2012 Fall: The First Midterm of Digital Communications

1. (Chapter 2)

   (a) (5%) Is the system with input $x(t)$, output $y(t)$ and input-output relation

   $$y(t) = \mathbf{Re}\left\{x(t)e^{\imath 2\pi f_0 t}\right\}$$

   a linear system? If the answer is positive, prove it. If negative, give a counterexample.

   Hint: Superposition principle.

   (b) (5%) Is the system in (a) time-invariant? If the answer is positive, prove it. If negative, give a counterexample.

   Hint: For a time-invariant system, input $x(t-\tau)$ should induce output $y(t-\tau)$ if input $x(t)$ induces output $y(t)$.

   (c) (5%) Give a counterexample (i.e., give an example of f_0 and $X_\ell(f)$) that fails the linear-time-invariant (LTI) system below:

   Input $x(t) = \mathbf{Re}\left\{x_\ell(t)e^{\imath 2\pi f_0 t}\right\}$

   Output $y(t) = \mathbf{Im}\left\{x_\ell(t)e^{\imath 2\pi f_0 t}\right\}$

   Transfer function $H(f) = -\imath \cdot \mathrm{sgn}(f)$

   Hint: Consider the relation between f_0 and the bandwidth W of $x_\ell(t)$.

   (d) (5%) Prove that the power spectrum density of a WSS process $\boldsymbol{x}(t)$ is always real-valued.

   Hint: The autocorrelation function $R_{\boldsymbol{x}}(\tau)$ satisfies $R_{\boldsymbol{x}}(-\tau) = R^*_{\boldsymbol{x}}(\tau)$.

   (e) (5%) Show that $x(t) = \mathbf{Re}\left\{x_\ell(t)e^{\imath 2\pi f_0 t}\right\}$ implies

   $$X(f) = \frac{1}{2}\left[X_\ell(f-f_0) + X^*_\ell(-f-f_0)\right].$$

Solutions.

(a) The answer is YES since $a \cdot y_1(t) + b \cdot y_2(t) = \mathbf{Re}\left\{(a \cdot x_1(t) + b \cdot x_2(t))e^{\imath 2\pi f_0 t}\right\}$ for any **real-valued** scalars a and b and any outputs

$$y_1(t) = \mathbf{Re}\left\{x_1(t)e^{\imath 2\pi f_0 t}\right\} \quad \text{and} \quad y_2(t) = \mathbf{Re}\left\{x_2(t)e^{\imath 2\pi f_0 t}\right\},$$

and zero input produces zero output.

The answer could also be NO (or not necessary) since

$$a \cdot y_1(t) + b \cdot y_2(t) = \mathbf{Re}\left\{(a \cdot x_1(t) + b \cdot x_2(t))e^{\imath 2\pi f_0 t}\right\}$$

is not well defined for complex-valued a and b. Indeed, the problem should be rephrased to clearly indicate that the **ground field** (i.e., the field that a and b belong to) is the real line.

Since this subproblem is not well-stated, it cannot be accounted; so every student gets 5 points from this subproblem.

(b) The answer is NO! Let $f_0 = 1$ for convenience. Then, input $x(t) = e^{\imath 2\pi t}$ induces output $y(t) = \cos(4\pi t)$, and input $x(t-\tau) = e^{\imath 2\pi(t-\tau)}$ induces output $\cos(2\pi(2t-\tau))$. Apparently, $\cos(2\pi(2t-\tau)) \neq y(t-\tau) = \cos(4\pi(t-\tau))$.

(c) First note that the system is valid when $x_\ell(t)$ is band-limited to W (with $W > 0$) and $f_0 > W$. So in its extreme case, letting $f_0 = 0$ and $x_\ell(t) = \imath \cdot z(t)$ for some non-zero real-valued waveform $z(t)$, we have $x(t) = 0$ and $y(t) = z(t)$. Apparently, with zero input, $H(f)$ can only produce zero output; this is contradicted to $y(t) = z(t) \neq$ zero waveform signal.

(d) See slide 2-45.

(e)

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t)e^{-\imath 2\pi f t}dt \\ &= \int_{-\infty}^{\infty} \mathbf{Re}\left\{x_\ell(t)e^{\imath 2\pi f_0 t}\right\} e^{-\imath 2\pi f t}dt \\ &= \int_{-\infty}^{\infty} \frac{1}{2}\left[x_\ell(t)e^{\imath 2\pi f_0 t} + \left(x_\ell(t)e^{\imath 2\pi f_0 t}\right)^*\right] e^{-\imath 2\pi f t}dt \\ &= \frac{1}{2}\int_{-\infty}^{\infty} x_\ell(t)e^{-\imath 2\pi (f-f_0)t}dt + \frac{1}{2}\int_{-\infty}^{\infty} x_\ell^*(t)e^{-\imath 2\pi (f+f_0)t}dt \\ &= \frac{1}{2}\left[X_\ell(f-f_0) + X_\ell^*(-f-f_0)\right] \end{aligned}$$

2. (Chapter 3)

(a) (5%) Suppose $\boldsymbol{s}(t)$ is a cyclostationary random process with period T. Let random vector $\vec{\boldsymbol{x}}(t)$ be defined as:

$$\vec{\boldsymbol{x}}(t) = \begin{bmatrix} \boldsymbol{s}(t) \\ \boldsymbol{s}(t-\tau) \end{bmatrix},$$

where τ is a constant. Is $\vec{\boldsymbol{x}}(t)$ also cyclostationary? If your answer is positive, prove it. If negative, show a counterexample.

