2012 Fall: The Second Midterm of Digital Communications

1. (Chapter 5)

   (a) (8%) For a system defined by

   $$r_\ell(t) = s_\ell(t;\phi) + n_\ell(t),$$

   where $s_\ell(t;\phi) = e^{\imath\phi} s_\ell(t)$, and $n_\ell(t)$ is additive Gaussian white with two-sided power spectrum density σ_ℓ^2, the likelihood function is given by

   $$\Lambda(\phi) = \exp\left\{-\frac{1}{\sigma_\ell^2}\int_0^{T_0} \left|r_\ell(t) - s_\ell(t;\phi)\right|^2 dt\right\}.$$

   Prove that

   $$\arg\max_\phi \Lambda(\phi) = \arg\max_\phi \exp\left\{\frac{2}{\sigma_\ell^2}\int_0^{T_0} \mathbf{Re}\left\{r_\ell(t)s_\ell^*(t;\phi)\right\} dt\right\}.$$

   (b) (6%) Continue from (a). If $s_\ell(t) = A$ and $r_\ell(t) = 1+\imath$, where A is a positive real-valued constant, what is the best estimate $\hat{\phi} = \arg\max_\phi \Lambda(\phi)$ in $[0, 2\pi)$?

   (c) (8%) Continue from (a). If $s_\ell(t) = \sum_{n=-\infty}^{\infty} I_n g(t-nT)$ and $r_\ell(t) = 1+\imath$ for $0 \le t < T_0$, where

   $$I_n \in \{\pm 1\}, \quad T_0 = 2T, \quad \text{and} \quad g(t) = \begin{cases} 1, & 0 \le t < T \\ 0, & \text{otherwise} \end{cases},$$

   what are the four best estimates $\hat{\phi}$ in $[0, 2\pi)$, respectively corresponding to directed decisions $(I_0, I_1) = (-1,-1)$, $(-1,+1)$, $(+1,-1)$, $(+1,+1)$? (Each answer earns you 2%.)

   (d) (6%) For a non-decision directed loop, the criterion for estimate $\hat{\phi}$ is changed to

   $$\hat{\phi} = \arg\max_\phi \mathbb{E}[\Lambda(\phi)].$$

   Given that $s_\ell(t) = \sum_{n=-\infty}^{\infty} I_n g(t-nT)$ and $r_\ell(t) = 1+\imath$ for $0 \le t < T_0$, where

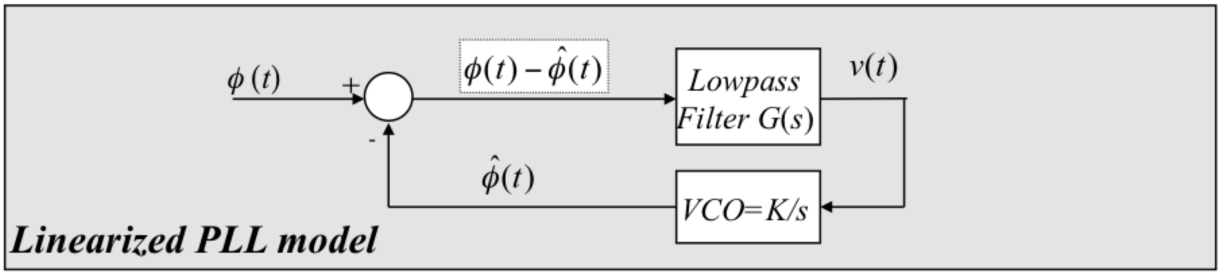
   $$\{I_n\} \text{ i.i.d.}, \quad \Pr[I_n = -1] = \Pr[I_n = 1] = \frac{1}{2}, \quad T_0 = 2T, \quad \text{and} \quad g(t) = \begin{cases} 1, & 0 \le t < T \\ 0, & \text{otherwise} \end{cases},$$

   what is the best estimate $\hat{\phi}$ in $[0, 2\pi)$?

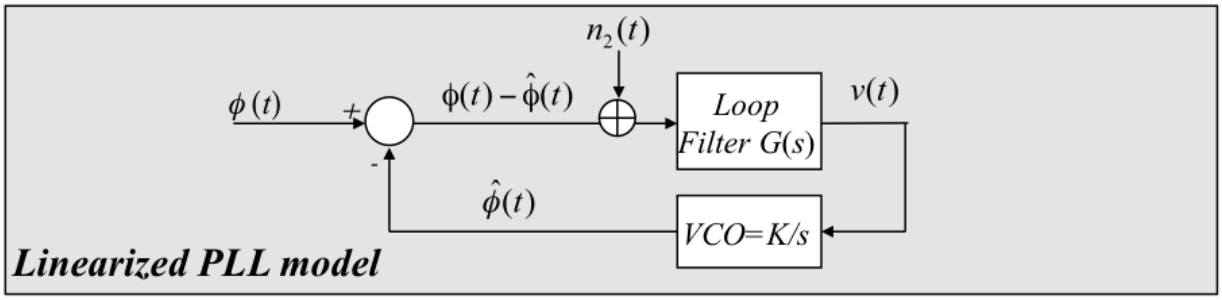
   Note: $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$ is an even function, and has minimum at $x = 0$, and monotonically increases to infinity when $|x| \to \infty$.

