2012 Fall: The Final Exam

1. (Chapter 11)

   (a) (6%) There are two kinds of convolutions in literature:

$$\begin{cases}\text{linear convolution} & x_k \star I_k = \sum_{n=0}^{P-1} x_n \cdot I_{k-n} \\ N\text{-point circular convolution} & x_k \circledast I_k = \sum_{n=0}^{P-1} x_{n \,\texttt{mod}\, N} \cdot I_{(k-n) \texttt{mod}\, N} = \sum_{n=0}^{P-1} x_n \cdot I_{(k-n)\texttt{mod}\, N}\end{cases}$$

   where $\{x_n\}$ is a discrete causal filter satisfying that $x_n \neq 0$ only at $0 \leq n \leq P-1$, and "`mod`" stands for modulo operation. Denote

$$y_k = x_k \star I_k \quad \text{and} \quad \tilde{y}_k = x_k \circledast I_k$$

   for $k = 0, 1, \ldots, N-1$. Is $y_k = \tilde{y}_k$ for every $k = 0, 1, \ldots, N-1$? Construct a counterexample if the answer is negative.

   Hint: You may wish to compare the two convolutions at small N and P such as $N = 2$ and $P = 2$.

   (b) (6%) Follow (a) and suppose $N = 5$ and $P = 3$. For a block of data sequence of size N (i.e., $I_0, I_1, \ldots, I_4$), the cycle prefix (CP) technique will prefix the sequence with $I_{-P}, I_{-P+1}, \ldots, I_{-1}$ satisfying that

$$I_{-P} = I_{N-P}, \quad I_{-P+1} = I_{N-P+1}, \ldots, I_{-1} = I_{N-1}. \tag{1}$$

   Show that for unknown (or varying) $\{x_n\}_{n=1}^{P-1}$, the condition $y_k = \tilde{y}_k$ for every $k = 0, 1, \ldots, N-1$ holds if, and only if, CP condition (1) holds.

   Hint: For unknowns $x_0, x_1, \ldots, x_{P-1}$, the linear equations below are solvable for non-zeros unknowns if, and only if, $a_1 = a_2 = \cdots = a_{P-1} = 0$.

$$\begin{cases} x_1 \cdot a_1 + x_2 \cdot a_2 + x_3 \cdot a_3 + \cdots + x_{P-1} \cdot a_{P-1} = 0 \\ x_2 \cdot a_1 + x_3 \cdot a_2 + \cdots + x_{P-1} \cdot a_{P-2} = 0 \\ \vdots \\ x_{P-2} \cdot a_1 + x_{P-1} \cdot a_2 = 0 \\ x_{P-1} \cdot a_1 = 0 \end{cases}$$

   Note: The statement in (b) holds actually for any N and P with $N > P$. Thus, CP is the one and only technique that can guarantee $y_k = \tilde{y}_k$ for every $k = 0, 1, \ldots, N-1$. However, to lower your load, I only demand the proof for a specific case of $N = 5$ and $P = 3$.

   (c) (6%) Under $N > P$ and based on the discrete Fourier transform (DFT) pair given below,

$$\begin{cases} \text{DFT} \quad X_k = \displaystyle\sum_{m=0}^{N-1} x_m e^{-\imath 2\pi \frac{mk}{N}} \quad k = 0, 1, \ldots, N-1 \\ \text{iDFT} \quad x_m = \dfrac{1}{N} \displaystyle\sum_{k=0}^{N-1} X_k e^{\imath 2\pi \frac{mk}{N}} \quad m = 0, 1, \ldots, N-1 \end{cases} \tag{2}$$

   prove that $\tilde{y}_k = x_k \circledast I_k$ for $k = 0, 1, \ldots, N-1$ implies $\tilde{Y}_k = X_k \cdot \mathfrak{I}_k$ for $k = 0, 1, \ldots, N-1$, where $Y_k = \text{DFT}\{y_k\}$, $X_k = \text{DFT}\{x_k\}$ and $\mathfrak{I}_k = \text{DFT}\{I_k\}$.