(b) (5%) If the autocorrelation function of $\boldsymbol{s}(t)$ is

$$R_{\boldsymbol{s}}(t_1, t_2) = \sum_{m=-\infty}^{\infty} g(t_1 - mT)g^*(t_2 - mT),$$

where $g(t)$ is a given continuous waveform. Prove that the time-average autocorrelation function of $\boldsymbol{s}(t)$ is given by:

$$\overline{R}_{\boldsymbol{s}}(\tau) = \frac{1}{T}\int_{-\infty}^{\infty} g(u+\tau)g^*(u)du.$$

(c) (5%) Following (b), further prove that the time-average power spectrum density of $\boldsymbol{s}(t)$ is equal to:

$$\overline{S}_{\boldsymbol{s}}(f) = \frac{1}{T}|G(f)|^2.$$

(d) (6%) Below is the passband signal of OQPSK modulation.

$$s_{\text{OQPSK}}(t) = \sum_{n=-\infty}^{\infty} I_{2n}g(t-2nT)\cos(2\pi f_c t) - \sum_{n=-\infty}^{\infty} I_{2n+1}g(t-(2n+1)T)\sin(2\pi f_c t)$$

where $g(t) = \begin{cases} 1, & 0 \le t < 2T \\ 0, & \text{otherwise} \end{cases}$ and $I_n \in \{\pm 1\}$. Assume that T is a multiple of $1/f_c$. Define the inner product of two signals, $a(t)$ and $b(t)$, to be $\int_0^T a(t)b(t)dt$. Now by considering all possible signal waveforms that could appear during $[0, T)$, determine the dimension of the OQPSK modulation (3%). Also, give an orthonormal basis of these waveforms (3%) .

Hint: $s_{\text{OQPSK}}(t) = I_0 g(t)\cos(2\pi f_c t) - I_{-1} g(t+T)\sin(2\pi f_c t)$.

(e) (6%) Re-do subproblem (d) by re-defining $g(t)$ to be $g(t) = \begin{cases} \sin\left(\pi \frac{t}{2T}\right), & 0 \le t < 2T \\ 0, & \text{otherwise} \end{cases}$.

Hint: You may let $f_1 = f_c - \frac{1}{4T}$ and $f_2 = f_c + \frac{1}{4T}$ for notational convenience.

Solutions.

(a)

$$E[\vec{\boldsymbol{x}}(t)] = \begin{bmatrix} E[\boldsymbol{s}(t)] \\ E[\boldsymbol{s}(t-\tau)] \end{bmatrix} = \begin{bmatrix} m_{\boldsymbol{s}}(t) \\ m_{\boldsymbol{s}}(t-\tau) \end{bmatrix}$$

and

$$\begin{aligned} E[\vec{\boldsymbol{x}}(t_1)\vec{\boldsymbol{x}}^\dagger(t_2)] &= E\left[\begin{bmatrix} \boldsymbol{s}(t_1) \\ \boldsymbol{s}(t_1-\tau) \end{bmatrix} \begin{bmatrix} \boldsymbol{s}^*(t_2) & \boldsymbol{s}^*(t_2-\tau) \end{bmatrix}\right] \\ &= \begin{bmatrix} E[\boldsymbol{s}(t_1)\boldsymbol{s}^*(t_2)] & E[\boldsymbol{s}(t_1)\boldsymbol{s}^*(t_2-\tau)] \\ E[\boldsymbol{s}(t_1-\tau)\boldsymbol{s}^*(t_2)] & E[\boldsymbol{s}(t_1-\tau)\boldsymbol{s}^*(t_2-\tau)] \end{bmatrix} \\ &= \begin{bmatrix} R_{\boldsymbol{ss}}(t_1, t_2) & R_{\boldsymbol{ss}}(t_1, t_2-\tau) \\ R_{\boldsymbol{ss}}(t_1-\tau, t_2) & R_{\boldsymbol{ss}}(t_1-\tau, t_2-\tau) \end{bmatrix} \end{aligned}$$

With these expressions, $\boldsymbol{x}(t)$ should be also a cyclostationary random process with period T.

