Sample Problems for the Seventh Quiz

- Please carefully read DSB-C, DSB-SC, SSB, VSB and FM modulations. Our term project requires you to implement software demodulators for these modulation schemes.

1. Draw the block diagram of an envelop detector.

   Solution. See Slide 2-16.

2.

   (a) Draw the block diagram of a coherent detector.

   (b) Determine the output of a coherent detector for DSB-C modulated signal, and explain what quadrature null effect is.

   Solution.

   (a) See Slide 2-25.

   (b) The output of a coherent detector is $\frac{1}{2}A_c A'_c \cos(\phi)m(t)$, for which the derivation can be found on Slide 2-26. If $\phi = \pi/2$ or $-\pi/2$, the output of a coherent detector for DSB-SC is nullified. This is referred to as *quadrature null effect* of the coherent detector.

3. Draw the block diagram of a Costas receiver.

   Solution. See Slide 2-28.

4. Explain why for the generation of a SSB modulated signal to be possible, the message spectrum must have an energy gap centered at the origin.

   Solution. This is because an ideal bandpass filter is not feasible in practice; so an energy gap centered at the origin is necessary for the practical design of a highly selective filter.

5.

   (a) Suppose

   $$m_{\text{VSB}}(f) = m(f)\frac{1 - \imath H_Q(f)}{2} \text{ and } m(f) = m^*(-f).$$

   Prove that

   $$m_{\text{VSB}}(f) + m^*_{\text{VSB}}(-f) = m(f),$$

if $H_Q(f)$ satisfies $H_Q(-f) = H_Q^*(f)$ and

$$\frac{1}{\imath}H_Q(f) = \begin{cases} 1, & f \le -f_\text{v}; \\ \in (0,1), & -f_\text{v} < f < 0; \\ 0, & f = 0. \end{cases}$$

(b) Further suppose

$$\begin{cases} s_\text{DSB}(f) = \frac{1}{2}[m(f+f_c) + m^*(-f+f_c)]; \\ s_\text{VSB}(f) = \frac{1}{2}[m_\text{VSB}(f+f_c) + m_\text{VSB}^*(-f+f_c)]. \end{cases}$$

and

$$s_\text{VSB}(f) = s_\text{DSB}(f)H(f).$$

Prove that

$$H(f) = 1 - \imath\frac{1}{2}[H_Q(f+f_c) + H_Q(-f+f_c)].$$

(c) Show that $H(f)$ in (b) satisfies (i) $|H(f_c - f)| + |H(f_c + f)| = 1$ for $-f_\text{v} < f < f_\text{v}$, and (ii) $H(f - f_c) + H(f + f_c) = 1$ for $-W < f < W$, provided that $f_c > W > f_\text{v}$.

Solution.

(a) For convenience, let $L_Q(f) = \frac{1}{\imath}H_Q(f)$, which is real-valued according to the definition of $H_Q(f)$. It can be shown that $L_Q(-f) = -L_Q(f)$ (by the derivation on Slide 2-38). Hence, the proof can be done via the derivation on Slide 2-39.

(b) See Slides 2-40 $\sim$ 2-41.

(c) For $-f_\text{v} < f < f_\text{v}$, we have

$$2f_c - f > 2f_c - f_\text{v} > f_\text{v},$$

which implies

$$\frac{1}{\imath}H_Q(-(2f_c - f)) = 1 \Rightarrow H_Q(2f_c - f) = H_Q^*(-(2f_c - f)) = -\imath.$$