Hint: Perform

$$\begin{aligned}\tilde{Y}_k &= \sum_{m=0}^{N-1} \tilde{y}_m e^{-\imath 2\pi \frac{mk}{N}} \\ &= \sum_{m=0}^{N-1} \left(\sum_{n=0}^{P-1} x_n \cdot I_{(m-n) \texttt{mod}\, N} \right) e^{-\imath 2\pi \frac{mk}{N}} \\ &= \sum_{n=0}^{P-1} x_n \sum_{m=0}^{N-1} I_{(m-n) \texttt{mod}\, N} e^{-\imath 2\pi \frac{mk}{N}}\end{aligned}$$

and remove the modulo index by performing

$$\sum_{m=0}^{N-1} I_{(m-n) \texttt{mod}\, N} e^{-\imath 2\pi \frac{mk}{N}} = \sum_{m=0}^{n-1} I_{(m-n) \texttt{mod}\, N} e^{-\imath 2\pi \frac{mk}{N}} + \sum_{m=n}^{N-1} I_{(m-n) \texttt{mod}\, N} e^{-\imath 2\pi \frac{mk}{N}}.$$

(d) (6%) Follow (c). Is $y_k = x_k \star I_k$ implying $Y_k = X_k \cdot \mathfrak{I}_k$? Construct a counterexample if the answer is negative.

Hint: Think of why we need to introduce CP! Also consider the case that $y_k \neq \tilde{y}_k$ in (a).

Solutions.

(a) Let $N = 2$ and $P = 2$. Then, we have

$$\begin{cases} \text{linear convolution} & y_k = x_0 \cdot I_k + x_1 \cdot I_{k-1} \\ \text{circular convolution} & \tilde{y}_k = x_0 \cdot I_{k \,\texttt{mod}\, 2} + x_1 \cdot I_{(k-1) \texttt{mod}\, 2} \end{cases}$$

Hence,

$$\begin{cases} y_0 = x_0 \cdot I_0 + x_1 \cdot I_{-1} & \text{and} \quad y_1 = x_0 \cdot I_1 + x_1 \cdot I_0 \\ \tilde{y}_0 = x_0 \cdot I_0 + x_1 \cdot I_1 & \text{and} \quad \tilde{y}_1 = x_0 \cdot I_1 + x_1 \cdot I_0 \end{cases}$$

which implies that if $I_0 \neq I_{-1}$, then $y_0 \neq \tilde{y}_0$.

(b)

$$\begin{cases} \text{linear convolution} & y_k = x_0 \cdot I_k + x_1 \cdot I_{k-1} + x_2 \cdot I_{k-2} \\ \text{circular convolution} & \tilde{y}_k = x_0 \cdot I_{k \,\texttt{mod}\, 5} + x_1 \cdot I_{(k-1) \texttt{mod}\, 5} + x_2 \cdot I_{(k-2) \texttt{mod}\, 5} \end{cases}$$

Hence, for $2 \leq k \leq 4$, $y_k = \tilde{y}_k$ is always valid. So we only need to consider the cases for $0 \leq k \leq 1$, which gives

$$\begin{cases} y_0 = x_0 \cdot I_0 + x_1 \cdot I_{-1} + x_2 \cdot I_{-2} = x_0 \cdot I_0 + x_1 \cdot I_4 + x_2 \cdot I_3 = \tilde{y}_0 \\ y_1 = x_0 \cdot I_1 + x_1 \cdot I_0 + x_2 \cdot I_{-1} = x_0 \cdot I_1 + x_1 \cdot I_0 + x_2 \cdot I_4 = \tilde{y}_1 \end{cases}$$

We then obtain

$$\begin{cases} x_1 \cdot (I_{-1} - I_4) + x_2 \cdot (I_{-2} - I_3) = 0 \\ x_2 \cdot (I_{-1} - I_4) = 0 \end{cases}$$

By treating $\{x_n\}$ as unknowns, these equations are solvable for non-zero $\{x_n\}$ if, and only if,

$$I_{-2} = I_3 \quad \text{and} \quad I_{-1} = I_4.$$

(c)