(b)

$$\begin{aligned} \overline{R}_{\boldsymbol{s}}(\tau) &= \frac{1}{T}\int_0^T \sum_{m=-\infty}^{\infty} g(t+\tau-mT)g^*(t-mT)dt \\ &= \frac{1}{T}\sum_{m=-\infty}^{\infty}\int_0^T g(t+\tau-mT)g^*(t-mT)dt \\ &\overset{u=t-mT}{=} \frac{1}{T}\sum_{m=-\infty}^{\infty}\int_{-mT}^{-(m-1)T} g(u+\tau)g^*(u)du \\ &= \frac{1}{T}\int_{-\infty}^{\infty} g(u+\tau)g^*(u)du \end{aligned}$$

(c)

$$\begin{aligned}
\overline{S}_{\boldsymbol{s}}(f) &= \int_{-\infty}^{\infty} \overline{R}_{\boldsymbol{s}}(\tau) e^{-\imath 2\pi f \tau} d\tau \\
&= \frac{1}{T}\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(u+\tau) g^*(u) du \right) e^{-\imath 2\pi f\tau} d\tau \\
&= \frac{1}{T}\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(u+\tau) e^{-\imath 2\pi f\tau} d\tau \right) g^*(u) du \\
&\stackrel{v=u+\tau}{=} \frac{1}{T}\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(v) e^{-\imath 2\pi f(v-u)} dv \right) g^*(u) du \\
&= \frac{1}{T}\left(\int_{-\infty}^{\infty} g(v) e^{-\imath 2\pi f v} dv \right) \left(\int_{-\infty}^{\infty} g^*(u) e^{\imath 2\pi f u} du \right) \\
&= \frac{1}{T}|G(f)|^2
\end{aligned}$$

(d) For $0 \le t < T$,

$$\begin{aligned}
s_{\text{OQPSK}}(t) &= \sum_{n=-\infty}^{\infty} I_{2n} g(t-2nT)\cos(2\pi f_c t) - \sum_{n=-\infty}^{\infty} I_{2n+1} g(t-(2n+1)T)\sin(2\pi f_c t) \\
&= I_0 g(t)\cos(2\pi f_c t) - I_{-1} g(t+T)\sin(2\pi f_c t) \\
&= \begin{cases} -\cos(2\pi f_c t) + \sin(2\pi f_c t), & (I_{-1}, I_0) = (-1,-1) \\ \cos(2\pi f_c t) + \sin(2\pi f_c t), & (I_{-1}, I_0) = (-1,+1) \\ -\cos(2\pi f_c t) - \sin(2\pi f_c t), & (I_{-1}, I_0) = (+1,-1) \\ \cos(2\pi f_c t) - \sin(2\pi f_c t), & (I_{-1}, I_0) = (+1,+1) \end{cases}
\end{aligned}$$

As a result, the dimension of the above four signals is two with orthonormal basis $\phi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t)$ and $\phi_2(t) = \sqrt{\frac{2}{T}}\sin(2\pi f_c t)$.

(e) Observe that $g(t) = \sin(\pi t/(2T))$ and $g(t+T) = \cos(\pi t/(2T))$. Then, for $0 \le t < T$,

$$\begin{aligned}
s(t) &= I_0 g(t)\cos(2\pi f_c t) - I_{-1} g(t+T)\sin(2\pi f_c t) \\
&= \begin{cases} -\sin(\pi t/(2T))\cos(2\pi f_c t) + \cos(\pi t/(2T))\sin(2\pi f_c t), & (I_{-1}, I_0) = (-1,-1) \\ \sin(\pi t/(2T))\cos(2\pi f_c t) + \cos(\pi t/(2T))\sin(2\pi f_c t), & (I_{-1}, I_0) = (-1,+1) \\ -\sin(\pi t/(2T))\cos(2\pi f_c t) - \cos(\pi t/(2T))\sin(2\pi f_c t), & (I_{-1}, I_0) = (+1,-1) \\ \sin(\pi t/(2T))\cos(2\pi f_c t) - \cos(\pi t/(2T))\sin(2\pi f_c t), & (I_{-1}, I_0) = (+1,+1) \end{cases} \\
&= \begin{cases} \sin(2\pi f_1 t), & (I_{-1}, I_0) = (-1,-1) \\ \sin(2\pi f_2 t), & (I_{-1}, I_0) = (-1,+1) \\ -\sin(2\pi f_2 t), & (I_{-1}, I_0) = (+1,-1) \\ -\sin(2\pi f_1 t), & (I_{-1}, I_0) = (+1,+1) \end{cases}
\end{aligned}$$

As a result, the dimension of the above four signals is two with basis $\phi_1 = \sqrt{\frac{2}{T}}\sin(2\pi f_1 t)$ and $\phi_2 = \sqrt{\frac{2}{T}}\sin(2\pi f_2 t)$.

