Principles of Communication Engineering II

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Model Answers to Exercise 12 of 19 May 2009

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Problem 1 — *Budget Link Calculation*

With $\mathcal{M}$ being the security margin, we have

$$\frac{\mathcal{E}_\text{b}}{\mathcal{N}_0} = \mathcal{M} \cdot \left(\frac{\mathcal{E}_\text{b}}{\mathcal{N}_0}\right)_\text{req}$$

and

$$\frac{\mathcal{E}_\text{b}}{\mathcal{N}_0} = \frac{\mathsf{T}_\text{b}\mathsf{P}_\text{r}}{\mathcal{N}_0} = \frac{1}{\mathsf{R}_\text{b}} \cdot \frac{\mathsf{P}_\text{r}}{\mathcal{N}_0}.$$

Moreover, we have

$$\begin{aligned} \mathsf{P}_\text{r} &= \text{EIRP} \cdot \mathsf{G}_\text{r} \left(\frac{\lambda}{4\pi d}\right)^2 = \text{EIRP} \cdot \mathsf{G}_\text{r} \left(\frac{c_0}{4\pi f_\text{c} d}\right)^2; \\ \mathsf{G}_\text{r} &= \frac{4\pi\eta_\text{eff} \cdot \mathsf{A}_\text{r}}{\lambda^2} = \frac{4\pi\eta_\text{eff} f_\text{c}^2}{c_0^2} \cdot \pi \left(\frac{\mathsf{D}}{2}\right)^2; \\ \mathcal{N}_0 &= k_\text{B}\mathsf{T}, \end{aligned}$$

where d denotes the distance between satellite and earth.
Plugging that all together and solving for the antenna dish diameter now gives

$$\begin{aligned} \mathsf{D} &= \sqrt{\frac{c_0^2}{\eta_\text{eff}\pi^2 f_\text{c}^2} \cdot \frac{\mathsf{P}_\text{r}}{\mathcal{N}_0} \cdot k_\text{B}\mathsf{T} \cdot \frac{1}{\text{EIRP}} \cdot \left(\frac{4\pi f_\text{c} d}{c_0}\right)^2} \\ &= \sqrt{\frac{16d^2}{\eta_\text{eff}} \cdot \frac{\mathsf{P}_\text{r}}{\mathcal{N}_0} \cdot k_\text{B}\mathsf{T} \cdot \frac{1}{\text{EIRP}}} \\ &= \sqrt{\frac{16d^2}{\eta_\text{eff}} \cdot \mathsf{R}_\text{b}\frac{\mathcal{E}_\text{b}}{\mathcal{N}_0} \cdot k_\text{B}\mathsf{T} \cdot \frac{1}{\text{EIRP}}} \\ &= \sqrt{\frac{16d^2}{\eta_\text{eff}} \cdot \mathsf{R}_\text{b}\mathcal{M}\left(\frac{\mathcal{E}_\text{b}}{\mathcal{N}_0}\right)_\text{req} \cdot k_\text{B}\mathsf{T} \cdot \frac{1}{\text{EIRP}}} \\ &= 36.5 \text{ cm}. \end{aligned}$$

Problem 2 *GSM*

a) There are

$$\frac{25 \text{ MHz}}{200 \text{ kHz}} = 125$$

radio channel, where each channel supports 8 users. Hence, we get a total of 1000 simultaneous users.

b) We get the following:

i) The time duration of a bit is

$$\mathsf{T}_\text{b} = \frac{1}{270.833 \text{ kbps}} = 3.692 \ \mu\text{s}.$$

ii) The time duration of a slot is

$$\mathsf{T}_\text{slot} = 156.25 \cdot \mathsf{T}_\text{b} = 0.577 \text{ ms}.$$

iii) The time duration of a frame is

$$\mathsf{T}_\text{frame} = 8 \cdot \mathsf{T}_\text{slot} = 4.615 \text{ ms}.$$

iv) A user has to wait from one frame duration, *i.e.*, 4.615 ms, for its next transmission.

v) We have 156.25 bits per time slot, *i.e.*, ignoring overhead we get a raw bit rate per user of

$$\mathsf{R}_\text{raw} = \frac{156.25}{\mathsf{T}_\text{frame}} \approx 33.86 \text{ kb/s}.$$

c) The total number of bits per frame is $8 \cdot 156.25 = 1250$ bits/frame. The number of overhead bits per frame is given by

$$b_\text{oh} = 8 \cdot (6 + 8.25 + 26) = 322 \text{ bits}.$$

Hence, the frame efficiency is given as

$$\eta_\text{frame} = 1 - \frac{322}{1250} = 0.7424 = 74.24\%.$$

The effective data rate per user is then

$$\mathsf{R}_\text{eff} = \frac{116}{\mathsf{T}_\text{frame}} \approx 25.14 \text{ kb/s}.$$