$$\begin{aligned}
\tilde{Y}_k &= \sum_{m=0}^{N-1} \tilde{y}_m e^{-\imath 2\pi \frac{mk}{N}} \\
&= \sum_{m=0}^{N-1} \left(\sum_{n=0}^{P-1} x_n \cdot I_{(m-n) \texttt{mod}\, N} \right) e^{-\imath 2\pi \frac{mk}{N}} \\
&= \sum_{n=0}^{P-1} x_n \sum_{m=0}^{N-1} I_{(m-n)\texttt{mod}\, N} e^{-\imath 2\pi \frac{mk}{N}} \\
&= \sum_{n=0}^{P-1} x_n \left(\sum_{m=0}^{n-1} I_{m-n+N} e^{-\imath 2\pi \frac{mk}{N}} + \sum_{m=n}^{N-1} I_{m-n} e^{-\imath 2\pi \frac{mk}{N}} \right) \\
&= \sum_{n=0}^{P-1} x_n \left(\sum_{\ell'=N-n}^{N-1} I_{\ell'} e^{-\imath 2\pi \frac{(\ell'+n-N)k}{N}} + \sum_{\ell=0}^{N-1-n} I_{\ell} e^{-\imath 2\pi \frac{(\ell+n)k}{N}} \right) \\
& \quad \text{(Let } \ell' = m+N-n \text{ and } \ell = m-n.) \\
&= \left(\sum_{n=0}^{P-1} x_n e^{-\imath 2\pi \frac{nk}{N}} \right) \left(\sum_{\ell=0}^{N-1} I_\ell e^{-\imath 2\pi \frac{\ell k}{N}} \right) \\
&= \left(\sum_{n=0}^{N-1} x_n e^{-\imath 2\pi \frac{nk}{N}} \right) \left(\sum_{\ell=0}^{N-1} I_\ell e^{-\imath 2\pi \frac{\ell k}{N}} \right) \\
&= X_k \cdot I_k
\end{aligned}$$

(d) Since DFT and iDFT are duality operations, subproblem (c) indicates that $\tilde{y}_k = x_k \circledast I_k$ for $k = 0, 1, \ldots, N-1$ if, and only if, $\tilde{Y}_k = X_k \cdot \mathfrak{I}_k$ for $k = 0, 1, \ldots, N-1$. Together with subproblem (a), one can infer that $\tilde{y}_k = y_k$ for $k = 0, 1, \ldots, N-1$ may not be true; so the answer to this question should be negative.

As for the counterexample, take $N = 2$ and $P = 2$. Also take $x_0 = x_1 = I_0 = I_1 = 1$ but $I_{-1} = -1$. Then

$$\begin{cases} y_0 = x_0 \cdot I_0 + x_1 \cdot I_{-1} = 0 \\ y_1 = x_0 \cdot I_1 + x_1 \cdot I_0 = 2 \\ Y_0 = \displaystyle\sum_{m=0}^{1} y_m = 2 \\ Y_1 = \displaystyle\sum_{m=0}^{1} y_m e^{-\imath 2\pi \frac{m}{2}} = 2e^{-\imath\pi} = -2 \end{cases}, \quad \begin{cases} I_0 = 1 \\ I_1 = 1 \\ \mathfrak{I}_0 = \displaystyle\sum_{m=0}^{1} I_m = 2 \\ \mathfrak{I}_1 = \displaystyle\sum_{m=0}^{1} I_m e^{-\imath 2\pi \frac{m}{2}} = 1 + e^{-\imath\pi} = 0 \end{cases}$$

and

$$\begin{cases} x_0 = 1 \\ x_1 = 1 \\ X_0 = \displaystyle\sum_{m=0}^{1} x_m = 2 \\ X_1 = \displaystyle\sum_{m=0}^{1} x_m e^{-\imath 2\pi \frac{m}{2}} = 1 + e^{-\imath\pi} = 0 \end{cases}$$

So

$$2 = Y_0 \neq X_0 \cdot \mathfrak{I}_0 = 4 \quad \text{and} \quad -2 = Y_1 \neq X_1 \cdot \mathfrak{I}_1 = 0.$$

(Note that if we change I_{-1} to 1. Then, Y_0 and Y_1 become 4 and 0, respectively; so the statement in (c) can be applied!)

2. (Chapter 11)

(a) (6%) An OFDM (baseband) signal can be formulated as

$$s_\ell(t) = \sum_{n=-\infty}^{\infty} \left(\sum_{k=0}^{Q-1} X_{k,n} e^{\imath 2\pi \frac{k}{T} t} \right) g(t - nT)$$

where $\{X_{k,n}\}$ are random in nature. If $\{X_{k,n}\}$ are zero-mean i.i.d. both in k and n, then its autocorrelation can be computed as:

$$\begin{aligned} R_{s_\ell}(t+\tau, t) &= \mathbb{E}\left[\left(\sum_{n=-\infty}^{\infty} \sum_{k=0}^{Q-1} X_{k,n} g(t+\tau-nT) e^{\imath 2\pi \frac{k}{T}(t+\tau)} \right) \right. \\ &\qquad \left. \left(\sum_{m=-\infty}^{\infty} \sum_{j=0}^{Q-1} X_{j,m}^* g^*(t-mT) e^{-\imath 2\pi \frac{j}{T} t} \right) \right] \\ &= \sigma^2 \sum_{k=0}^{Q-1} e^{\imath 2\pi \frac{k}{T}\tau} \sum_{n=-\infty}^{\infty} g(t+\tau-nT) g^*(t-nT) \end{aligned}$$

where σ^2 is the variance of $X_{k,n}$. Now if $\{X_{k,n}\}$ remains i.i.d. both in k and$n$ but are with a (complex-valued) mean μ (i.e., $\mathbb{E}[X_{k,n}] = \mu$) and variance σ^2, what will its autocorrelation become?

(b) (6%) Follow (a). Find the time-average autocorrelation function of $s_\ell(t)$ at $\mu = 0$, and prove that the time-average power spectrum density is

$$\bar{S}_{s_\ell}(f) = \frac{\sigma^2}{T} \sum_{k=0}^{Q-1} \left| G\left(f - \frac{k}{T} \right) \right|^2.$$

(c) (6%) Follow (a). If

$$x_{\ell,n} = \frac{1}{N} \sum_{k=0}^{Q-1} X_{k,n} e^{\imath 2\pi \frac{\ell k}{N}} \quad \ell = 0, 1, \ldots, N-1$$

and $N > Q$, prove that

$$s_\ell(t) = \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{N-1} x_{\ell,n} \frac{\sin(\pi t N/T)}{\sin(\pi(t/T - \ell/N))} e^{\imath\pi(t(N-1)/T + \ell/N)} g(t - nT)$$

Hint: Re-express $\{X_{k,n}\}$ in terms of $\{x_{\ell,n}\}$ using iDFT formula in (2).

(d) (6%) Continue from (d). What will be the value of $s_\ell(mT/N)$, where $0 \le m \le N-1$, if $g(t) = \begin{cases} 1, & 0 \le t < T \\ 0, & \text{otherwise} \end{cases}$?

Hint: $\lim_{t \uparrow \frac{m}{N}T} \frac{\sin(\pi t N/T)}{\sin(\pi(t/T - \ell/N))} e^{\imath\pi(t(N-1)/T+\ell/N)} = 0$ for $\ell \neq m$.

Solutions.

(a)

$$
\begin{aligned}
R_{s_\ell}(t+\tau,t) &= \mathbb{E}\Bigg[\left(\sum_{n=-\infty}^{\infty}\sum_{k=0}^{Q-1} X_{k,n}\, g(t+\tau-nT)e^{\imath 2\pi \frac{k}{T}(t+\tau)}\right) \\
&\qquad \left(\sum_{m=-\infty}^{\infty}\sum_{j=0}^{Q-1} X_{j,m}^{*}\, g^{*}(t-mT)e^{-\imath 2\pi \frac{j}{T}t}\right)\Bigg] \\
&= \sum_{n=-\infty}^{\infty}\sum_{k=0}^{Q-1}\sum_{m=-\infty}^{\infty}\sum_{j=0}^{Q-1} \mathbb{E}\left[X_{k,n}X_{j,m}^{*}\right] g(t+\tau-nT)e^{\imath 2\pi \frac{k}{T}(t+\tau)} g^{*}(t-mT)e^{-\imath 2\pi \frac{j}{T}t} \\
&= |\mu|^2 \sum_{k=0}^{Q-1} e^{\imath 2\pi \frac{k}{T}\tau} \sum_{j=0}^{Q-1} e^{\imath 2\pi \frac{(k-j)}{T}t} \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} g(t+\tau-nT)g^{*}(t-mT) \\
&\quad + \sigma^2 \sum_{k=0}^{Q-1} e^{\imath 2\pi \frac{k}{T}\tau} \sum_{n=-\infty}^{\infty} g(t+\tau-nT)g^{*}(t-nT)
\end{aligned}
$$