3. (Chapter 4) Consider two signals defined as

$$s_1(t) = \begin{cases} A, & 0 \le t < T \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad s_2(t) = \begin{cases} A, & 0 \le t < \tau \\ -A, & \tau \le t < T \\ 0, & \text{otherwise} \end{cases}$$

(a) (5%) Find signal space representations of $s_1(t)$ and $s_2(t)$ based on the basis

$$\phi_1(t) = \begin{cases} \frac{1}{\sqrt{\tau}}, & 0 \le t < \tau \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \phi_2(t) = \begin{cases} \frac{1}{\sqrt{T-\tau}}, & \tau \le t < T \\ 0, & \text{otherwise} \end{cases}$$

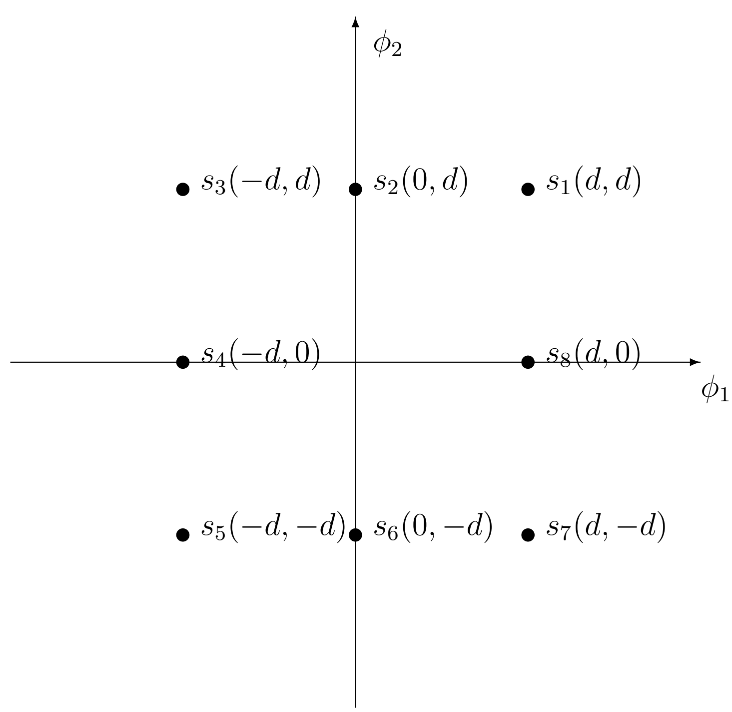
Note that the inner product between two signals $a(t)$ and $b(t)$ is defined as

$$\int_0^T a(t)b(t)dt.$$

(b) (5%) Use the two signals to carry binary information over the AWGN channel with one-sided power spectrum density N_0 (in other words, the power spectrum density of the additive while noise is equal to $N_0/2$). What are the optimal decision regions for the two signals. Suppose $s_1(t)$ will be used with probability p.

(c) (5%) Find the optimal error probability in (b).
Hint: $\Pr\{\mathcal{N}(m, \sigma^2) < r\} = Q\left(\frac{m-r}{\sigma}\right)$.

(d) (6%) Now let $\tau = T/2$ for $\phi_1(t)$ and $\phi_2(t)$ in (a). Consider the following constellation. Assume equal prior probability. Using the fact that the binary error probability between points s_i and s_j can be expressed as $Q(d_{i,j}/\sqrt{2N_0})$, find the union bound expression in terms of every $d_{i,j}$ and Q-function for the error probability of this constellation, where $d_{i,j}$ is the Euclidean distance between points s_i and s_j.



Hint: The distance enumerator function of this constellation is given by

$$T(X) = 16X^{d^2} + 8X^{2d^2} + 12X^{4d^2} + 16X^{5d^2} + 4X^{8d^2}.$$

(e) (6%) Following (d), find the union bound expression in terms of $d_{\min} = \min_{i \neq j} d_{i,j}$ and Q-function for the error probability of this constellation.

Solutions.

(a) $\boldsymbol{s}_1 = \begin{bmatrix} A\sqrt{\tau} \\ A\sqrt{T-\tau} \end{bmatrix}$ and $\boldsymbol{s}_2 = \begin{bmatrix} A\sqrt{\tau} \\ -A\sqrt{T-\tau} \end{bmatrix}$

(b)

$$\begin{aligned}
\mathcal{D}_1 &= \{\boldsymbol{r} \in \Re^2 : \Pr\{\boldsymbol{s}_1 \text{ sent}\} f(\boldsymbol{r}|\boldsymbol{s}_1) > \Pr\{\boldsymbol{s}_2 \text{ sent}\} f(\boldsymbol{r}|\boldsymbol{s}_2)\} \\
&= \left\{\boldsymbol{r} \in \Re^2 : N_0 \log p - \|\boldsymbol{s}_1\|^2 + 2\boldsymbol{r}^T \boldsymbol{s}_1 > N_0 \log(1-p) - \|\boldsymbol{s}_2\|^2 + 2\boldsymbol{r}^T \boldsymbol{s}_2\right\} \\
&= \left\{\boldsymbol{r} \in \Re^2 : r_2 > \frac{N_0}{4A\sqrt{T-\tau}} \log \frac{(1-p)}{p}\right\}
\end{aligned}$$

and

$$\mathcal{D}_2 = \mathcal{D}_1^c = \left\{\boldsymbol{r} \in \Re^2 : r_2 \leq \frac{N_0}{4A\sqrt{T-\tau}} \log \frac{(1-p)}{p}\right\}.$$

