$$\begin{cases} s_o(t) = \displaystyle\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} s(u)e^{j2\pi f(t-u)}du\right)H(f;t)df, \text{ where } u = t-\tau \\ \quad\;\; = \displaystyle\int_{-\infty}^{\infty} S(f)H(f;t)e^{j2\pi ft}df \\ S_o(f) = S(f)H(f;t) \end{cases}$$

Then

$$\begin{aligned}
\tilde{S}(f)\tilde{H}(f;t) &= S_+(f+f_c)H_+(f+f_c;t) \\
&= (2u(f+f_c)S(f+f_c))(2u(f+f_c)H(f+f_c;t)) \\
&= 4u(f+f_c)S(f+f_c)H(f+f_c) \\
&= 4u(f+f_c)S_o(f+f_c) \\
&= 2[2u(f+f_c)S_o(f+f_c)] \\
&= 2\tilde{S}_o(f) \qquad \Rightarrow \tilde{s}_o(t) = \frac{1}{2}\int_{-\infty}^{\infty}\tilde{h}(\tau;t)\tilde{s}(t-\tau)d\tau
\end{aligned}$$

Note that in a time-varying environment,

$$\begin{aligned}
H(f;t) &= \int_{-\infty}^{\infty} h(\tau;t)e^{-j2\pi f\tau}d\tau \\
&= \int_{-\infty}^{\infty} \text{Re}\left\{\tilde{h}(\tau;t)\exp(j2\pi f_c\tau)\right\}e^{-j2\pi f\tau}d\tau \\
&= \frac{1}{2}\int_{-\infty}^{\infty}\left(\tilde{h}(\tau;t)e^{j2\pi f_c\tau} + \tilde{h}^*(\tau;t)e^{-j2\pi f_c\tau}\right)e^{-j2\pi f\tau}d\tau \\
&= \frac{1}{2}\int_{-\infty}^{\infty}\tilde{h}(\tau;t)e^{-j2\pi(f-f_c)\tau}d\tau + \frac{1}{2}\left(\int_{-\infty}^{\infty}\tilde{h}(\tau;t)e^{-j2\pi(-f-f_c)\tau}d\tau\right)^* \\
&= \frac{1}{2}\left[\tilde{H}(f-f_c;t) + \tilde{H}^*(-f-f_c;t)\right]
\end{aligned}$$

So, we still have

$$\tilde{H}(f;t) = H_+(f+f_c;t) = (2u(f+f_c)H(f+f_c;t))$$