(b)

$$
\begin{aligned}
\bar{R}_{s_\ell}(\tau) &= \frac{1}{T}\int_0^T R_{s_\ell}(t+\tau,t)\,dt \\
&= \frac{\sigma^2}{T}\sum_{k=0}^{Q-1} e^{\imath 2\pi \frac{k}{T}\tau} \sum_{n=-\infty}^{\infty} \int_0^T g(t+\tau-nT)g^{*}(t-nT)\,dt \\
&= \frac{\sigma^2}{T}\sum_{k=0}^{Q-1} e^{\imath 2\pi \frac{k}{T}\tau} \sum_{n=-\infty}^{\infty} \int_{-nT}^{-(n-1)T} g(u+\tau)g^{*}(u)\,du \\
&= \frac{\sigma^2}{T}\sum_{k=0}^{Q-1} e^{\imath 2\pi \frac{k}{T}\tau} \int_{-\infty}^{\infty} g(t+\tau)g^{*}(t)\,dt
\end{aligned}
$$

$$
\begin{aligned}
\bar{S}_{s_\ell}(f) &= \int_{-\infty}^{\infty} \bar{R}_{s_\ell}(\tau) e^{-\imath 2\pi f \tau}\, d\tau \\
&= \frac{\sigma^2}{T} \sum_{k=0}^{Q-1} \int_{-\infty}^{\infty} g^*(t) \left(\int_{-\infty}^{\infty} g(t+\tau) e^{-\imath 2\pi \left(f - \frac{k}{T}\right)\tau}\, d\tau \right) dt \\
&= \frac{\sigma^2}{T} \sum_{k=0}^{Q-1} \int_{-\infty}^{\infty} g^*(t) \left(\int_{-\infty}^{\infty} g(u) e^{-\imath 2\pi \left(f - \frac{k}{T}\right)u}\, du \right) e^{\imath 2\pi \left(f - \frac{k}{T}\right)t} dt \\
&= \frac{\sigma^2}{T} \sum_{k=0}^{Q-1} G\left(f - \frac{k}{T}\right) \left(\int_{-\infty}^{\infty} g(t) e^{-\imath 2\pi \left(f - \frac{k}{T}\right)t}\, dt \right)^* \\
&= \frac{\sigma^2}{T} \sum_{k=0}^{Q-1} \left| G\left(f - \frac{k}{T}\right) \right|^2
\end{aligned}
$$

(c) Since

$$
X_{k,n} = \sum_{\ell=0}^{N-1} x_{\ell,n} e^{-\imath 2\pi \frac{\ell k}{N}} \quad k = 0, 1, \ldots, Q-1
$$

we obtain

$$
\begin{aligned}
s_\ell(t) &= \sum_{n=-\infty}^{\infty} \left(\sum_{k=0}^{Q-1} X_{k,n} e^{\imath 2\pi \frac{k}{T} t} \right) g(t - nT) \\
&= \sum_{n=-\infty}^{\infty} \left(\sum_{k=0}^{N-1} \left[\sum_{\ell=0}^{N-1} x_{\ell,n} e^{-\imath 2\pi \frac{\ell k}{N}} \right] e^{\imath 2\pi \frac{k}{T} t} \right) g(t - nT) \\
&= \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{N-1} x_{\ell,n} \left(\sum_{k=0}^{N-1} e^{-\imath 2\pi \left(\frac{\ell}{N} - \frac{t}{T}\right) k} \right) g(t - nT) \\
&= \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{N-1} x_{\ell,n} \frac{1 - e^{-\imath 2\pi (\ell/N - t/T) N}}{1 - e^{-\imath 2\pi (\ell/N - t/T)}} g(t - nT) \\
&= \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{N-1} x_{\ell,n} \frac{1 - e^{\imath 2\pi t N/T}}{1 - e^{-\imath 2\pi (\ell/N - t/T)}} g(t - nT) \\
&= \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{N-1} x_{\ell,n} \left(\frac{e^{-\imath \pi t N/T} - e^{\imath \pi t N/T}}{e^{\imath \pi (\ell/N - t/T)} - e^{-\imath \pi (\ell/N - t/T)}} \right) \left(\frac{e^{\imath \pi t N/T}}{e^{-\imath \pi (\ell/N - t/T)}} \right) g(t - nT) \\
&= \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{N-1} x_{\ell,n} \left(\frac{-\imath 2 \sin(\pi t N/T)}{\imath 2 \sin(\pi(\ell/N - t/T))} \right) e^{\imath \pi t N/T + \imath \pi (\ell/N - t/T)} g(t - nT) \\
&= \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{N-1} x_{\ell,n} \left(- \frac{\sin(\pi t N/T)}{\sin(\pi(\ell/N - t/T))} \right) e^{\imath \pi t N/T + \imath \pi \ell/N - \imath \pi t/T} g(t - nT) \\
&= \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{N-1} x_{\ell,n} \frac{\sin(\pi t N/T)}{\sin(\pi(t/T - \ell/N))} e^{\imath \pi (t(N-1)/T + \ell/N)} g(t - nT)
\end{aligned}
$$

