

2012 Fall: The Final Exam

1. (Chapter 11)

(a) (6%) There are two kinds of convolutions in literature:

$$\begin{cases} \text{linear convolution} & x_k \star I_k = \sum_{n=0}^{P-1} x_n \cdot I_{k-n} \\ N\text{-point circular convolution} & x_k \otimes I_k = \sum_{n=0}^{P-1} x_{n \bmod N} \cdot I_{(k-n) \bmod N} = \sum_{n=0}^{P-1} x_n \cdot I_{(k-n) \bmod N} \end{cases}$$

where $\{x_n\}$ is a discrete causal filter satisfying that $x_n \neq 0$ only at $0 \leq n \leq P-1$, and “mod” stands for modulo operation. Denote

$$y_k = x_k \star I_k \quad \text{and} \quad \tilde{y}_k = x_k \otimes I_k$$

for $k = 0, 1, \dots, N-1$. Is $y_k = \tilde{y}_k$ for every $k = 0, 1, \dots, N-1$? Construct a counterexample if the answer is negative.

Hint: You may wish to compare the two convolutions at small N and P such as $N = 2$ and $P = 2$.

(b) (6%) Follow (a) and suppose $N = 5$ and $P = 3$. For a block of data sequence of size N (i.e., I_0, I_1, \dots, I_4), the cycle prefix (CP) technique will prefix the sequence with $I_{-P}, I_{-P+1}, \dots, I_{-1}$ satisfying that

$$I_{-P} = I_{N-P}, \quad I_{-P+1} = I_{N-P+1}, \dots, I_{-1} = I_{N-1}. \quad (1)$$

Show that for unknown (or varying) $\{x_n\}_{n=1}^{P-1}$, the condition $y_k = \tilde{y}_k$ for every $k = 0, 1, \dots, N-1$ holds if, and only if, CP condition (1) holds.

Hint: For unknowns x_0, x_1, \dots, x_{P-1} , the linear equations below are solvable for non-zeros unknowns if, and only if, $a_1 = a_2 = \dots = a_{P-1} = 0$.

$$\begin{cases} x_1 \cdot a_1 + x_2 \cdot a_2 + x_3 \cdot a_3 + \dots + x_{P-1} \cdot a_{P-1} = 0 \\ x_2 \cdot a_1 + x_3 \cdot a_2 + \dots + x_{P-1} \cdot a_{P-2} = 0 \\ \vdots \\ x_{P-2} \cdot a_1 + x_{P-1} \cdot a_2 = 0 \\ x_{P-1} \cdot a_1 = 0 \end{cases}$$

Note: The statement in (b) holds actually for any N and P with $N > P$. Thus, CP is the one and only technique that can guarantee $y_k = \tilde{y}_k$ for every $k = 0, 1, \dots, N-1$. However, to lower your load, I only demand the proof for a specific case of $N = 5$ and $P = 3$.

(c) (6%) Under $N > P$ and based on the discrete Fourier transform (DFT) pair given below,

$$\begin{cases} \text{DFT} & X_k = \sum_{m=0}^{N-1} x_m e^{-i2\pi \frac{mk}{N}} \quad k = 0, 1, \dots, N-1 \\ \text{iDFT} & x_m = \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{i2\pi \frac{mk}{N}} \quad m = 0, 1, \dots, N-1 \end{cases} \quad (2)$$

prove that $\tilde{y}_k = x_k \otimes I_k$ for $k = 0, 1, \dots, N-1$ implies $\tilde{Y}_k = X_k \cdot \mathfrak{I}_k$ for $k = 0, 1, \dots, N-1$, where $Y_k = \text{DFT}\{y_k\}$, $X_k = \text{DFT}\{x_k\}$ and $\mathfrak{I}_k = \text{DFT}\{I_k\}$.

Hint: Perform

$$\begin{aligned}
 \tilde{Y}_k &= \sum_{m=0}^{N-1} \tilde{y}_m e^{-i2\pi \frac{mk}{N}} \\
 &= \sum_{m=0}^{N-1} \left(\sum_{n=0}^{P-1} x_n \cdot I_{(m-n) \bmod N} \right) e^{-i2\pi \frac{mk}{N}} \\
 &= \sum_{n=0}^{P-1} x_n \sum_{m=0}^{N-1} I_{(m-n) \bmod N} e^{-i2\pi \frac{mk}{N}}
 \end{aligned}$$

and remove the modulo index by performing

$$\sum_{m=0}^{N-1} I_{(m-n) \bmod N} e^{-i2\pi \frac{mk}{N}} = \sum_{m=0}^{n-1} I_{(m-n) \bmod N} e^{-i2\pi \frac{mk}{N}} + \sum_{m=n}^{N-1} I_{(m-n) \bmod N} e^{-i2\pi \frac{mk}{N}}.$$

- (d) (6%) Follow (c). Is $y_k = x_k \star I_k$ implying $Y_k = X_k \cdot \mathfrak{I}_k$? Construct a counterexample if the answer is negative.