We can similarly argue that $H_Q(2f_c + f) = -\imath$. This gives that for $-f_{\rm v} < f < f_{\rm v}$,

$$
\begin{aligned}
&|H(f_c - f)| + |H(f_c + f)| \\
&= \left|1 - \imath\frac{1}{2}[H_Q((f_c - f) + f_c) + H_Q(-(f_c - f) + f_c)]\right| \\
&\quad + \left|1 - \imath\frac{1}{2}[H_Q((f_c + f) + f_c) + H_Q(-(f_c + f) + f_c)]\right| \\
&= \left|1 - \imath\frac{1}{2}[H_Q(2f_c - f) + H_Q(f)]\right| + \left|1 - \imath\frac{1}{2}[H_Q(2f_c + f) + H_Q(-f)]\right| \\
&= \left|1 - \imath\frac{1}{2}[-\imath + H_Q(f)]\right| + \left|1 - \imath\frac{1}{2}[-\imath + H_Q(-f)]\right| \\
&= \frac{1}{2}\left|1 + \frac{1}{\imath}H_Q(f)]\right| + \frac{1}{2}\left|1 + \frac{1}{\imath}H_Q(-f)]\right| \\
&= \frac{1}{2}\left|1 + L_Q(f)\right| + \frac{1}{2}\left|1 + L_Q(-f)\right| \\
&= \begin{cases} \frac{1}{2}\left|1 - L_Q(-f)\right| + \frac{1}{2}\left|1 + L_Q(-f)\right|, & \text{if } f > 0; \\ \frac{1}{2}\left|1 + 0\right| + \frac{1}{2}\left|1 - 0\right|, & \text{if } f = 0; \ \text{(since } L_Q(f) = -L_Q(-f)\text{)} \\ \frac{1}{2}\left|1 + L_Q(f)\right| + \frac{1}{2}\left|1 - L_Q(f)\right|, & \text{if } f < 0; \end{cases} \\
&= \begin{cases} \frac{1}{2}\left(1 - L_Q(-f)\right) + \frac{1}{2}\left(1 + L_Q(-f)\right), & \text{if } f > 0; \\ \frac{1}{2}\left(1 + 0\right) + \frac{1}{2}\left(1 - 0\right), & \text{if } f = 0; \ \text{(since } 0 < L_Q(f) < 1 \\ \frac{1}{2}\left(1 + L_Q(f)\right) + \frac{1}{2}\left(1 - L_Q(f)\right), & \text{if } f < 0; \quad \text{for } \ -f_{\rm v} < f < 0) \end{cases} \\
&= 1
\end{aligned}
$$

On the other hand, we can similarly argue that for $-W < f < W$,

$$2f_c - f > 2f_c - W > W > f_{\rm v} \text{ and } 2f_c + f > 2f_c - W > W > f_{\rm v}$$

and obtain that

$$H_Q(2f_c + f) = H_Q(2f_c - f) = -\imath.$$

This brings that

$$
\begin{aligned}
& H(f-f_c)+H(f+f_c) \\
&= \left(1-\imath\frac{1}{2}[H_Q((f-f_c)+f_c)+H_Q(-(f-f_c)+f_c)]\right) \\
&\quad + \left(1-\imath\frac{1}{2}[H_Q((f+f_c)+f_c)+H_Q(-(f+f_c)+f_c)]\right) \\
&= \left(1-\imath\frac{1}{2}[H_Q(f)+H_Q(2f_c-f)]\right) \\
&\quad + \left(1-\imath\frac{1}{2}[H_Q(2f_c+f)+H_Q(-f)]\right) \\
&= \left(1-\imath\frac{1}{2}[H_Q(f)-\imath]\right)+\left(1-\imath\frac{1}{2}[-\imath+H_Q(-f)]\right) \\
&= 1+\frac{1}{2}\left(\frac{1}{\imath}H_Q(f)+\frac{1}{\imath}H_Q(-f)\right) \\
&= 1+\frac{1}{2}\left(L_Q(f)+L_Q(-f)\right) \\
&= 1.
\end{aligned}
$$

6. Formulate the signals of AM, PM and FM modulations as:

$$s_{\text{DSB-C}}(t) = s_{\text{AM}}(t) = A_c[1+k_{\text{a}}m(t)]\cos(2\pi f_c t);$$

$$s_{\text{PM}}(t) = A_c\cos[2\pi f_c t + k_{\text{p}}m(t)]$$

$$s_{\text{FM}}(t) = A_c\cos\left[2\pi f_c t + 2\pi k_{\text{f}}\int_0^t m(\tau)d\tau\right]$$

and suppose $m(t) = A_m\cos(2\pi f_m t)$. Answer the following questions.