(d) For $0 \leq m \leq N-1$,

$$
\begin{aligned}
\lim_{t \uparrow \frac{m}{N}T} s_\ell(t) &= \lim_{t \uparrow \frac{m}{N}T} \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{N-1} x_{\ell,n} \frac{\sin(\pi t N/T)}{\sin(\pi(t/T - \ell/N))} e^{\imath\pi(t(N-1)/T + \ell/N)} g(t - nT) \\
&= \lim_{t \uparrow \frac{m}{N}T} \sum_{\ell=0}^{N-1} x_{\ell,n} \frac{\sin(\pi t N/T)}{\sin(\pi(t/T - \ell/N))} e^{\imath\pi(t(N-1)/T + \ell/N)} \\
&= x_{m,n} \lim_{t \uparrow \frac{m}{N}T} \frac{\sin(\pi t N/T)}{\sin(\pi(t/T - m/N))} e^{\imath\pi(t(N-1)/T + m/N)} \\
&= x_{m,n} e^{\imath\pi m} \lim_{t \uparrow \frac{m}{N}T} \frac{\sin(\pi t N/T)}{\sin(\pi(t/T - m/N))} \\
&= x_{m,n} e^{\imath\pi m} \lim_{t \uparrow \frac{m}{N}T} \frac{(\pi N/T)\cos(\pi t N/T)}{(\pi/T)\cos(\pi(t/T - m/N))} \\
&= x_{m,n} e^{\imath\pi m} N \cos(m\pi) \\
&= N x_{m,n}
\end{aligned}
$$

3. (Chapter 13)

(a) (6%) Let $c(\tau;t) = \sum_{i=1}^{3} \alpha_i \cdot \delta(\tau - \beta_i)$, where $\beta_1 = 0 < \beta_2 < \beta_3$ are constants, and $\{\alpha_i\}_{i=1}^{3}$ are independent non-negative random variables. Assume that $\beta_i f_c$ is an integer for all i, where f_c is the carrier frequency. Find its low-pass equivalent channel response $c_\ell(\tau;t)$.

Note: $c_\ell(\tau;t)$ should be a function of carrier frequency f_c.

(b) (6%) Follow (a). Is $c_\ell(\tau;t)$ WSS in t.

(c) (6%) Follow (a). Show that delay power spectrum $R_{c_\ell}(\tau;\Delta t = 0)$ of $c_\ell(\tau;t)$ is equal to

$$
R_{c_\ell}(\tau;\Delta t = 0) = \sum_{j=1}^{3} \left(\mu_j \left(\sum_{i=1}^{3} \mu_i \right) + \sigma_j^2 \right) \delta(\tau - \beta_j)
$$

provided that $\mathbb{E}[\alpha_i] = \mu_i$ and $\mathbb{E}[\alpha_i^2] = \sigma_i^2 + \mu_i^2$.

Hint: Here, we extend the definition of delay power spectrum to be

$$
R_{c_\ell}(\tau;\Delta t) = \int_{-\infty}^{\infty} R_{c_\ell}(\bar{\tau}, \tau; \Delta t) d\bar{\tau}.
$$

(d) (5%) What is the (maximum) delay spread of the channel in (c)?

(e) (5%) Which category does the channel in (c) belongs to, overspread or underspread?

Hint: What is the Doppler spread of a time-invariant channel?

Solutions.

(a) From Slide 13-5,

$$\begin{aligned} c_\ell(\tau;t) &= c(\tau;t)e^{-\imath 2\pi f_c \tau} \\ &= \sum_{i=1}^{3} \alpha_i \cdot \delta(\tau-\beta_i)e^{-\imath 2\pi f_c \tau} \\ &= \sum_{i=1}^{3} \alpha_i \cdot \delta(\tau-\beta_i)e^{-\imath 2\pi f_c \beta_i} \\ &= \sum_{i=1}^{3} \alpha_i \cdot \delta(\tau-\beta_i) \end{aligned}$$

(b) The answer is YES. In fact, $c_\ell(\tau;t)=c_\ell(\tau)$ is time-invariant and not a function of t. So its mean and autocorrelation function are of course not a function of t; hence, they certainly only depend on the time difference.