(c) Following slide 4-39, we obtain

$$P_e = pQ\left(\frac{A\sqrt{T-\tau} - \frac{N_0}{4A\sqrt{T-\tau}} \log \frac{(1-p)}{p}}{\sqrt{N_0/2}}\right) + (1-p)Q\left(\frac{A\sqrt{T-\tau} + \frac{N_0}{4A\sqrt{T-\tau}} \log \frac{(1-p)}{p}}{\sqrt{N_0/2}}\right)$$

(d)

$$\begin{aligned}
P_e &\leq \frac{1}{8} \sum_{m=1}^{8} \sum_{\substack{1 \leq m' \leq 8 \\ m' \neq m}} Q\left(\sqrt{\frac{d_{m,m'}^2}{2N_0}}\right) \\
&= \frac{1}{8}\left[16Q\left(\sqrt{\frac{d^2}{2N_0}}\right) + 8Q\left(\sqrt{\frac{2d^2}{2N_0}}\right) + 12Q\left(\sqrt{\frac{4d^2}{2N_0}}\right) + 16Q\left(\sqrt{\frac{5d^2}{2N_0}}\right) + 4Q\left(\sqrt{\frac{8d^2}{2N_0}}\right)\right] \\
&= 2Q\left(\sqrt{\frac{d^2}{2N_0}}\right) + Q\left(\sqrt{\frac{2d^2}{2N_0}}\right) + \frac{3}{2}Q\left(\sqrt{\frac{4d^2}{2N_0}}\right) + 2Q\left(\sqrt{\frac{5d^2}{2N_0}}\right) + \frac{1}{2}Q\left(\sqrt{\frac{8d^2}{2N_0}}\right).
\end{aligned}$$

(e) Notice that $d_{\min} = d$; hence,

$$P_e \leq 7Q\left(\sqrt{\frac{d_{\min}^2}{2N_0}}\right).$$

4. (Chapter 4)

(a) (11%) Consider four equal-probable signals,

$$\boldsymbol{s}_1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad \boldsymbol{s}_2 = \begin{bmatrix} -1 \\ +1 \end{bmatrix}, \quad \boldsymbol{s}_3 = \begin{bmatrix} +1 \\ -1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{s}_4 = \begin{bmatrix} +1 \\ +1 \end{bmatrix},$$

sending through an additive noisy channel with $\boldsymbol{n} = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$ with joint pdf

$$f(\boldsymbol{n}) = \begin{cases} \exp(-n_1 - n_2), & \text{if } n_1, n_2 \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Find MAP rule for the four symbols (8%), and symbol error rate P_e (3%).
Hint: For your convenience,

$$\int_{-1}^{1}\int_{-1}^{1} e^{-r_1-r_2} = (e - e^{-1})^2, \quad \int_{1}^{\infty}\int_{-1}^{1} e^{-r_1-r_2} = 1 - e^{-2}, \quad \text{and} \quad \int_{1}^{\infty}\int_{1}^{\infty} e^{-r_1-r_2} = e^{-2}.$$

Note that

$$f(\boldsymbol{r}|\boldsymbol{s}_1) = e^{-r_1-r_2-2} \cdot \mathbf{1}\{r_1 \geq -1, r_2 \geq -1\}, \quad f(\boldsymbol{r}|\boldsymbol{s}_2) = e^{-r_1-r_2} \cdot \mathbf{1}\{r_1 \geq -1, r_2 \geq 1\}$$

$$f(\boldsymbol{r}|\boldsymbol{s}_3) = e^{-r_1-r_2} \cdot \mathbf{1}\{r_1 \geq 1, r_2 \geq -1\}, \quad f(\boldsymbol{r}|\boldsymbol{s}_4) = e^{-r_1-r_2+2} \cdot \mathbf{1}\{r_1 \geq 1, r_2 \geq 1\}.$$

(b) (6%) Determine the corresponding bit error rate (respectively for the two bits) if the receiver does the following bit mapping after the symbol detection in (a).

$$\boldsymbol{s}_1 \mapsto \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{s}_2 \mapsto \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{s}_3 \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{s}_4 \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Note: For notational convenience, you may denote the bit error rates respectively for the first and second bits as $P_{b,1}$ and $P_{b,2}$.

(c) (4%) Re-do subproblem (b) for the new bit mapping below.

$$\boldsymbol{s}_1 \mapsto \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{s}_2 \mapsto \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{s}_3 \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{s}_4 \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

(d) (5% bonus) Explain why one of the bit mappings in (b) and (c) performs better than the other one.

Solutions.

