Fundamental Formulas in Communications

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Turn Formulas into Concrete Concepts

- $b(t) = \int_{-\infty}^{\infty} h(\tau)a(t-\tau)d\tau = a(t) * h(t)$

$$\Rightarrow B(f) = H(f)A(f)$$

- $b(t) = \int_{-\infty}^{\infty} h(\tau)a(t+\tau)d\tau = \int_{-\infty}^{\infty} h(-\tau)a(t-\tau)d\tau = a(t) * h(-t)$

$$\Rightarrow B(f) = H(-f)A(f)$$

- $b(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\tau_1)h_2(\tau_2)a(t-\tau_1+\tau_2)d\tau_1 d\tau_2 = a(t) * h_1(t) * h_2(-t)$

$$\Rightarrow B(f) = H_1(f)H_2(-f)A(f)$$

Turn Formulas into Concrete Concepts

- $Y(t) = X(t) * h(t)$

$$\Rightarrow \boxed{S_Y(f) = H(f)H(-f)S_X(f)}$$

Fundamental Formulas of Fourier Transforms (Slide 1-108)

- $\delta(t) \xrightarrow{Fourier} 1$
- $\Pi\left(\frac{t}{T}\right) = \text{rect}\left(\frac{t}{T}\right) \xrightarrow{Fourier} T\,\text{sinc}(fT)$
- $\Delta\left(\frac{t}{T}\right) \xrightarrow{Fourier} T\,\text{sinc}^2(fT)$

For example, on Slide 1-136 to Slide 1-137, we obtain $A^2\Delta\left(\frac{t}{T}\right) \xrightarrow{Fourier} A^2T\,\text{sinc}^2(fT)$.

If $g(t) \xrightarrow{Fourier} G(f)$, then $G(t) \xrightarrow{Fourier} g(-f)$.

Turn Formulas into Concrete Concepts

- $\delta(-f) = \delta(f) \overset{Fourier}{\longleftarrow} 1$
- $\Pi\left(-\frac{f}{T}\right) = \Pi\left(\frac{f}{T}\right) \overset{Fourier}{\longleftarrow} T\,\mathrm{sinc}(tT)$
- $\Delta\left(-\frac{f}{T}\right) = \Delta\left(\frac{f}{T}\right) \overset{Fourier}{\longleftarrow} T\,\mathrm{sinc}^2(tT)$

Example. Determine the Fourier transform of $\mathrm{sinc}(2Wt)$.

Answer: Noting $T\,\mathrm{sinc}(tT) \overset{Fourier}{\longrightarrow} \Pi\left(\frac{t}{T}\right)$ with $T = 2W$, we obtain

$$\mathrm{sinc}(tT) \overset{Fourier}{\longrightarrow} \frac{1}{T}\Pi\left(\frac{t}{T}\right).$$

- $\frac{d}{dt}g(t) \overset{Fourier}{\longrightarrow} j2\pi f\,G(f)$
- $\int_{-\infty}^{t} g(s)ds \overset{Fourier}{\longrightarrow} \frac{1}{j2\pi f}G(f) + \frac{G(0)}{2}\delta(f)$

Example. Determine the Fourier transform of $\text{sgn}(t)$.

Answer: Noting $2\delta(t) = \frac{d}{dt}\text{sgn}(t) \xrightarrow{Fourier} G(f) = 2$, we obtain

$$\text{sgn}(t) \xrightarrow{Fourier} \frac{1}{j2\pi f}G(f) = \frac{1}{j\pi f}.$$

□

Turn Formulas into Concrete Concepts

$$
\begin{aligned}
&\lim_{a \downarrow 0} \int_{-\infty}^{\infty} e^{-a|t|} \mathrm{sgn}(t) e^{-j2\pi f t} dt \\
&= \lim_{a \downarrow 0} \left(\int_{-\infty}^{0} (-1) e^{(a - j2\pi f)t} dt + \int_{0}^{\infty} (+1) e^{(-a - j2\pi f)t} dt \right) \\
&= \lim_{a \downarrow 0} \left(\frac{1}{-a + j2\pi f} e^{(a - j2\pi f)t} \Big|_{-\infty}^{0} + \frac{1}{-a - j2\pi f} e^{(-a - j2\pi f)t} \Big|_{0}^{\infty} \right) \\
&= \lim_{a \downarrow 0} \left(\frac{1}{-a + j2\pi f} + \frac{1}{a + j2\pi f} \right) = \lim_{a \downarrow 0} \left(- \frac{j4\pi f}{4\pi^2 f^2 + a^2} \right) \\
&= \begin{cases} \frac{1}{j\pi f}, & \text{if } f \neq 0 \\ 0, & \text{if } f = 0 \end{cases}
\end{aligned}
$$