(c)

$$\begin{aligned} R_{c_\ell}(\bar{\tau},\tau;\Delta t) &= \mathbb{E}\left[\sum_{i=1}^{3}\alpha_i\cdot\delta(\bar{\tau}-\beta_i)\sum_{j=1}^{3}\alpha_j\cdot\delta(\tau-\beta_j)\right] \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3}\mathbb{E}\left[\alpha_i\alpha_j\right]\cdot\delta(\bar{\tau}-\beta_i)\delta(\tau-\beta_j) \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3}\mu_i\mu_j\delta(\bar{\tau}-\beta_i)\delta(\tau-\beta_j)+\sum_{j=1}^{3}\sigma_j^2\delta(\bar{\tau}-\beta_j)\delta(\tau-\beta_j) \end{aligned}$$

Hence,

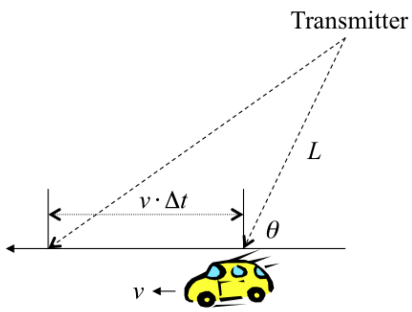
$$\begin{aligned} R_{c_\ell}(\tau;\Delta t=0) &= \int_{-\infty}^{\infty} R_{c_\ell}(\bar{\tau},\tau;\Delta t=0)\,d\bar{\tau} \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3}\mu_i\mu_j\delta(\tau-\beta_j)+\sum_{j=1}^{3}\sigma_j^2\delta(\tau-\beta_j) \\ &= \sum_{j=1}^{3}\left(\mu_j\sum_{i=1}^{3}\mu_i+\sigma_j^2\right)\delta(\tau-\beta_j) \end{aligned}$$

(d) $T_m=\beta_3$.

(e) The Doppler spread of this channel is $B_d=0$; hence, $B_dT_m=0$. The channel is an underspread channel.

4. (Chapter 13) (6%) Suppose that the transmitter sends a single tone of 5 GHz to a receiver inside the car as shown in the figure below. What will be the Doppler shift if the light speed is equal to 10^7 times the car speed?

Hint: Doppler shift is equal to $\lambda_m=\lim_{\Delta t\to 0}\frac{1}{2\pi}\frac{\Delta\phi}{\Delta t}=\lim_{\Delta t\to 0}\frac{1}{2\pi}\frac{\Delta L/\lambda}{\Delta t}$, where λ is the wave length of the single tone.



Solution. From Slides 13-35 and 13-36, the answer is $500\cos(\theta)$ Hz.

5. (Chapter 13) Suppose the channel can be molded as:

$$\boldsymbol{r}_\ell = \alpha e^{\imath\theta}\boldsymbol{s}_\ell + \boldsymbol{n}_\ell,$$

where $\boldsymbol{n}_\ell$ is a zero-mean Gaussian vector with marginal variance σ^2, α is a non-negative real number, and $\theta \in [0, 2\pi)$. From Chapter 4, we learn that for any binary signaling (denoted by $\boldsymbol{s}_{1,\ell}$ and $\boldsymbol{s}_{2,\ell}$) transmitted over this channel, the error rate is given by

$$Q\left(\sqrt{\frac{d_{12}^2}{4\sigma^2}}\right), \tag{3}$$

where $d_{12} = \|\alpha e^{\imath\theta}\boldsymbol{s}_{1,\ell} - \alpha e^{\imath\theta}\boldsymbol{s}_{2,\ell}\|$.

(a) (6%) Denote $\gamma_{12} = \|\boldsymbol{s}_{1,\ell} - \boldsymbol{s}_{2,\ell}\|$. Let $\Pr[\alpha = 0] = 1 - p$ and $\Pr[\alpha = 1] = p$. Assume θ is uniformly distributed over $[0, 2\pi)$. Then, find the error rate under this fading channel.