Hint: Think of why we need to introduce CP! Also consider the case that $y_k \neq \tilde{y}_k$ in (a).

2. (Chapter 11)

- (a) (6%) An OFDM (baseband) signal can be formulated as

$$s_\ell(t) = \sum_{n=-\infty}^{\infty} \left(\sum_{k=0}^{Q-1} X_{k,n} e^{i2\pi \frac{k}{T} t} \right) g(t - nT)$$

where $\{X_{k,n}\}$ are random in nature. If $\{X_{k,n}\}$ are zero-mean i.i.d. both in k and n , then its autocorrelation can be computed as:

$$\begin{aligned}
 R_{s_\ell}(t + \tau, t) &= \mathbb{E} \left[\left(\sum_{n=-\infty}^{\infty} \sum_{k=0}^{Q-1} X_{k,n} g(t + \tau - nT) e^{i2\pi \frac{k}{T} (t+\tau)} \right) \right. \\
 &\quad \left. \left(\sum_{m=-\infty}^{\infty} \sum_{j=0}^{Q-1} X_{j,m}^* g^*(t - mT) e^{-i2\pi \frac{j}{T} t} \right) \right] \\
 &= \sigma^2 \sum_{k=0}^{Q-1} e^{i2\pi \frac{k}{T} \tau} \sum_{n=-\infty}^{\infty} g(t + \tau - nT) g^*(t - nT)
 \end{aligned}$$

where σ^2 is the variance of $X_{k,n}$. Now if $\{X_{k,n}\}$ remains i.i.d. both in k and n but are with a (complex-valued) mean μ (i.e., $\mathbb{E}[X_{k,n}] = \mu$) and variance σ^2 , what will its autocorrelation become?

- (b) (6%) Follow (a). Find the time-average autocorrelation function of $s_\ell(t)$ at $\mu = 0$, and prove that the time-average power spectrum density is

$$\bar{S}_{s_\ell}(f) = \frac{\sigma^2}{T} \sum_{k=0}^{Q-1} \left| G \left(f - \frac{k}{T} \right) \right|^2.$$

(c) (6%) Follow (a). If

$$x_{\ell,n} = \frac{1}{N} \sum_{k=0}^{Q-1} X_{k,n} e^{i2\pi \frac{\ell k}{N}} \quad \ell = 0, 1, \dots, N-1$$

and $N > Q$, prove that

$$s_{\ell}(t) = \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{N-1} x_{\ell,n} \frac{\sin(\pi t N/T)}{\sin(\pi(t/T - \ell/N))} e^{i\pi(t(N-1)/T + \ell/N)} g(t - nT)$$

Hint: Re-express $\{X_{k,n}\}$ in terms of $\{x_{\ell,n}\}$ using iDFT formula in (2).

(d) (6%) Continue from (c). What will be the value of $s_{\ell}(mT/N)$, where $0 \leq m \leq N-1$, if

$$g(t) = \begin{cases} 1, & 0 \leq t < T \\ 0, & \text{otherwise} \end{cases} ?$$

Hint: $\lim_{t \uparrow \frac{m}{N}T} \frac{\sin(\pi t N/T)}{\sin(\pi(t/T - \ell/N))} e^{i\pi(t(N-1)/T + \ell/N)} = 0$ for $\ell \neq m$.

3. (Chapter 13)

(a) (6%) Let $c(\tau; t) = \sum_{i=1}^3 \alpha_i \cdot \delta(\tau - \beta_i)$, where $\beta_1 = 0 < \beta_2 < \beta_3$ are constants, and $\{\alpha_i\}_{i=1}^3$ are independent non-negative random variables. Assume that $\beta_i f_c$ is an integer for all i , where f_c is the carrier frequency. Find its low-pass equivalent channel response $c_{\ell}(\tau; t)$.

Note: $c_{\ell}(\tau; t)$ should be a function of carrier frequency f_c .

(b) (6%) Follow (a). Is $c_{\ell}(\tau; t)$ WSS in t .