Problem 3 *Maximum Ratio Combiner*

a) The output of the linear combiner is given by

$$\begin{aligned} X(t) &= \sum_{\ell=1}^{\mathsf{L}} \alpha_\ell X_\ell(t) \\ &= \sum_{\ell=1}^{\mathsf{L}} \alpha_\ell \left(S_\ell(t) + N_\ell(t) \right) \end{aligned}$$

$$= \underbrace{\sum_{\ell=1}^{\mathsf{L}} \alpha_\ell S_\ell(t)}_{\text{signal}} + \underbrace{\sum_{\ell=1}^{\mathsf{L}} \alpha_\ell N_\ell(t)}_{\text{noise}}.$$

The output signal-to-noise ratio is therefore

$$\begin{aligned}
\text{SNR}_{\text{out}} &= \frac{\text{average signal power}}{\text{average noise power}} \\
&= \frac{\mathsf{E}\left[\left(\sum_{\ell=1}^{\mathsf{L}} \alpha_\ell z_\ell M(t)\right)^2\right]}{\mathsf{E}\left[\left(\sum_{\ell=1}^{\mathsf{L}} \alpha_\ell N_\ell(t)\right)^2\right]} \\
&= \frac{\mathsf{E}\left[\sum_{\ell=1}^{\mathsf{L}} \sum_{\ell'=1}^{\mathsf{L}} \alpha_\ell \alpha_{\ell'} z_\ell z_{\ell'} M^2(t)\right]}{\mathsf{E}\left[\sum_{\ell=1}^{\mathsf{L}} \sum_{\ell'=1}^{\mathsf{L}} \alpha_\ell \alpha_{\ell'} N_\ell(t) N_{\ell'}(t)\right]} \\
&= \frac{\sum_{\ell=1}^{\mathsf{L}} \sum_{\ell'=1}^{\mathsf{L}} \alpha_\ell \alpha_{\ell'} z_\ell z_{\ell'} \mathsf{E}\left[M^2(t)\right]}{\sum_{\ell=1}^{\mathsf{L}} \sum_{\ell'=1}^{\mathsf{L}} \alpha_\ell \alpha_{\ell'} \mathsf{E}\left[N_\ell(t) N_{\ell'}(t)\right]} \\
&= \frac{\sum_{\ell=1}^{\mathsf{L}} \sum_{\ell'=1}^{\mathsf{L}} \alpha_\ell \alpha_{\ell'} z_\ell z_{\ell'}}{\sum_{\ell=1}^{\mathsf{L}} \alpha_\ell^2 \sigma_\ell^2} \\
&= \frac{\left(\sum_{\ell=1}^{\mathsf{L}} \alpha_\ell z_\ell\right)^2}{\sum_{\ell=1}^{\mathsf{L}} \alpha_\ell^2 \sigma_\ell^2},
\end{aligned} \tag{1}$$

where we have used that the message has unit power

$$\mathsf{E}\left[M^2(t)\right] = 1 \qquad \forall\, t$$

and that the noise components have zero mean and are statistically independent.

b) Equation (1) can be rewritten in the equivalent form

$$\text{SNR}_{\text{out}} = \frac{\left(\sum_{\ell=1}^{\mathsf{L}} \mu_\ell \nu_\ell\right)^2}{\sum_{\ell=1}^{\mathsf{L}} \mu_\ell^2}.$$

We now invoke the Schwarz inequality, which, in discrete form the the problem at hand is stated as follows:

$$\left(\sum_{\ell=1}^{\mathsf{L}} \mu_\ell \nu_\ell\right)^2 \le \left(\sum_{\ell=1}^{\mathsf{L}} \mu_\ell^2\right)\left(\sum_{\ell=1}^{\mathsf{L}} \nu_\ell^2\right). \tag{2}$$

Hence, we get

$$\text{SNR}_{\text{out}} \le \frac{\left(\sum_{\ell=1}^{\mathsf{L}} \mu_\ell^2\right)\left(\sum_{\ell=1}^{\mathsf{L}} \nu_\ell^2\right)}{\sum_{\ell=1}^{\mathsf{L}} \mu_\ell^2} = \sum_{\ell=1}^{\mathsf{L}} \nu_\ell^2 = \sum_{\ell=1}^{\mathsf{L}} \text{SNR}_\ell.$$

c) The Schwarz inequality of (2) is satisfied with equality if $\mu_\ell = \nu_\ell$, *i.e.*, in our case if

$$\alpha_\ell = \frac{z_\ell}{\sigma_\ell^2},$$

which corresponds to the optimum values of the combiner's coefficients.

a) Using the density $f_{|H_0|^2}(t) = e^{-t}$ we get

$$\begin{aligned}
\Pr\left[|H_0|^2 < \frac{1}{\text{SNR}}\right] &= \int_0^{\frac{1}{\text{SNR}}} e^{-t}\,\mathrm{d}t \\
&= -e^{-t}\Big|_{t=0}^{t=\frac{1}{\text{SNR}}} \\
&= 1 - e^{-\frac{1}{\text{SNR}}} \\
&\approx \frac{1}{\text{SNR}}
\end{aligned}$$

where we have used that $e^x \approx 1 + x$ for $|x| \ll 1$. This corresponds exactly to the asymptotic behavior of the error probability apart from a constant factor.

