2012 Fall: The First Midterm of Digital Communications

1. (Chapter 2)

   (a) (5%) Is the system with input $x(t)$, output $y(t)$ and input-output relation

   $$y(t) = \mathbf{Re}\left\{x(t)e^{\imath 2\pi f_0 t}\right\}$$

   a linear system? If the answer is positive, prove it. If negative, give a counterexample.
   Hint: Superposition principle.

   (b) (5%) Is the system in (a) time-invariant? If the answer is positive, prove it. If negative, give a counterexample.
   Hint: For a time-invariant system, input $x(t-\tau)$ should induce output $y(t-\tau)$ if input $x(t)$ induces output $y(t)$.

   (c) (5%) Give a counterexample (i.e., give an example of f_0 and $X_\ell(f)$) that fails the linear-time-invariant (LTI) system below:
   Input $x(t) = \mathbf{Re}\left\{x_\ell(t)e^{\imath 2\pi f_0 t}\right\}$
   Output $y(t) = \mathbf{Im}\left\{x_\ell(t)e^{\imath 2\pi f_0 t}\right\}$
   Transfer function $H(f) = -\imath \cdot \mathrm{sgn}(f)$
   Hint: Consider the relation between f_0 and the bandwidth W of $x_\ell(t)$.

   (d) (5%) Prove that the power spectrum density of a WSS process $\boldsymbol{x}(t)$ is always real-valued.
   Hint: The autocorrelation function $R_{\boldsymbol{x}}(\tau)$ satisfies $R_{\boldsymbol{x}}(-\tau) = R^*_{\boldsymbol{x}}(\tau)$.

   (e) (5%) Show that $x(t) = \mathbf{Re}\left\{x_\ell(t)e^{\imath 2\pi f_0 t}\right\}$ implies

   $$X(f) = \frac{1}{2}\left[X_\ell(f-f_0) + X^*_\ell(-f-f_0)\right].$$

2. (Chapter 3)

   (a) (5%) Suppose $\boldsymbol{s}(t)$ is a cyclostationary random process with period T. Let random vector $\vec{\boldsymbol{x}}(t)$ be defined as:

   $$\vec{\boldsymbol{x}}(t) = \begin{bmatrix} \boldsymbol{s}(t) \\ \boldsymbol{s}(t-\tau) \end{bmatrix},$$

   where τ is a constant. Is $\vec{\boldsymbol{x}}(t)$ also cyclostationary? If your answer is positive, prove it. If negative, show a counterexample.

   (b) (5%) If the autocorrelation function of $\boldsymbol{s}(t)$ is

   $$R_{\boldsymbol{s}}(t_1, t_2) = \sum_{m=-\infty}^{\infty} g(t_1 - mT)g^*(t_2 - mT),$$

   where $g(t)$ is a given continuous waveform. Prove that the time-average autocorrelation function of $\boldsymbol{s}(t)$ is given by:

   $$\overline{R}_{\boldsymbol{s}}(\tau) = \frac{1}{T}\int_{-\infty}^{\infty} g(u+\tau)g^*(u)du.$$

(c) (5%) Following (b), further prove that the time-average power spectrum density of $\boldsymbol{s}(t)$ is equal to:

$$\overline{S}_{\boldsymbol{s}}(f) = \frac{1}{T}|G(f)|^2.$$

(d) (6%) Below is the passband signal of OQPSK modulation.

$$s_{\text{OQPSK}}(t) = \sum_{n=-\infty}^{\infty} I_{2n} g(t-2nT)\cos(2\pi f_c t) - \sum_{n=-\infty}^{\infty} I_{2n+1} g(t-(2n+1)T)\sin(2\pi f_c t)$$

where $g(t) = \begin{cases} 1, & 0 \le t < 2T \\ 0, & \text{otherwise} \end{cases}$ and $I_n \in \{\pm 1\}$. Assume that T is a multiple of $1/f_c$. Define the inner product of two signals, $a(t)$ and $b(t)$, to be $\int_0^T a(t)b(t)dt$. Now by considering all possible signal waveforms that could appear during $[0, T)$, determine the dimension of the OQPSK modulation (3%). Also, give an orthonormal basis of these waveforms (3%) .
Hint: $s_{\text{OQPSK}}(t) = I_0 g(t)\cos(2\pi f_c t) - I_{-1} g(t+T)\sin(2\pi f_c t)$.

(e) (6%) Re-do subproblem (d) by re-defining $g(t)$ to be $g(t) = \begin{cases} \sin\left(\pi \frac{t}{2T}\right), & 0 \le t < 2T \\ 0, & \text{otherwise} \end{cases}$.
Hint: You may let $f_1 = f_c - \frac{1}{4T}$ and $f_2 = f_c + \frac{1}{4T}$ for notational convenience.

3. (Chapter 4) Consider two signals defined as

$$s_1(t) = \begin{cases} A, & 0 \le t < T \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad s_2(t) = \begin{cases} A, & 0 \le t < \tau \\ -A, & \tau \le t < T \\ 0, & \text{otherwise} \end{cases}$$

(a) (5%) Find signal space representations of $s_1(t)$ and $s_2(t)$ based on the basis

$$\phi_1(t) = \begin{cases} \frac{1}{\sqrt{\tau}}, & 0 \le t < \tau \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \phi_2(t) = \begin{cases} \frac{1}{\sqrt{T-\tau}}, & \tau \le t < T \\ 0, & \text{otherwise} \end{cases}$$