(c) (6%) Follow (a). Show that delay power spectrum $R_{c_{\ell}}(\tau; \Delta t = 0)$ of $c_{\ell}(\tau; t)$ is equal to

$$R_{c_{\ell}}(\tau; \Delta t = 0) = \sum_{j=1}^3 \left(\mu_j \left(\sum_{i=1}^3 \mu_i \right) + \sigma_j^2 \right) \delta(\tau - \beta_j)$$

provided that $\mathbb{E}[\alpha_i] = \mu_i$ and $\mathbb{E}[\alpha_i^2] = \sigma_i^2 + \mu_i^2$.

Hint: Here, we extend the definition of delay power spectrum to be

$$R_{c_{\ell}}(\tau; \Delta t) = \int_{-\infty}^{\infty} R_{c_{\ell}}(\bar{\tau}, \tau; \Delta t) d\bar{\tau}.$$

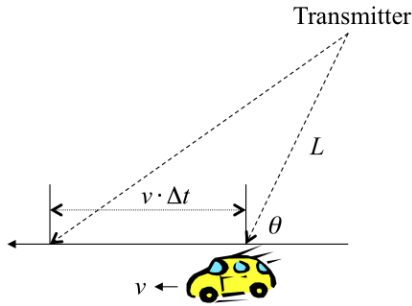
(d) (5%) What is the (maximum) delay spread of the channel in (c)?

(e) (5%) Which category does the channel in (c) belongs to, overspread or underspread?

Hint: What is the Doppler spread of a time-invariant channel?

4. (Chapter 13) (6%) Suppose that the transmitter sends a single tone of 5 GHz to a receiver inside the car as shown in the figure below. What will be the Doppler shift if the light speed is equal to 10^7 times the car speed?

Hint: Doppler shift is equal to $\lambda_m = \lim_{\Delta t \rightarrow 0} \frac{1}{2\pi} \frac{\Delta \phi}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{1}{2\pi} \frac{\Delta L/\lambda}{\Delta t}$, where λ is the wave length of the single tone.



5. (Chapter 13) Suppose the channel can be modeled as:

$$\mathbf{r}_\ell = \alpha e^{i\theta} \mathbf{s}_\ell + \mathbf{n}_\ell,$$

where \mathbf{n}_ℓ is a zero-mean Gaussian vector with marginal variance σ^2 , α is a non-negative real number, and $\theta \in [0, 2\pi)$. From Chapter 4, we learn that for any binary signaling (denoted by $\mathbf{s}_{1,\ell}$ and $\mathbf{s}_{2,\ell}$) transmitted over this channel, the error rate is given by

$$Q\left(\sqrt{\frac{d_{12}^2}{4\sigma^2}}\right), \quad (3)$$

where $d_{12} = \|\alpha e^{i\theta} \mathbf{s}_{1,\ell} - \alpha e^{i\theta} \mathbf{s}_{2,\ell}\|$.

- (6%) Denote $\gamma_{12} = \|\mathbf{s}_{1,\ell} - \mathbf{s}_{2,\ell}\|$. Let $\Pr[\alpha = 0] = 1 - p$ and $\Pr[\alpha = 1] = p$. Assume θ is uniformly distributed over $[0, 2\pi)$. Then, find the error rate under this fading channel.
- (6%) Follow (a). Now suppose θ can be perfectly estimated at the receiver; so it can be removed. One then adopts the maximal-ratio-combining diversity technique to improve the error rate as follows:

$$\mathbf{r}_\ell = \sum_{k=1}^L \alpha_k^2 \mathbf{s}_\ell + \sum_{k=1}^L \alpha_k \mathbf{n}_{k,\ell}, \quad (4)$$

where $\{\alpha_k\}$ are i.i.d. with the same distribution defined in (a), and $\{\mathbf{n}_{k,\ell}\}$ are i.i.d. Prove that the error rate under this fading channel with maximal-ratio-combining diversity technique is

$$P_e = \sum_{k=0}^L Q\left(\sqrt{\frac{k\gamma_{12}^2}{4\sigma^2}}\right) \binom{L}{k} p^k (1-p)^{L-k}$$

Hint: (4) can be written as $\mathbf{r}_\ell = \tilde{\alpha}^2 \mathbf{s}_\ell + \tilde{\mathbf{n}}_\ell$, where $\tilde{\alpha}^2 = \sum_{k=1}^L \alpha_k^2$ and $\tilde{\mathbf{n}}_\ell = \sum_{k=1}^L \alpha_k \mathbf{n}_{k,\ell}$. Hence, the error rate formula in (3) can be used by replacing σ^2 with the marginal variance of $\tilde{\mathbf{n}}_\ell$.

- (6%) Use the upper-bound $Q(x) \leq \frac{1}{2} e^{-x^2/2}$ to show that the diversity technique can make the error rate decrease exponentially with respect to L , i.e., P_e can be bounded above by Cq^L for some constant C and q with $0 < q < 1$.