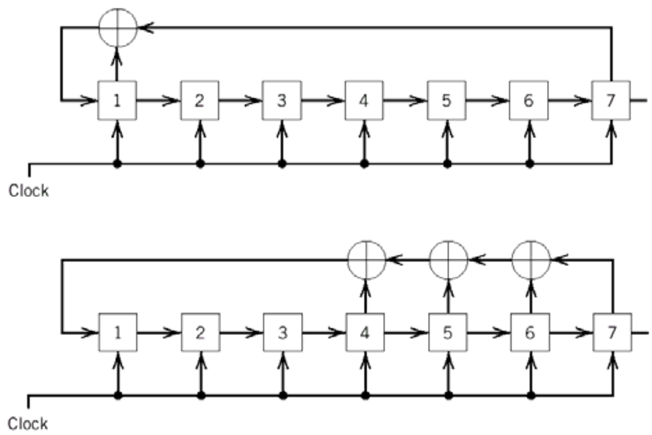
7.8 Computer experiments: Maximum-length and gold codes

- Gold's theorem
  - The cross-correlation between any pair in the $2^m + 1$ sequences also lies in

$$\{-1, -t(m), t(m)-2\},$$
$$\text{where } t(m) = \begin{cases} 2^{(m+1)/2}+1, & m \text{ odd} \\ 2^{(m+2)/2}+1, & m \text{ even} \end{cases}$$

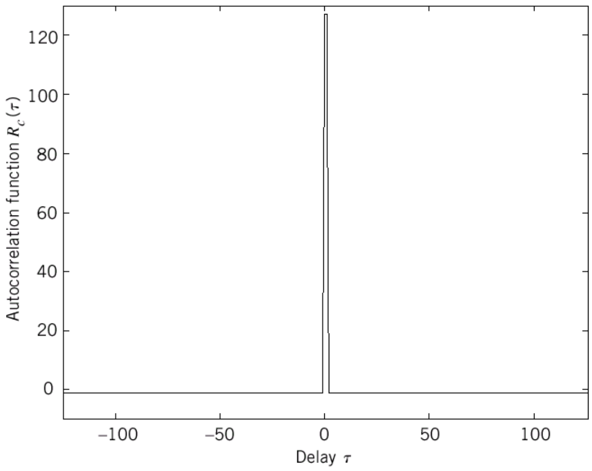
Further discussion on $g_1(x)$ and $g_2(x)$ is defered to Chapter 8.

☐ Experiment 1: Correlation properties of PN sequences

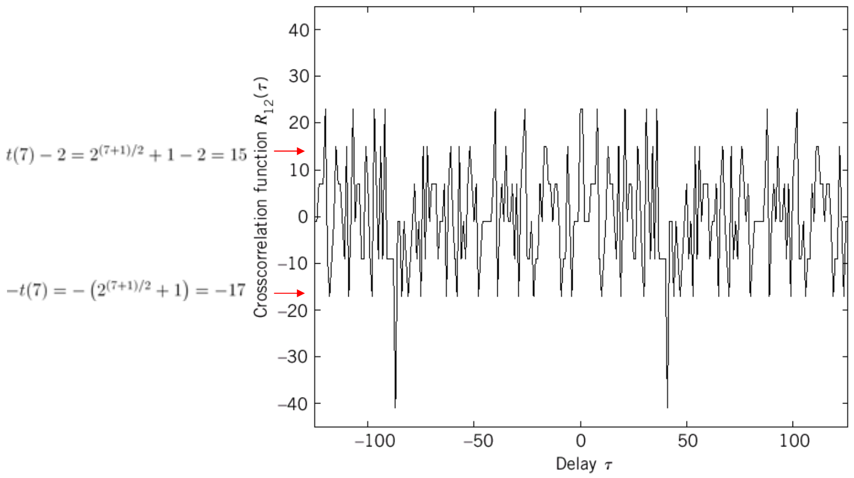


■ Autocorrelation

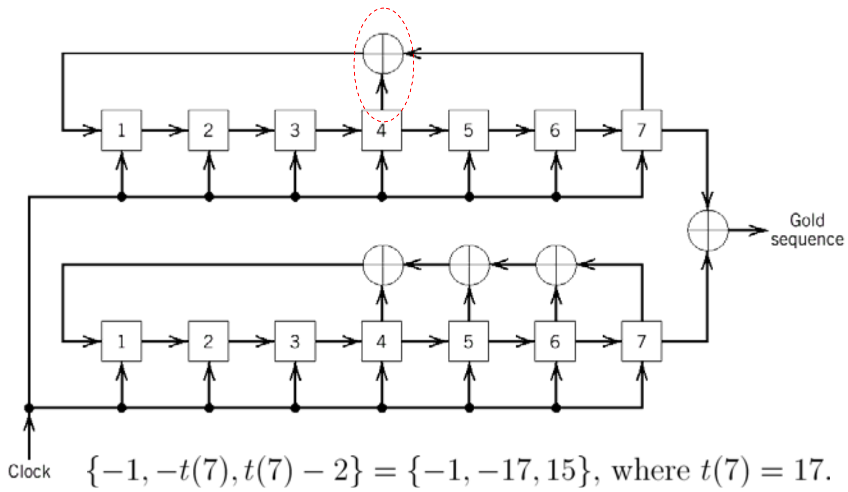
$2^7 - 1 = 127$



■ Cross-correlation (of general PN sequences, not necessarily Gold sequences)



□ Experiment 2: Correlation properties of Gold sequences



$\{-1, -t(7), t(7) - 2\} = \{-1, -17, 15\}$, where $t(7) = 17$.

■ Cross-correlation