2. (Chapter 5)

   (a) (8%) Fine the (Laplace-transform-based) close-loop system transfer function $H(s) = \dfrac{\hat{\phi}(s)}{\phi(s)}$ of the phase-lock loop below.



(b) (8%) When adding a noise $n_2(t)$ as shown in the figure below, prove that the system becomes $\hat{\phi}(t) = [\phi(t) + n_2(t)] \star h(t)$, where $h(t)$ is the close-loop system impulse response corresponding to $H(s)$ in (a).



Hint: $\dfrac{\hat{\phi}(s)}{\phi(s) + n_2(s)}$.

3. (Chapter 6)

(a) (6%) Section 6.6 shows that for M-ary orthogonal signals, the error probability P_e can be bounded above by:

$$P_e \;\leq\; Q\left(\sqrt{2k\gamma_b} - \sqrt{2k\log(2)}\right) + \frac{Me^{-k\gamma_b/2}}{\sqrt{2}} Q\left(\frac{\sqrt{2k\log(2)} - \sqrt{k\gamma_b/2}}{\sqrt{1/2}}\right)$$

where $Q(\cdot)$ is the Q-function, $M = 2^k$, and γ_b is the signal to noise ratio per information bit. Using $Q(x) \leq \frac{1}{2}e^{-x^2/2}$, prove that when $\gamma_b > \log(2)$, $\lim_{k\to\infty} P_e = 0$.

(b) (6%) In 1948, Shannon proved that

- if $R < C$, then P_e can be made arbitrarily small (by extending the code size);
- if $R > C$, then P_e is bounded away from zero,

where $C = \max_{P_X} I(X;Y)$ is the channel capacity, and R is the code rate. For AWGN channels,

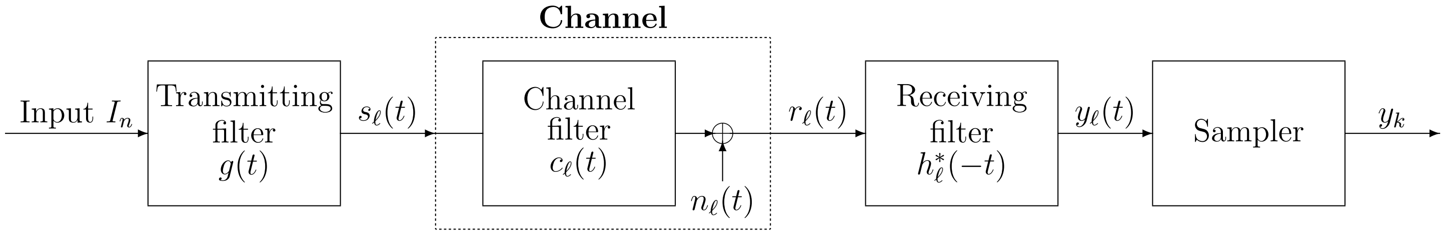
$$C = W \log_2\left(1 + \frac{P}{N_0 W}\right) \quad \text{bit/second},$$

where the units of W, N_0 and P are Hz, Joule and Watt, respectively. Use this theorem to prove that for M-ary orthogonal FSK, if

$$\gamma_b < \lim_{k\to\infty} \frac{2^{2k/2^k} - 1}{2k/2^k} = \log(2),$$

then P_e is bounded away from zero.
Hint: $\gamma_b = \mathcal{E}_b/N_0$, $P = R\mathcal{E}_b$, $W = \frac{M}{2T}$ and $R = \frac{\log_2(M)}{T}$.

4. (Chapter 9)



(a) (8%) Suppose that T is the symbol period, $s_\ell(t) = \sum_{n=-\infty}^{\infty} I_n g(t-nT)$, $c_\ell(t) = \delta(t)$, $n_\ell(t) = 0$ (noise-free), $h_\ell(t) = g(t) \star c_\ell(t) = g(t)$, and

$$g(t) = \begin{cases} \frac{1}{\sqrt{u}}, & 0 \le t < u \\ 0, & \text{otherwise} \end{cases}$$

Assume that the sampler samples at time instant kT for integer k. Does the system have ISI if $u = \frac{T}{2}$ (4%)? Answer the same question if $u = 2T$ (4%). You should justify your answers.