b) Using the density $f_T(t) = \frac{1}{(\mathsf{L}-1)!} t^{\mathsf{L}-1} e^{-t}$ we get

$$\begin{aligned}
\Pr\left[\sum_{i=1}^{\mathsf{L}} |H_i|^2 < \frac{1}{\text{SNR}}\right] &= \int_0^{\frac{1}{\text{SNR}}} \frac{1}{(\mathsf{L}-1)!} t^{\mathsf{L}-1} e^{-t}\,\mathrm{d}t \\
&= \frac{1}{(\mathsf{L}-1)!}\left(-t^{\mathsf{L}-1}e^{-t}\Big|_{t=0}^{t=\frac{1}{\text{SNR}}} + (\mathsf{L}-1)\int_0^{\frac{1}{\text{SNR}}} t^{\mathsf{L}-2} e^{-t}\,\mathrm{d}t\right) \\
&= -\frac{1}{(\mathsf{L}-1)!}\left(\frac{1}{\text{SNR}}\right)^{\mathsf{L}-1} e^{-\frac{1}{\text{SNR}}} + \int_0^{\frac{1}{\text{SNR}}} \frac{1}{(\mathsf{L}-2)!} t^{\mathsf{L}-2} e^{-t}\,\mathrm{d}t \\
&= -\frac{1}{(\mathsf{L}-1)!}\left(\frac{1}{\text{SNR}}\right)^{\mathsf{L}-1} e^{-\frac{1}{\text{SNR}}} \\
&\quad - \frac{1}{(\mathsf{L}-2)!}\left(\frac{1}{\text{SNR}}\right)^{\mathsf{L}-2} e^{-\frac{1}{\text{SNR}}} + \int_0^{\frac{1}{\text{SNR}}} \frac{1}{(\mathsf{L}-3)!} t^{\mathsf{L}-3} e^{-t}\,\mathrm{d}t \qquad (3) \\
&= -\sum_{j=1}^{\mathsf{L}-1} \frac{1}{j!}\left(\frac{1}{\text{SNR}}\right)^{j} e^{-\frac{1}{\text{SNR}}} + \int_0^{\frac{1}{\text{SNR}}} e^{-t}\,\mathrm{d}t \qquad (4) \\
&= -\sum_{j=1}^{\mathsf{L}-1} \frac{1}{j!}\left(\frac{1}{\text{SNR}}\right)^{j} e^{-\frac{1}{\text{SNR}}} + 1 - e^{-\frac{1}{\text{SNR}}} \\
&= -\sum_{j=1}^{\mathsf{L}-1} \frac{1}{j!}\left(\frac{1}{\text{SNR}}\right)^{j} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\left(\frac{1}{\text{SNR}}\right)^{k} - \sum_{k=1}^{\infty} \frac{(-1)^k}{k!}\left(\frac{1}{\text{SNR}}\right)^{k} \\
&= \sum_{k=1}^{\infty}\left(-\sum_{j=1}^{\min\{k,\mathsf{L}-1\}} \frac{1}{j!}\frac{1}{(k-j)!}(-1)^{k-j} - \frac{(-1)^k}{k!}\right)\left(\frac{1}{\text{SNR}}\right)^{k} \\
&= \sum_{k=\mathsf{L}}^{\infty}\left(-\sum_{j=1}^{\mathsf{L}-1} \frac{1}{j!}\frac{1}{(k-j)!}(-1)^{k-j} - \frac{(-1)^k}{k!}\right)\left(\frac{1}{\text{SNR}}\right)^{k} \\
&\approx \frac{1}{\mathsf{L}!}\left(\frac{1}{\text{SNR}}\right)^{\mathsf{L}}.
\end{aligned}$$

Here in (3) we applied the same steps as before for a second time (*i.e.*, we use integration by parts); in (4) we repeated the integration in total $\mathsf{L}-1$ times; and in the last step we have used that

$$\sum_{j=1}^{k} \frac{1}{j!} \frac{1}{(k-j)!} (-1)^{k-j} = -\frac{(-1)^k}{k!}.$$

Hence, we have a diversity gain of L.

c) Using the density $f_T(t) = \mathsf{L} t^{\mathsf{L}-1}$ we get

$$\begin{aligned} \Pr\left[\max_{i \in \{1,\ldots,\mathsf{L}\}} |H_i|^2 < \frac{1}{\mathrm{SNR}} \right] &\approx \int_0^{\frac{1}{\mathrm{SNR}}} \mathsf{L} t^{\mathsf{L}-1} \, \mathrm{d}t \\ &= t^{\mathsf{L}} \Big|_{t=0}^{t=\frac{1}{\mathrm{SNR}}} \\ &= \left(\frac{1}{\mathrm{SNR}} \right)^{\mathsf{L}}, \end{aligned}$$

which yields the same diversity, however we have a factor $\mathsf{L}!$ larger probability of error. Hence, we loose in coding gain, but can keep the diversity gain the same.