(a) Since $P\{\boldsymbol{s}_1\} = P\{\boldsymbol{s}_2\} = P\{\boldsymbol{s}_3\} = P\{\boldsymbol{s}_4\} = \frac{1}{4}$, MAP and ML rules coincide.
Given $\boldsymbol{r} = \boldsymbol{s} + \boldsymbol{n}$, we would choose $\boldsymbol{s}_1$ if

$$\begin{aligned}
& f(\boldsymbol{r}|\boldsymbol{s}_1) > \max\{f(\boldsymbol{r}|\boldsymbol{s}_2), f(\boldsymbol{r}|\boldsymbol{s}_3), f(\boldsymbol{r}|\boldsymbol{s}_4)\} \\
\Leftrightarrow\ & e^{-(r_1-(-1))-(r_2-(-1))} \cdot \mathbf{1}(r_1 \geq -1, r_2 \geq -1) > \\
& \max\Big\{e^{-(r_1-(-1))-(r_2-1)} \cdot \mathbf{1}(r_1 \geq -1, r_2 \geq 1),\ e^{-(r_1-1)-(r_2-(-1))} \cdot \mathbf{1}(r_1 \geq 1, r_2 \geq -1), \\
& e^{-(r_1-1)-(r_2-1)} \cdot \mathbf{1}(r_1 \geq 1, r_2 \geq 1)\Big\} \\
\Leftrightarrow\ & \mathbf{1}(r_1 \geq -1, r_2 \geq -1) > \\
& e^2 \cdot \max\Big\{\mathbf{1}(r_1 \geq -1, r_2 \geq 1), \mathbf{1}(r_1 \geq 1, r_2 \geq -1), e^2 \cdot \mathbf{1}(r_1 \geq 1, r_2 \geq 1)\Big\}
\end{aligned}$$

where $\mathbf{1}(\cdot)$ is the set indicator function. Hence

$$\mathcal{D}_1 = \left\{\boldsymbol{r} \in \Re^2 : -1 \le r_1 < 1, -1 \le r_2 < 1\right\}.$$

Similarly, we would choose $\boldsymbol{s}_2$ if

$$\begin{aligned}
& f(\boldsymbol{r}|\boldsymbol{s}_2) > \max\{f(\boldsymbol{r}|\boldsymbol{s}_1), f(\boldsymbol{r}|\boldsymbol{s}_3), f(\boldsymbol{r}|\boldsymbol{s}_4)\} \\
\Leftrightarrow\ & e^{-(r_1-(-1))-(r_2-1)} \cdot \mathbf{1}(r_1 \ge -1, r_2 \ge 1) > \\
& \max\left\{e^{-(r_1-(-1))-(r_2-(-1))} \cdot \mathbf{1}(r_1 \ge -1, r_2 \ge -1),\ e^{-(r_1-1)-(r_2-(-1))} \cdot \mathbf{1}(r_1 \ge 1, r_2 \ge -1),\right. \\
& \left. e^{-(r_1-1)-(r_2-1)} \cdot \mathbf{1}(r_1 \ge 1, r_2 \ge 1)\right\} \\
\Leftrightarrow\ & \mathbf{1}(r_1 \ge -1, r_2 \ge 1) > \\
& \max\left\{e^{-2} \cdot \mathbf{1}(r_1 \ge -1, r_2 \ge -1), \mathbf{1}(r_1 \ge 1, r_2 \ge -1), e^2 \cdot \mathbf{1}(r_1 \ge 1, r_2 \ge 1)\right\}.
\end{aligned}$$

Hence

$$\mathcal{D}_2 = \left\{\boldsymbol{r} \in \Re^2 : -1 \le r_1 < 1, r_2 \ge 1\right\}.$$

Following similar procedure, we know

$$\mathcal{D}_3 = \left\{\boldsymbol{r} \in \Re^2 : r_1 \ge 1, -1 \le r_2 < 1\right\}$$

and

$$\mathcal{D}_4 = \left\{\boldsymbol{r} \in \Re^2 : r_1 \ge 1, r_2 \ge 1\right\}.$$

Note that the region that $\{\boldsymbol{r} \in \Re : r_1 < -1 \text{ or } r_2 < -1\}$ can be arbitrarily assigned without affecting the error performance.

We next compute the probability of correct demodulation as follows.

