Sample Problems for the Fourth Quiz

1. Give an independent and identically distributed sequence of random variables X_1, X_2, X_3, $\cdots$. Suppose X is uniformly distributed over $\mathcal{X} \triangleq \{-2, -1, 0, 1, 2\}$.

   (a) Find the *ensemble average* of X_1.

   (b) What is the probability of the *time average* of X_1 and X_2 equal to the *ensemble average*.

   Solution.

   (a) $E[X] = 0$.

   (b)

$$\begin{aligned}
&\Pr\left[\frac{X_1+X_2}{2}=0\right] = \Pr[X_1+X_2=0] \\
&= \Pr[X_1=-2 \text{ and } X_2=2] + \Pr[X_1=-1 \text{ and } X_2=1] \\
&\quad + \Pr[X_1=0 \text{ and } X_2=0] + \Pr[X_1=1 \text{ and } X_2=-1] \\
&\quad + \Pr[X_1=2 \text{ and } X_2=-2] \\
&= \Pr[X_1=-2]\cdot\Pr[X_2=2] + \Pr[X_1=-1]\cdot\Pr[X_2=1] \\
&\quad + \Pr[X_1=0]\cdot\Pr[X_2=0] + \Pr[X_1=1]\cdot\Pr[X_2=-1] \\
&\quad + \Pr[X_1=2]\cdot\Pr[X_2=-2] \\
&= \frac{1}{5}\cdot\frac{1}{5} + \frac{1}{5}\cdot\frac{1}{5} + \frac{1}{5}\cdot\frac{1}{5} + \frac{1}{5}\cdot\frac{1}{5} + \frac{1}{5}\cdot\frac{1}{5} = \frac{1}{5}.
\end{aligned}$$

2. Give an example of random variables X and Y that are orthogonal but not uncorrelated. Next, give an example of random variables X and Y that are not orthogonal but uncorrelated. Then, give an example of random variables X and Y that are not orthogonal and also not uncorrelated. Finally, give an example of random variables X and Y that are both orthogonal and uncorrelated.

   Solution. Suppose

$$\Pr[(X,Y)=(x,y)] = \begin{cases} p, & (x,y)=(-1,-1) \\ q, & (x,y)=(-1,+1) \\ 1-p-q, & (x,y)=(+1,+1) \end{cases}$$

Then

$$\begin{cases} \mu_X = 1 - 2p - 2q \\ \mu_Y = 1 - 2p \\ E[XY] = p + (1 - p - q) - q = 1 - 2q \\ \text{Cov}[X,Y] = E[XY] - \mu_X \mu_Y = (1 - 2q) - (1 - 2p)(1 - 2p - 2q) \end{cases}$$

	Uncorrelated $\text{Cov}[X,Y]$	Orthogonal $E[XY]$
$p = 1/2$ and $q = 1/2$	0	0
$p = 1/2$ and $q = 1/2$	1/4	0
$p = 3/4$ and $q = 1/4$	0	1/2
$p = 1/4$ and $q = 1/4$	1/2	1/2

As a result, all four combinations are possible!

3. Prove that for a stable LTI filter, a WSS input induces a WSS output.

Solution. First, continuing from the derivation in Slide 1-83, we obtain

$$\mu_Y(t) = \int_{-\infty}^{\infty} h(\tau)\mu_X(t-\tau)d\tau.$$

Thus if X is WSS, $\mu_X(t)$ is a constant, which implies $\mu_Y(t) = \mu_X \int_{-\infty}^{\infty} h(\tau)d\tau$ is also a constant.

Secondly, continuing the derivation in Slide 1-85, we obtain

$$R_Y(t,u) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(t-\tau_1, u-\tau_2)d\tau_2 d\tau_1.$$

Thus, if X is WSS, $R_X(t-\tau_1, u-\tau_2)$ can be briefed as $R_X((t-u) - \tau_1 + \tau_2)$, which implies $R_Y(t,u)$ is only a function of $(t-u)$.

Accordingly, a WSS input X induces a WSS output Y.

4. Write down the formula of the Fourier transform pair.

Solution. See Slide 1-87.

5.

(a) If $g_T(t)$ is a bounded periodic function satisfying Dirichlet condition, then express it as a linear combination of $\{\exp(\imath 2\pi \frac{n}{T} t)\}_{n=-\infty}^{\infty}$.

(b) Let $g(t)$ be the generating function of $g_T(t)$ and let $G(f)$ be the Fourier transform of $g(t)$. How we can obtain the coefficients of the linear combination in (a) in terms of $G(f)$.

Solution.

(a) See Slide 1-96.

(b) See Slides 1-98 $\sim$ 1-99.

6. Prove that $y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$ implies $y(f) = H(f)x(f)$, where $x(f)$, $H(f)$ and $y(f)$ are Fourier transforms of $x(t)$, $h(t)$ and $y(t)$.

   Solution. See Slides 1-103 $\sim$ 1-104.

7. The time-averaged autocorrelation functions of LTI system input $X(t)$ and LTI system output $Y(t)$ follow

$$\bar{R}_Y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h^*(\tau_2)\bar{R}_X(\tau - \tau_1 + \tau_2)d\tau_2 d\tau_1,$$

   where $h(\tau)$ is the impulse response of the LTI system. Prove that

$$\bar{S}_Y(f) = |H(f)|^2 \bar{S}_X(f).$$

   Solution. See Slide 1-119.

8. Prove that the PSD is non-negative.

   Solution. See Slides 1-127 $\sim$ 1-129.

9. (Bonus Problems) If $G(f)$ is the Fourier transform of $g(t)$, write the Fourier transforms of the below functions without derivation.

   (a) $\delta(t)$, $\Pi(t/T)$ and $\Delta(t/T)$

   (b) $g(t-t_0)$ and $g(t)e^{\imath 2\pi f_0 t}$

   (c) $g'(t)$ and $\int_{-\infty}^{t} g(s)ds$

   (d) $g^*(t)$ and $G(t)$

   (e) $\cos(2\pi f_0 t)$ and $\sin(2\pi f_0 t)$

   (f) $g(at)$

   (g) $\sum_{n=-\infty}^{\infty} \delta(t - nT)$

Note: I suggest you to get familiar with these rules. They will help you build up a strong intuitive sense on Fourier transform pairs. I may select one or two rules to offer bonus points for the next quiz.

Solution.

(a) 1, $T\text{sinc}(fT)$ and $T\text{sinc}^2(fT)$

(b) $G(f)e^{-\imath 2\pi f t_0}$ and $G(f-f_0)$

(c) $\imath 2\pi f G(f)$ and $\frac{1}{\imath 2\pi f}G(f) + \frac{G(0)}{2}\delta(f)$

(d) $G^*(-f)$ and $g(-f)$

(e) $\frac{1}{2}[\delta(f-f_0)+\delta(f+f_0)]$ and $\frac{1}{2\imath}[\delta(f-f_0)-\delta(f+f_0)]$

(f) $\frac{1}{|a|}G\left(\frac{f}{a}\right)$

(g) $\frac{1}{T}\sum_{n=-\infty}^{\infty}\delta(f-\frac{n}{T})$