1. Question: On Slide 1-224, how to prove the fifth property, i.e., $S_{\tilde{N}}(f)$ is real.

   Answer: Since

$$\begin{aligned} R_{\tilde{N}}(\tau) &\triangleq E[\tilde{N}(t+\tau)\tilde{N}^*(t)] \\ &= E[\tilde{N}^*(t)\tilde{N}(t+\tau)] \\ &= \left(E[\tilde{N}(t)\tilde{N}^*(t+\tau)]\right)^* \\ &= (R_{\tilde{N}}(-\tau))^* = R^*_{\tilde{N}}(-\tau), \end{aligned}$$

   we derive

$$\begin{aligned} S_{\tilde{N}}(f) &= \int_{-\infty}^{\infty} R_{\tilde{N}}(\tau)e^{-\imath 2\pi f\tau}d\tau \\ &= \int_{-\infty}^{\infty} R^*_{\tilde{N}}(-\tau)e^{-\imath 2\pi f\tau}d\tau \\ &= \left(\int_{-\infty}^{\infty} R_{\tilde{N}}(-\tau)e^{\imath 2\pi f\tau}d\tau\right)^* \\ &= \left(\int_{\infty}^{-\infty} R_{\tilde{N}}(s)e^{-\imath 2\pi f s}(-ds)\right)^* \quad (s=-\tau) \\ &= \left(\int_{-\infty}^{\infty} R_{\tilde{N}}(s)e^{-\imath 2\pi f s}ds\right)^* \\ &= S^*_{\tilde{N}}(f). \end{aligned}$$

   So $S_{\tilde{N}}(f)$ is equal to its complex conjugate, which implies it's real-valued.

2. Question: On Slide 1-225, why do we put the second term on the right-hand-side in the form of $S_{\tilde{N}}(-f-f_c)$? Why not $S_{\tilde{N}}(f+f_c)$?

   Answer: $S^*_{\tilde{N}}(-f-f_c)$ is in general not equal to $S_{\tilde{N}}(f+f_c)$. In other words, we never guarantee to have

$$S^*_{\tilde{N}}(-f) = S_{\tilde{N}}(f)$$

   since $R_{\tilde{N}}(\tau)$ is in general complex-valued.

3. Question: On Slide 1-233, why $H_I(f)H_Q(f) = |H(f)|^2$?

   Answer: Sorry that I did not make it clear. Here, we simply take $H_I(f) = H_Q(f) = H(f)$ and assume $h(\tau)$ is real. Then $H_I(f)H_Q(-f) = H(f)H(-f) = H(f)H^*(f) = |H(f)|^2$.

4. Question: On Slide 1-251, how to derive the variance of Y_I?

Answer: $\{(A_k, \Theta_k)\}_{k=1}^{\infty}$ is i.i.d. with A_k uniform over $[-1, 1)$ and Θ_k uniform over $[0, 2\pi)$. Also, here we assume that A_k is independent of Θ_k. Hence,

$$E[A_k \cos(\Theta_k)] = E[A_k]\, E[\cos(\Theta_k)] = 0$$

and

$$\begin{aligned}
\text{Var}[A_k \cos(\Theta_k)] &= E[(A_k \cos(\Theta_k))^2] \\
&= E[A_k^2]\, E[\cos^2(\Theta_k)] \\
&= \left(\int_{-1}^{1} \frac{1}{2} x^2 dx \right) \cdot \left(\int_{0}^{2\pi} \frac{1}{2\pi} \cos^2(\theta) d\theta \right) \\
&= \frac{1}{3} \cdot \frac{1}{2} \\
&= \frac{1}{6}.
\end{aligned}$$

Consequently,

$$E[Y_I^2] = \text{Var}[Y_I] = N \cdot \text{Var}[A_k \cos(\Theta_k)] = \frac{N}{6}.$$