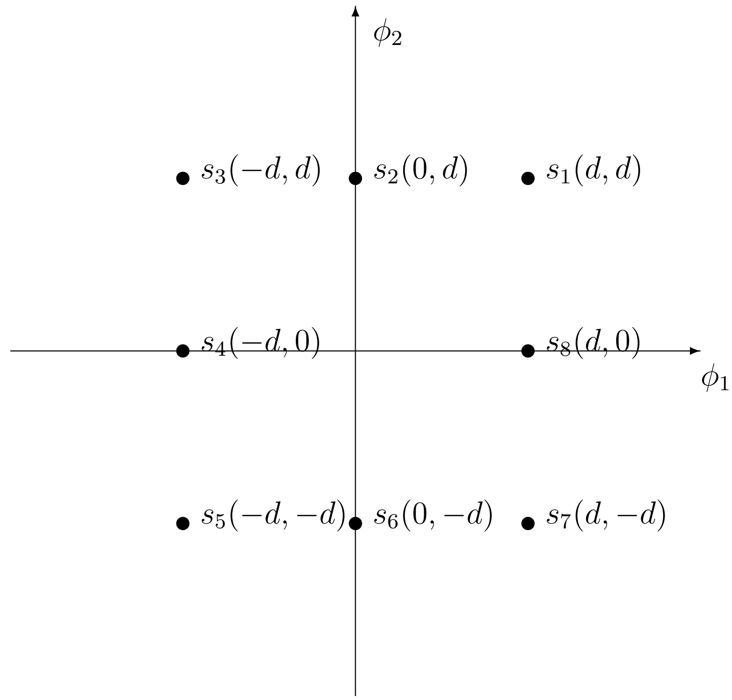
Note that the inner product between two signals $a(t)$ and $b(t)$ is defined as

$$\int_0^T a(t)b(t)dt.$$

(b) (5%) Use the two signals to carry binary information over the AWGN channel with one-sided power spectrum density N_0 (in other words, the power spectrum density of the additive while noise is equal to $N_0/2$). What are the optimal decision regions for the two signals. Suppose $s_1(t)$ will be used with probability p.

(c) (5%) Find the optimal error probability in (b).
Hint: $\Pr\{\mathcal{N}(m, \sigma^2) < r\} = Q\left(\frac{m-r}{\sigma}\right)$.

(d) (6%) Now let $\tau = T/2$ for $\phi_1(t)$ and $\phi_2(t)$ in (a). Consider the following constellation. Assume equal prior probability. Using the fact that the binary error probability between points s_i and s_j can be expressed as $Q(d_{i,j}/\sqrt{2N_0})$, find the union bound expression in terms of every $d_{i,j}$ and Q-function for the error probability of this constellation, where $d_{i,j}$ is the Euclidean distance between points s_i and s_j.



Hint: The distance enumerator function of this constellation is given by

$$T(X) = 16X^{d^2} + 8X^{2d^2} + 12X^{4d^2} + 16X^{5d^2} + 4X^{8d^2}.$$

(e) (6%) Following (d), find the union bound expression in terms of $d_{\min} = \min_{i \neq j} d_{i,j}$ and Q-function for the error probability of this constellation.

4. (Chapter 4)

(a) (11%) Consider four equal-probable signals,

$$\boldsymbol{s}_1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad \boldsymbol{s}_2 = \begin{bmatrix} -1 \\ +1 \end{bmatrix}, \quad \boldsymbol{s}_3 = \begin{bmatrix} +1 \\ -1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{s}_4 = \begin{bmatrix} +1 \\ +1 \end{bmatrix},$$

sending through an additive noisy channel with $\boldsymbol{n} = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$ with joint pdf

$$f(\boldsymbol{n}) = \begin{cases} \exp(-n_1 - n_2), & \text{if } n_1, n_2 \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Find MAP rule for the four symbols (8%), and symbol error rate P_e (3%).
Hint: For your convenience,

$$\int_{-1}^{1}\int_{-1}^{1} e^{-r_1-r_2} = (e - e^{-1})^2, \quad \int_{1}^{\infty}\int_{-1}^{1} e^{-r_1-r_2} = 1 - e^{-2}, \quad \text{and} \quad \int_{1}^{\infty}\int_{1}^{\infty} e^{-r_1-r_2} = e^{-2}.$$

Note that

$$f(\boldsymbol{r}|\boldsymbol{s}_1) = e^{-r_1-r_2-2} \cdot \mathbf{1}\{r_1 \geq -1, r_2 \geq -1\}, \quad f(\boldsymbol{r}|\boldsymbol{s}_2) = e^{-r_1-r_2} \cdot \mathbf{1}\{r_1 \geq -1, r_2 \geq 1\}$$

$$f(\boldsymbol{r}|\boldsymbol{s}_3) = e^{-r_1-r_2} \cdot \mathbf{1}\{r_1 \geq 1, r_2 \geq -1\}, \quad f(\boldsymbol{r}|\boldsymbol{s}_4) = e^{-r_1-r_2+2} \cdot \mathbf{1}\{r_1 \geq 1, r_2 \geq 1\}.$$

(b) (6%) Determine the corresponding bit error rate (respectively for the two bits) if the receiver does the following bit mapping after the symbol detection in (a).

$$\boldsymbol{s}_1 \mapsto \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{s}_2 \mapsto \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{s}_3 \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{s}_4 \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Note: For notational convenience, you may denote the bit error rates respectively for the first and second bits as $P_{b,1}$ and $P_{b,2}$.

(c) (4%) Re-do subproblem (b) for the new bit mapping below.

$$\boldsymbol{s}_1 \mapsto \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{s}_2 \mapsto \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{s}_3 \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{s}_4 \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

(d) (5% bonus) Explain why one of the bit mappings in (b) and (c) performs better than the other one.