(a) Which parameter is referred to as *amplitude sensitivity*? Also, which parameter is referred to as *phase sensitivity* and which is *frequency sensitivity*?

(b) Find the instantaneous frequency $f_i(t)$ at time t of the FM signal. What is the frequency deviation (i.e., the largest deviation of $f_i(t)$ from f_c)? What is the modulation index of the FM signal (i.e., the largest phase change away from zero phase)?

Note: The modulation of the PM signal is also the largest phase change away from zero phase, which is $k_{\text{p}}A_m$.

(c) The spectrum of $s_{\mathrm{FM}}(t)$ is equal to

$$S_{\mathrm{FM}}(f) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(\beta) \left[\delta(f - f_c - nf_m) + \delta(f + f_c + nf_m)\right].$$

Show that

$$\begin{aligned} S_{\mathrm{FM},2T}(f) &= A_c T \sum_{n=-\infty}^{\infty} J_n(\beta) \left[\mathrm{sinc}(2T(f - f_c - nf_m)) \right. \\ &\qquad \left. + \mathrm{sinc}(2T(f + f_c + nf_m))\right], \end{aligned}$$

where $S_{\mathrm{FM},2T}(f)$ is the Fourier transform of $s_{\mathrm{FM}}(t) \cdot \mathbf{1}\{|t| < T\}$ and $\mathbf{1}\{\cdot\}$ is the set indicator function.
Hint: $\mathcal{F}\left\{ \mathbf{1}\{|t| < T\} \right\} = 2T\,\mathrm{sinc}(2Tf)$.

(d) The time-averaged PSD of the FM signal can be obtained via

$$\begin{aligned} &\overline{\mathrm{PSD}}_{\mathrm{FM}}(f) \\ &= \lim_{T\to\infty} \frac{1}{2T} S_{\mathrm{FM}}(f) S^*_{\mathrm{FM},2T}(f) \\ &= \lim_{T\to\infty} \frac{1}{2T} \left(\frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(\beta) \left[\delta(f - f_c - nf_m) + \delta(f + f_c + nf_m)\right] \right) \\ &\quad \left(A_c T \sum_{k=-\infty}^{\infty} J_k(\beta) \left[\mathrm{sinc}(2T(f - f_c - kf_m)) + \mathrm{sinc}(2T(f + f_c + kf_m))\right] \right) \\ &= \lim_{T\to\infty} \frac{A_c^2}{4} \left(\sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} J_n(\beta) J_k(\beta) \big[\delta(f - f_c - nf_m) + \delta(f + f_c + nf_m)\big] \right. \\ &\quad \left. \big[\mathrm{sinc}(2T(f - f_c - kf_m)) + \mathrm{sinc}(2T(f + f_c + kf_m))\big] \right) \\ &= \lim_{T\to\infty} \frac{A_c^2}{4} \left(\sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} J_n(\beta) J_k(\beta) \delta(f - f_c - nf_m) \mathrm{sinc}(2T(f - f_c - kf_m)) \right. \\ &\quad + \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} J_n(\beta) J_k(\beta) \delta(f - f_c - nf_m) \mathrm{sinc}(2T(f + f_c + kf_m)) \\ &\quad + \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} J_n(\beta) J_k(\beta) \delta(f + f_c + nf_m) \mathrm{sinc}(2T(f - f_c - kf_m)) \\ &\quad \left. + \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} J_n(\beta) J_k(\beta) \delta(f + f_c + nf_m) \mathrm{sinc}(2T(f + f_c + kf_m)) \right) \end{aligned}$$

Prove that

$$\sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} J_n(\beta)J_k(\beta)\delta(f-f_c-nf_m)\text{sinc}(2T(f-f_c-kf_m))$$
$$= \sum_{n=-\infty}^{\infty} J_n^2(\beta)\delta(f-f_c-nf_m)$$

and

$$\sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} J_n(\beta)J_k(\beta)\delta(f-f_c-nf_m)\text{sinc}(2T(f+f_c+kf_m))$$
$$= \sum_{n=-\infty}^{\infty} J_n(\beta)J_{-n-2f_c/f_m}(\beta)\delta(f-f_c-nf_m)$$

provided that $4Tf_c$ and $2Tf_m$ are both integers.