Example. Determine the Fourier transform of $u(t) = \frac{1}{2}\text{sgn}(t) + \frac{1}{2}$.

Answer: Noting $\text{sgn}(t) \stackrel{Fourier}{\longrightarrow} \frac{1}{j\pi f}$ and $1 \stackrel{Fourier}{\longrightarrow} \delta(f)$, we obtain

$$u(t) \stackrel{Fourier}{\longrightarrow} \frac{1}{j2\pi f} + \frac{1}{2}\delta(f).$$

□

Example. Determine the Fourier transform of $\frac{1}{\pi f}$.

Answer: Noting $g(t) = \text{sgn}(t) \stackrel{Fourier}{\longrightarrow} G(f) = \frac{1}{j\pi f}$ and $G(t) \stackrel{Fourier}{\longrightarrow} g(-f)$, we obtain

$$\frac{1}{j\pi t} \stackrel{Fourier}{\longrightarrow} \text{sgn}(-f) = -\text{sgn}(f)$$

and hence,

$\frac{1}{\pi t} \stackrel{Fourier}{\longrightarrow} -j\text{sgn}(f)$. □

Turn Formulas into Concrete Concepts

- $\exp(j2\pi f_0 t) \overset{Fourier}{\longleftarrow} \delta(f - f_0)$
- $\cos(2\pi f_0 t) \overset{Fourier}{\longleftarrow} \frac{1}{2}[\delta(f - f_0) + \delta(f + f_0)]$
- $\sin(2\pi f_0 t) \overset{Fourier}{\longleftarrow} \frac{1}{2j}[\delta(f - f_0) - \delta(f + f_0)]$
- $\exp\{-\pi t^2\} \overset{Fourier}{\longleftarrow} \exp\{-\pi f^2\}$
- $\exp\{-a|t|\} \overset{Fourier}{\longleftarrow} \frac{2a}{a^2+(2\pi f)^2}$ for $a > 0$
- $\exp\{-at\}u(t) \overset{Fourier}{\longleftarrow} \frac{1}{a+j2\pi f}$ for $a > 0$
- $\sum_{n=-\infty}^{\infty} \delta(t - nT) \overset{Fourier}{\longleftarrow} \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right)$

Turn Formulas into Concrete Concepts

- $g(at) \overset{Fourier}{\longleftarrow} \frac{1}{|a|} G\left(\frac{f}{a}\right)$
- $g(t - t_0) \overset{Fourier}{\longleftarrow} G(f)e^{-j2\pi f t_0}$
- $g(t)e^{j2\pi f_0 t} \overset{Fourier}{\longleftarrow} G(f - f_0)$
- If $g(t) \overset{Fourier}{\longrightarrow} G(f)$, then $g^*(t) \overset{Fourier}{\longrightarrow} G^*(-f)$.

Convolution

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} Y(f)\exp(j2\pi ft)df \\
&= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} H(s)X(f-s)ds\right]\exp(j2\pi ft)df \\
&= \int_{-\infty}^{\infty} H(s)\left[\int_{-\infty}^{\infty} X(f-s)\exp(j2\pi ft)df\right]ds \\
&= \int_{-\infty}^{\infty} H(s)\left[\int_{-\infty}^{\infty} X(u)\exp(j2\pi(u+s)t)df\right]ds, u = f-s \\
&= \left[\int_{-\infty}^{\infty} H(s)\exp(j2\pi st)ds\right]\left[\int_{-\infty}^{\infty} X(u)\exp(j2\pi ut)df\right] \\
&= h(t)x(t)
\end{aligned}
$$