$$\begin{aligned}
P_{c|1} &= \Pr\{\text{decision} = \boldsymbol{s}_1 | \boldsymbol{s}_1 \text{ transmitted}\} \\
&= \int_{\mathcal{D}_1} f(\boldsymbol{r}|\boldsymbol{s}_1)\, d\boldsymbol{r} = \int_{-1}^{1}\int_{-1}^{1} e^{-r_1-r_2-2} \cdot \mathbf{1}\{r_1 \ge -1, r_2 \ge -1\}\, dr_1 dr_2 \;=\; e^{-2}(e - e^{-1})^2 \\
P_{c|2} &= \Pr\{\text{decision} = \boldsymbol{s}_2 | \boldsymbol{s}_2 \text{ transmitted}\} \\
&= \int_{\mathcal{D}_2} f(\boldsymbol{r}|\boldsymbol{s}_2)\, d\boldsymbol{r} = \int_{1}^{\infty}\int_{-1}^{1} e^{-r_1-r_2} \cdot \mathbf{1}\{r_1 \ge -1, r_2 \ge 1\} dr_1 dr_2 \;=\; 1 - e^{-2} \\
P_{c|3} &= \Pr\{\text{decision} = \boldsymbol{s}_3 | \boldsymbol{s}_3 \text{ transmitted}\} \\
&= \int_{\mathcal{D}_3} f(\boldsymbol{r}|\boldsymbol{s}_3) \cdot \mathbf{1}\{r_1 \ge 1, r_2 \ge -1\} d\boldsymbol{r} = \int_{-1}^{1}\int_{1}^{\infty} e^{-r_1-r_2}\, dr_1 dr_2 \;=\; 1 - e^{-2} \\
P_{c|4} &= \Pr\{\text{decision} = \boldsymbol{s}_4 | \boldsymbol{s}_4 \text{ transmitted}\} \\
&= \int_{\mathcal{D}_4} f(\boldsymbol{r}|\boldsymbol{s}_4)\, d\boldsymbol{r} = \int_{1}^{\infty}\int_{1}^{\infty} e^{-r_1-r_2+2} \cdot \mathbf{1}\{r_1 \ge 1, r_2 \ge 1\} dr_1 dr_2 \;=\; 1.
\end{aligned}$$

Hence,

$$\begin{aligned}
P_c &= \frac{1}{4}\left(P_{c|1} + P_{c|2} + P_{c|3} + P_{c|4}\right) \\
&= 1 - e^{-2} + \frac{1}{4}e^{-4}
\end{aligned}$$

and

$$P_e = 1 - P_c = e^{-2} - \frac{1}{4}e^{-4}.$$

(b) We know that

$$\begin{aligned}
P_{2|1} &= \Pr\{\text{decision} = \boldsymbol{s}_2|\boldsymbol{s}_1 \text{ transmitted}\} \\
&= \int_{\mathcal{D}_2} f(\boldsymbol{r}|\boldsymbol{s}_1)\, d\boldsymbol{r} = \int_1^\infty \int_{-1}^1 e^{-r_1-r_2-2} \cdot \mathbf{1}\{r_1 \geq -1, r_2 \geq -1\} dr_1 dr_2 \;=\; e^{-2}(1-e^{-2}) \\
P_{3|1} &= \Pr\{\text{decision} = \boldsymbol{s}_3|\boldsymbol{s}_1 \text{ transmitted}\} \\
&= \int_{\mathcal{D}_3} f(\boldsymbol{r}|\boldsymbol{s}_1)\, d\boldsymbol{r} = \int_{-1}^1 \int_1^\infty e^{-r_1-r_2-2} \cdot \mathbf{1}\{r_1 \geq -1, r_2 \geq -1\} dr_1 dr_2 \;=\; e^{-2}(1-e^{-2}) \\
P_{4|1} &= \Pr\{\text{decision} = \boldsymbol{s}_4|\boldsymbol{s}_1 \text{ transmitted}\} \\
&= \int_{\mathcal{D}_4} f(\boldsymbol{r}|\boldsymbol{s}_1)\, d\boldsymbol{r} = \int_1^\infty \int_1^\infty e^{-r_1-r_2-2} \cdot \mathbf{1}\{r_1 \geq -1, r_2 \geq -1\} dr_1 dr_2 \;=\; e^{-2} \cdot e^{-2} \\
P_{1|2} &= \Pr\{\text{decision} = \boldsymbol{s}_1|\boldsymbol{s}_2 \text{ transmitted}\} \\
&= \int_{\mathcal{D}_1} f(\boldsymbol{r}|\boldsymbol{s}_2)\, d\boldsymbol{r} = \int_{-1}^1 \int_{-1}^1 e^{-r_1-r_2} \cdot \mathbf{1}\{r_1 \geq -1, r_2 \geq 1\} dr_1 dr_2 \;=\; 0 \\
P_{3|2} &= \Pr\{\text{decision} = \boldsymbol{s}_3|\boldsymbol{s}_2 \text{ transmitted}\} \\
&= \int_{\mathcal{D}_3} f(\boldsymbol{r}|\boldsymbol{s}_2)\, d\boldsymbol{r} = \int_{-1}^1 \int_1^\infty e^{-r_1-r_2} \cdot \mathbf{1}\{r_1 \geq -1, r_2 \geq 1\} dr_1 dr_2 \;=\; 0 \\
P_{4|2} &= \Pr\{\text{decision} = \boldsymbol{s}_4|\boldsymbol{s}_2 \text{ transmitted}\} \\
&= \int_{\mathcal{D}_4} f(\boldsymbol{r}|\boldsymbol{s}_2)\, d\boldsymbol{r} = \int_1^\infty \int_1^\infty e^{-r_1-r_2} \cdot \mathbf{1}\{r_1 \geq -1, r_2 \geq 1\} dr_1 dr_2 \;=\; e^{-2} \\
P_{1|3} &= \Pr\{\text{decision} = \boldsymbol{s}_1|\boldsymbol{s}_3 \text{ transmitted}\} \\
&= \int_{\mathcal{D}_1} f(\boldsymbol{r}|\boldsymbol{s}_3)\, d\boldsymbol{r} = \int_{-1}^1 \int_{-1}^1 e^{-r_1-r_2} \cdot \mathbf{1}\{r_1 \geq 1, r_2 \geq -1\} dr_1 dr_2 \;=\; 0 \\
P_{2|3} &= \Pr\{\text{decision} = \boldsymbol{s}_2|\boldsymbol{s}_3 \text{ transmitted}\} \\
&= \int_{\mathcal{D}_2} f(\boldsymbol{r}|\boldsymbol{s}_3)\, d\boldsymbol{r} = \int_1^\infty \int_{-1}^1 e^{-r_1-r_2} \cdot \mathbf{1}\{r_1 \geq 1, r_2 \geq -1\} dr_1 dr_2 \;=\; 0 \\
P_{4|3} &= \Pr\{\text{decision} = \boldsymbol{s}_4|\boldsymbol{s}_3 \text{ transmitted}\} \\
&= \int_{\mathcal{D}_4} f(\boldsymbol{r}|\boldsymbol{s}_3)\, d\boldsymbol{r} = \int_1^\infty \int_1^\infty e^{-r_1-r_2} \cdot \mathbf{1}\{r_1 \geq 1, r_2 \geq -1\} dr_1 dr_2 \;=\; e^{-2} \\
P_{1|4} &= P_{2|4} = P_{3|4} = 0
\end{aligned}$$