Solution.

(a) It is obvious that k_a, k_p and k_f are respectively the amplitude, phase and frequency sensitivities.

(b) See Slide 2-63.

(c)

$$\begin{aligned}
&\mathcal{F}\left\{s_\text{FM}(t)\cdot\mathbf{1}\{|t|<T\}\right\}\\
&= \mathcal{F}\left\{s_\text{FM}(t)\right\}\star\mathcal{F}\left\{\mathbf{1}\{|t|<T\}\right\}\\
&= \left(\frac{A_c}{2}\sum_{n=-\infty}^{\infty} J_n(\beta)\left[\delta(f-f_c-nf_m)+\delta(f+f_c+nf_m)\right]\right)\star(2T\text{sinc}(2Tf))\\
&= \frac{A_c}{2}\sum_{n=-\infty}^{\infty} J_n(\beta)\Big[\delta(f-f_c-nf_m)\star(2T\text{sinc}(2Tf))\\
&\qquad\qquad +\delta(f+f_c+nf_m)\star(2T\text{sinc}(2Tf))\Big]\\
&= \frac{A_c}{2}\sum_{n=-\infty}^{\infty} J_n(\beta)\Big[2T\,\text{sinc}(2T(f-f_c-nf_m))\\
&\qquad\qquad +2T\,\text{sinc}(2T(f+f_c+nf_m))\Big],
\end{aligned}$$

where "$\star$" denotes the convolution operation.

(d)

$$\sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} J_n(\beta)J_k(\beta)\delta(f-f_c-nf_m)\text{sinc}(2T(f-f_c-kf_m))$$
$$= \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} J_n(\beta)J_k(\beta)\delta(f-f_c-nf_m)\text{sinc}(2T((f_c+nf_m)-f_c-kf_m))$$
$$(\text{since } \delta(f-f_c-nf_m)=0 \text{ for } f\neq f_c+nf_m)$$
$$= \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} J_n(\beta)J_k(\beta)\delta(f-f_c-nf_m)\text{sinc}(2T(n-k)f_m)$$
$$= \sum_{n=-\infty}^{\infty} J_n^2(\beta)\delta(f-f_c-nf_m) \; (\text{since sinc}(2T(n-k)f_m)=0 \text{ for } k\neq n)$$

and

$$\sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} J_n(\beta)J_k(\beta)\delta(f-f_c-nf_m)\text{sinc}(2T(f+f_c+kf_m))$$
$$= \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} J_n(\beta)J_k(\beta)\delta(f-f_c-nf_m)\text{sinc}(2T(f+f_c+kf_m))$$
$$= \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} J_n(\beta)J_k(\beta)\delta(f-f_c-nf_m)\text{sinc}(2T((f_c+nf_m)+f_c+kf_m))$$
$$(\text{since } \delta(f-f_c-nf_m)=0 \text{ for } f\neq f_c+nf_m)$$
$$= \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} J_n(\beta)J_k(\beta)\delta(f-f_c-nf_m)\text{sinc}(4Tf_c+2T(n+k)f_m))$$
$$= \sum_{n=-\infty}^{\infty} J_n(\beta)J_{-n-2f_c/f_m}(\beta)\delta(f-f_c-nf_m)$$

since

$$\text{sinc}(4Tf_c+2T(n+k)f_m)) = \begin{cases} 1, & 4Tf_c+2T(n+k)f_m=0; \\ 0, & \text{otherwise.} \end{cases}$$