(b) (6%) Follow (a). Now suppose θ can be perfectly estimated at the receiver; so it can be removed. One then adopts the maximal-ratio-combining diversity technique to improve the error rate as follows:

$$\boldsymbol{r}_\ell = \sum_{k=1}^{L} \alpha_k^2 \boldsymbol{s}_\ell + \sum_{k=1}^{L} \alpha_k \boldsymbol{n}_{k,\ell}, \tag{4}$$

where $\{\alpha_k\}$ are i.i.d. with the same distribution defined in (a), and $\{\boldsymbol{n}_{k,\ell}\}$ are i.i.d. Prove that the error rate under this fading channel with maximal-ratio-combining diversity technique is

$$P_e = \sum_{k=0}^{L} Q\left(\sqrt{\frac{k\gamma_{12}^2}{4\sigma^2}}\right)\binom{L}{k} p^k (1-p)^{L-k}$$

Hint: (4) can be written as $\boldsymbol{r}_\ell = \tilde{\alpha}^2\boldsymbol{s}_\ell + \tilde{\boldsymbol{n}}_\ell$, where $\tilde{\alpha}^2 = \sum_{k=1}^{L}\alpha_k^2$ and $\tilde{\boldsymbol{n}}_\ell = \sum_{k=1}^{L}\alpha_k\boldsymbol{n}_{k,\ell}$. Hence, the error rate formula in (3) can be used by replacing σ^2 with the marginal variance of $\tilde{\boldsymbol{n}}_\ell$.

(c) (6%) Use the upper-bound $Q(x) \le \frac{1}{2}e^{-x^2/2}$ to show that the diversity technique can make the error rate decrease exponentially with respect to L, i.e., P_e can be bounded above by Cq^L for some constant C and q with $0 < q < 1$.

Solutions.

(a)

$$d_{12} = \|\alpha e^{\imath\theta}\boldsymbol{s}_{1,\ell} - \alpha e^{\imath\theta}\boldsymbol{s}_{2,\ell}\| = |\alpha|\gamma_{12}.$$

Hence, the error rate under this fading channel is

$$\begin{aligned} P_e &= \sum_{\alpha} \Pr\{\text{error}|\alpha\} \Pr\{\alpha\} \\ &= \sum_{\alpha} Q\left(\sqrt{\frac{\alpha^2\gamma_{12}^2}{4\sigma^2}}\right) \Pr\{\alpha\} \\ &= (1-p)Q(0) + pQ\left(\sqrt{\frac{\gamma_{12}^2}{4\sigma^2}}\right) \\ &= \frac{1}{2}(1-p) + p\cdot Q\left(\sqrt{\frac{\gamma_{12}^2}{4\sigma^2}}\right). \end{aligned}$$

(b) For given $\{\alpha_k\}$, the marginal variance of $\tilde{\boldsymbol{n}}_\ell = \sum_{k=1}^{L}\alpha_k^2\sigma^2 = \tilde{\alpha}^2\sigma^2$; hence, the error rate under given $\{\alpha_k\}$ is equal to $Q\left(\sqrt{\frac{d_{12}^2}{4\tilde{\alpha}^2\sigma^2}}\right)$, where with maximal-ratio-combining diversity technique, $d_{12} = \tilde{\alpha}^2\gamma_{12}$. In other words, the error rate under given $\{\alpha_k\}$ is equal to

$$Q\left(\sqrt{\frac{\tilde{\alpha}^4\gamma_{12}^2}{4\tilde{\alpha}^2\sigma^2}}\right) = Q\left(\sqrt{\frac{\tilde{\alpha}^2\gamma_{12}^2}{4\sigma^2}}\right).$$

Now $\Pr[\tilde{\alpha}^2 = k] = \binom{L}{k}p^k(1-p)^{L-k}$. Hence, the error rate of this fading channel with maximal-ratio-combining diversity technique is

$$\begin{aligned} P_e &= \sum_{\tilde{\alpha}^2=0}^{L} \Pr\{\text{error}|\tilde{\alpha}^2\} \Pr\{\tilde{\alpha}^2\} \\ &= \sum_{k=0}^{L} Q\left(\sqrt{\frac{k\gamma_{12}^2}{4\sigma^2}}\right)\binom{L}{k}p^k(1-p)^{L-k} \end{aligned}$$

(c)

$$\begin{aligned} P_e &\le \sum_{k=0}^{L} \frac{1}{2} e^{-k\frac{\gamma_{12}^2}{8\sigma^2}} \binom{L}{k} p^k(1-p)^{L-k} \\ &= \frac{1}{2}\sum_{k=0}^{L}\binom{L}{k}\left(e^{-\frac{\gamma_{12}^2}{8\sigma^2}}p\right)^k(1-p)^{L-k} \\ &= \frac{1}{2}\left(e^{-\frac{\gamma_{12}^2}{8\sigma^2}}p + 1 - p\right)^L \end{aligned}$$