Hence,

$$\begin{aligned}
P_{b,1} &= \Pr\{\boldsymbol{s}_1\}(P_{3|1} + P_{4|1}) + \Pr\{\boldsymbol{s}_2\}(P_{3|2} + P_{4|2}) + \Pr\{\boldsymbol{s}_3\}(P_{1|3} + P_{2|3}) + \Pr\{\boldsymbol{s}_4\}(P_{1|4} + P_{2|4}) \\
&= \frac{1}{4}e^{-2} + \frac{1}{4}e^{-2} + 0 + 0 = \frac{1}{2}e^{-2}
\end{aligned}$$

and

$$\begin{aligned}
P_{b,2} &= \Pr\{\boldsymbol{s}_1\}(P_{2|1} + P_{4|1}) + \Pr\{\boldsymbol{s}_2\}(P_{1|2} + P_{3|2}) + \Pr\{\boldsymbol{s}_3\}(P_{2|3} + P_{4|3}) + \Pr\{\boldsymbol{s}_4\}(P_{1|4} + P_{3|4}) \\
&= \frac{1}{4}e^{-2} + 0 + \frac{1}{4}e^{-2} + 0 = \frac{1}{2}e^{-2}.
\end{aligned}$$

(c) For the new bit mapping, $P_{b,1}$ remains the same, but

$$\begin{aligned}
P_{b,2} &= \Pr\{\boldsymbol{s}_1\}(P_{2|1} + P_{3|1}) + \Pr\{\boldsymbol{s}_2\}(P_{1|2} + P_{4|2}) + \Pr\{\boldsymbol{s}_3\}(P_{1|3} + P_{4|3}) + \Pr\{\boldsymbol{s}_4\}(P_{2|4} + P_{3|4}) \\
&= \frac{1}{2}(e^{-2} - e^{-4}) + \frac{1}{4}e^{-2} + \frac{1}{4}e^{-2} + 0 = e^{-2} - \frac{1}{2}e^{-4}
\end{aligned}$$

(d) From the solutions in (b), we know that $\boldsymbol{s}_1$ when it is transmitted will be most possibly erroneously demodulated to either $\boldsymbol{s}_2$ or $\boldsymbol{s}_3$; $\boldsymbol{s}_2$ when it is transmitted will be most possibly erroneously demodulated to either $\boldsymbol{s}_4$; $\boldsymbol{s}_3$ when it is transmitted will be most possibly erroneously demodulated to either $\boldsymbol{s}_4$; the transmission of $\boldsymbol{s}_4$ will render no error in demodulation. Hence, according to the Gray mapping principle, it is better that the bit pattern of $\boldsymbol{s}_1$ has only one-bit difference from that of $\boldsymbol{s}_2$ and $\boldsymbol{s}_3$; the bit pattern of $\boldsymbol{s}_2$ has only one-bit difference from that of $\boldsymbol{s}_4$; the bit pattern of $\boldsymbol{s}_3$ has only one-bit difference from that of $\boldsymbol{s}_4$, which is exactly what the bit mapper in (b) does. So we can expect that the bit mapping in (b) performs better than that of (c).