(b) (6%) Continue from (a). Draw the eye diagram for $u = \frac{T}{2}$ over the duration $[-T/2, 5T/2]$, where $I_n \in \{\pm 1\}$.

(c) (8%) Denote $x_\ell(t) = g(t) \star c_\ell(t) \star h_\ell^*(-t)$. Let $x_\ell(t)$ be a band-limited signal with band $[-W, W]$ and $x_\ell(0) = 1$. Prove that

$$x_k \;=\; x_\ell(kT) \;=\; \int_{-\infty}^{\infty} x_\ell(t)\delta(t-kT)\,dt = \delta_k$$

if and only if

$$\frac{1}{T}\sum_{m=-\infty}^{\infty} X_\ell\left(f - \frac{m}{T}\right) \;=\; 1 \tag{1}$$

where $X_\ell(f) = \mathcal{F}\{x_\ell(t)\}$ and $\delta_k = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$ is the Kronecker delta function.

Hint: $\mathcal{F}\left\{\sum_{k=-\infty}^{\infty} \delta(t-kT)\right\} = \frac{1}{T}\sum_{m=-\infty}^{\infty} \delta\left(f - \frac{m}{T}\right)$.

(d) (6%) Continue from (c). If we now shoot for controlled ISI as

$$x_k = \delta_k + \delta_{k-1} + \delta_{k-2},$$

what will be the new condition replacing (1)? No derivation is required. You may answer the question directly.

(e) (8%) In order to satisfy (1), a choice is to let $X_\ell(f) = X_{\text{rc}}(f)$, where

$$X_{rc}(f) \;=\; \begin{cases} T, & 0 \le |f| \le \frac{1-\beta}{2T} \\ \frac{T}{2}\left\{1 + \cos\left[\frac{\pi T}{\beta}\left(|f| - \frac{1-\beta}{2T}\right)\right]\right\}, & \frac{1-\beta}{2T} \le |f| \le \frac{1+\beta}{2T} \\ 0, & \text{otherwise} \end{cases}$$

where $0 \le \beta \le 1$ is the roll-off factor. So we have

$$\begin{cases} s_\ell(t) = \sum_{n=-\infty}^{\infty} I_n g(t-nT) \\ r_\ell(t) = c_\ell(t) \star s_\ell(t) + n_\ell(t) \\ y_\ell(t) = s_\ell(t) \star c_\ell(t) \star h_\ell^*(-t) + n_\ell(t) \star h_\ell^*(-t) = \sum_{n=-\infty}^{\infty} I_n x_{\text{rc}}(t-nT) + z_\ell(t) \\ y_k = y_\ell(kT) \;=\; I_k + z_k \end{cases}$$

where $I_n \in \{\pm d\}$, $z_\ell(t) = n_\ell(t) \star h_\ell^*(-t)$, $z_k = z_\ell(kT)$ and n_ℓ is an additive while noise with two-sided power spectrum σ_ℓ^2. Now for fixed transmission power

$$P_{av,\ell} \;=\; \frac{d^2||g(t)||^2}{T}$$

prove that

$$\frac{d}{\sqrt{\mathbb{E}[z_k^2]}} = \sqrt{\frac{P_{av,\ell}T/\sigma_\ell^2}{\|g(t)\|^2\|h_\ell(t)\|^2}}.$$

Hint: In your proof, you may directly use the fact that $S_{z_\ell}(f) = S_{n_\ell}(f)|H_\ell(f)|^2$.

(f) (8%) Continue from (e). If we wish to minimize the error rate

$$P_e = Q\left(\frac{d}{\sqrt{\mathbb{E}[z_k^2]}}\right)$$

without ISI by setting $g(t) \star c_\ell(t) \star h_\ell^*(-t) = x_{\text{rc}}(t)$, then the system design desires to maximize $\|g(t)\|^2\|h_\ell(t)\|^2$ subject to $g(t) \star c_\ell(t) \star h_\ell^*(-t) = x_{\text{rc}}(t)$, or equivalently, to maximize $\|G(f)\|^2\|H_\ell(f)\|^2$ subject to $G(f)C_\ell(f)H_\ell^*(f) = X_{\text{rc}}(f)$. If $C_\ell(f)$ is known to both transmitter and receiver, what are choices of $G(f)$ and $H_\ell^*(f)$ such that $\|G(f)\|^2\|H_\ell(f)\|^2$ is maximized.

Hint: Cauchy-Schwartz inequality and $G(f)H_\ell^*(f) = X_{\text{rc}}(f)/C_\ell(f)$.