Amendment to Slides 1-59 $\sim$ 1-61

Based on the result from Slide 1-58, i.e.,

$$E[X(t_1)X(t_2)|t_d] = A^2 \sum_{n=-\infty}^{\infty} p(t_1 - nT - t_d)p(t_2 - nT - t_d),$$

we infer that

$$E[X(t_1)X(t_2)|t_d] = A^2 \sum_{n=-\infty}^{\infty} p(t_1 - nT - t_d)p(t_2 - nT - t_d) = A^2$$

if, and only if, there exists one integer n such that

$$p(t_1 - nT - t_d) = p(t_2 - nT - t_d) = 1. \tag{1}$$

Without loss of generality, we let $t_1 = mT + \xi$ and $t_2 = t_1 + \tau$, where m is an integer and $0 \le \xi < T$. Note that the derivations on Slides 1-60 and 1-61 only deal with the special case of $t_1 = 0$. We then examine the condition in (1) as follows:

$$\begin{aligned}
& p(t_1 - nT - t_d) = p(t_2 - nT - t_d) = 1 \\
\iff\ & 0 \le t_1 - nT - t_d < T \quad \text{and} \quad 0 \le t_2 - nT - t_d < T \\
\iff\ & \frac{t_1 - t_d}{T} - 1 < n \le \frac{t_1 - t_d}{T} \quad \text{and} \quad \frac{t_2 - t_d}{T} - 1 < n \le \frac{t_2 - t_d}{T} \\
\iff\ & \left\lfloor \frac{t_1 - t_d}{T} \right\rfloor = \left\lfloor \frac{t_2 - t_d}{T} \right\rfloor \\
\iff\ & \left\lfloor \frac{mT + \xi - t_d}{T} \right\rfloor = \left\lfloor \frac{mT + \xi + \tau - t_d}{T} \right\rfloor \\
\iff\ & \left\lfloor m + \frac{\xi - t_d}{T} \right\rfloor = \left\lfloor m + \frac{\xi + \tau - t_d}{T} \right\rfloor \\
\iff\ & m + \left\lfloor \frac{\xi - t_d}{T} \right\rfloor = m + \left\lfloor \frac{\xi + \tau - t_d}{T} \right\rfloor \\
\iff\ & \left\lfloor \frac{\xi - t_d}{T} \right\rfloor = \left\lfloor \frac{\xi + \tau - t_d}{T} \right\rfloor
\end{aligned}$$

$$\iff \left\lfloor \frac{\xi - t_d}{T} \right\rfloor = \left\lfloor \frac{\xi+\tau-t_d}{T} \right\rfloor = 0 \quad \text{or} \quad \left\lfloor \frac{\xi - t_d}{T} \right\rfloor = \left\lfloor \frac{\xi+\tau-t_d}{T} \right\rfloor = -1$$

$$\text{(Note that } \left\lfloor \frac{\xi - t_d}{T} \right\rfloor \text{ can only be either } -1 \text{ or } 0 \text{ since } 0 \le \xi, t_d < T.\text{)}$$

$$\iff (\xi - T < t_d \le \xi \quad \text{and} \quad \xi+\tau-T < t_d \le \xi+\tau)$$
$$\text{or} \quad (\xi < t_d \le \xi + T \quad \text{and} \quad \xi+\tau < t_d \le \xi+\tau+T)$$

$$\iff \max\{0, \xi - T, \xi+\tau-T\} < t_d \le \min\{T, \xi, \xi+\tau\}$$
$$\text{or} \quad \max\{0, \xi, \xi+\tau\} < t_d \le \min\{T, \xi+T, \xi+\tau+T\} \tag{2}$$

$$\iff \begin{cases} (\emptyset) \quad \text{or} \quad (\xi < t_d \le \xi+\tau+T), & \text{if } -T < \tau < -\xi \\ (0 \le t_d \le \xi+\tau) \quad \text{or} \quad (\xi < t_d < T), & \text{if } -\xi \le \tau < 0 \\ (0 \le t_d \le \xi) \quad \text{or} \quad (\xi+\tau < t_d < T), & \text{if } 0 \le \tau < T-\xi \\ (\xi+\tau-T < t_d \le \xi) \quad \text{or} \quad (\emptyset), & \text{if } T-\xi \le \tau < T \end{cases} \tag{3}$$

where the last step also incorporates $0 \le t_d < T$.[1]

[1] It is apparent that when $\tau \ge T$,

$$\max\{0, \xi-T, \xi+\tau-T\} \ge \xi+\tau-T \ge \xi \ge \min\{T, \xi, \xi+\tau\}$$

and

$$\max\{0, \xi, \xi+\tau\} \ge \xi+\tau \ge \xi+T \ge \min\{T, \xi+T, \xi+\tau+T\}.$$

Thus, no t_d can satisfy (2). Similar inference can be performed when $\tau \le -T$. It remains to consider the situation that $-T < \tau < T$. A systematic way to do the derivation from (2) to (3) is to remove "max" and "min" operations in (2), which requires to determine the threshold values of τ by equating the terms inside $\max\{\cdot\}$ and $\min\{\cdot\}$. This results in

$$\max\{0, \xi-T, \xi+\tau-T\} < t_d \le \min\{T, \xi, \xi+\tau\} \quad \Rightarrow \quad \begin{cases} 0 = \xi+\tau-T; \\ \xi - T = \xi+\tau-T; \\ T = \xi+\tau; \\ \xi = \xi+\tau \end{cases} \quad \Rightarrow \quad \begin{cases} \tau = T-\xi; \\ \tau = 0; \\ \tau = T-\xi; \\ \tau = 0 \end{cases}$$

Similarly,

$$\max\{0, \xi, \xi+\tau\} < t_d \le \min\{T, \xi+T, \xi+\tau+T\} \Rightarrow \begin{cases} 0 = \xi+\tau; \\ \xi = \xi+\tau; \\ T = \xi+\tau+T; \\ \xi+T = \xi+\tau+T \end{cases} \quad \Rightarrow \quad \begin{cases} \tau = -\xi; \\ \tau = 0; \\ \tau = -\xi; \\ \tau = 0 \end{cases}$$

We can then distinguish the ranges of $[-T < \tau < -\xi]$, $[-\xi \le \tau < 0]$, $[0 \le \tau < T-\xi]$ and $[T-\xi \le \tau < T]$ to remove "max" and "min" operations in (2), and forward to (3).

As a result,

$$
\begin{aligned}
E[X(t_1)X(t_2)] &= E[E[X(t_1)X(t_2)|t_d]] \\
&= \begin{cases}
\int_{\xi}^{\xi+\tau+T} A^2 \frac{1}{T} dt_d, & \text{if } -T < \tau < -\xi \\
\int_{0}^{\xi+\tau} A^2 \frac{1}{T} dt_d + \int_{\xi}^{T} A^2 \frac{1}{T} dt_d, & \text{if } -\xi \le \tau < 0 \\
\int_{0}^{\xi} A^2 \frac{1}{T} dt_d + \int_{\xi+\tau}^{T} A^2 \frac{1}{T} dt_d, & \text{if } 0 \le \tau < T-\xi \\
\int_{\xi+\tau-T}^{\xi} A^2 \frac{1}{T} dt_d, & \text{if } T-\xi \le \tau < T
\end{cases} \\
&= A^2 \left(1 - \frac{|\tau|}{T}\right) \quad \text{for } -T < \tau < T.
\end{aligned}
$$