2012 Fall: The Second Midterm of Digital Communications

1. (Chapter 5)

   (a) (8%) For a system defined by

   $$r_\ell(t) = s_\ell(t;\phi) + n_\ell(t),$$

   where $s_\ell(t;\phi) = e^{\imath\phi} s_\ell(t)$, and $n_\ell(t)$ is additive Gaussian white with two-sided power spectrum density σ_ℓ^2, the likelihood function is given by

   $$\Lambda(\phi) = \exp\left\{ -\frac{1}{\sigma_\ell^2} \int_0^{T_0} |r_\ell(t) - s_\ell(t;\phi)|^2 \, dt \right\}.$$

   Prove that

   $$\arg\max_\phi \Lambda(\phi) = \arg\max_\phi \exp\left\{ \frac{2}{\sigma_\ell^2} \int_0^{T_0} \mathbf{Re}\left\{ r_\ell(t) s_\ell^*(t;\phi) \right\} dt \right\}.$$

   (b) (6%) Continue from (a). If $s_\ell(t) = A$ and $r_\ell(t) = 1 + \imath$, where A is a positive real-valued constant, what is the best estimate $\hat{\phi} = \arg\max_\phi \Lambda(\phi)$ in $[0, 2\pi)$?

   (c) (8%) Continue from (a). If $s_\ell(t) = \sum_{n=-\infty}^{\infty} I_n g(t - nT)$ and $r_\ell(t) = 1 + \imath$ for $0 \le t < T_0$, where

   $$I_n \in \{\pm 1\}, \quad T_0 = 2T, \quad \text{and} \quad g(t) = \begin{cases} 1, & 0 \le t < T \\ 0, & \text{otherwise} \end{cases},$$

   what are the four best estimates $\hat{\phi}$ in $[0, 2\pi)$, respectively corresponding to directed decisions $(I_0, I_1) = (-1, -1)$, $(-1, +1)$, $(+1, -1)$, $(+1, +1)$? (Each answer earns you 2%.)

   (d) (6%) For a non-decision directed loop, the criterion for estimate $\hat{\phi}$ is changed to

   $$\hat{\phi} = \arg\max_\phi \mathbb{E}[\Lambda(\phi)].$$

   Given that $s_\ell(t) = \sum_{n=-\infty}^{\infty} I_n g(t - nT)$ and $r_\ell(t) = 1 + \imath$ for $0 \le t < T_0$, where

   $$\{I_n\} \text{ i.i.d.}, \quad \Pr[I_n = -1] = \Pr[I_n = 1] = \frac{1}{2}, \quad T_0 = 2T, \quad \text{and} \quad g(t) = \begin{cases} 1, & 0 \le t < T \\ 0, & \text{otherwise} \end{cases},$$

   what is the best estimate $\hat{\phi}$ in $[0, 2\pi)$?

   Note: $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$ is an even function, and has minimum at $x = 0$, and monotonically increases to infinity when $|x| \to \infty$.

Solutions.

(a)

$$\begin{aligned}
\arg\max_{\phi} \Lambda(\phi) &= \arg\max_{\phi} \exp\left\{-\frac{1}{\sigma_\ell^2}\int_0^{T_0} |r_\ell(t) - s_\ell(t;\phi)|^2 \, dt\right\} \\
&= \arg\max_{\phi} \exp\left(-\frac{1}{\sigma_\ell^2}\int_0^{T_0} \left[|r_\ell(t)|^2 - 2\mathbf{Re}\left\{r_\ell(t)s_\ell^*(t;\phi)\right\} + |s_\ell(t;\phi)|^2\right] dt\right) \\
&= \arg\max_{\phi} \exp\left(-\frac{1}{\sigma_\ell^2}\int_0^{T_0} \left[|r_\ell(t)|^2 - 2\mathbf{Re}\left\{r_\ell(t)s_\ell^*(t;\phi)\right\} + |s_\ell(t)|^2\right] dt\right) \\
&\qquad \text{since } |s_\ell(t;\phi)| = |e^{-\imath\phi}s_\ell(t)| = |s_\ell(t)| \\
&= \arg\max_{\phi} \exp\left(\frac{2}{\sigma_\ell^2}\int_0^{T_0} \mathbf{Re}\left\{r_\ell(t)s_\ell^*(t;\phi)\right\} dt\right),
\end{aligned}$$

where the last step follows because $|r_\ell(t)|$ and $|s_\ell(t)|$ are irrelevant to ϕ.

(b) Since

$$\hat{\phi} = \arg\max_{\phi} \exp\left\{\frac{2}{\sigma_\ell^2}\int_0^{T_0} \mathbf{Re}\left\{r_\ell(t)s_\ell^*(t;\phi)\right\} \, dt\right\} = \arg\max_{\phi} \int_0^{T_0} \mathbf{Re}\left\{r_\ell(t)s_\ell^*(t;\phi)\right\} \, dt.$$

and

$$\begin{aligned}
\int_0^{T_0} \mathbf{Re}\left\{r_\ell(t)s_\ell^*(t;\phi)\right\} dt &= \int_0^{T_0} \mathbf{Re}\left\{r_\ell(t)e^{-\imath\phi}A\right\} dt \\
&= A\int_0^{T_0} \mathbf{Re}\left\{r_\ell(t)e^{-\imath\phi}\right\} dt \\
&= A\cos(\phi)\int_0^{T_0} \mathbf{Re}\{r_\ell(t)\}dt + A\sin(\phi)\int_0^{T_0} \mathbf{Im}\{r_\ell(t)\}dt \\
&= A\cos(\phi)\int_0^{T_0} dt + A\sin(\phi)\int_0^{T_0} dt \\
&= AT_0\left[\cos(\phi) + \sin(\phi)\right] = AT_0\sqrt{2}\cos(\phi - \pi/4).
\end{aligned}$$

Hence, $\hat{\phi} = \pi/4$.

(c)

$$\begin{aligned}
\int_0^{T_0} \mathbf{Re}\left\{r_\ell(t)s_\ell^*(t;\phi)\right\} dt &= \int_0^{2T} \mathbf{Re}\left\{(1+\imath)e^{-\imath\phi}\sum_{n=-\infty}^{\infty} I_n g(t-nT)\right\} dt \\
&= \int_0^{2T} \mathbf{Re}\left\{(1+\imath)e^{-\imath\phi}\sum_{n=0}^{1} I_n g(t-nT)\right\} dt \\
&= (\cos(\phi) + \sin(\phi))\sum_{n=0}^{1} I_n \int_{nT}^{(n+1)T} g(t-nT)dt \\
&= T(\cos(\phi) + \sin(\phi))\sum_{n=0}^{1} I_n \\
&= T\sqrt{2}\cos(\phi - \pi/4)\sum_{n=0}^{1} I_n
\end{aligned}$$

Hence,

$$\begin{aligned}\hat{\phi} &= \arg\max_{\phi} \int_0^{T_0} \mathbf{Re}\left\{ r_\ell(t) s_\ell^*(t;\phi) \right\} dt \\ &= \arg\max_{\phi} T\sqrt{2}\cos(\phi - \pi/4) \sum_{n=0}^{1} I_n \\ &= \begin{cases} \pi/4, & (I_0, I_1) = (1,1) \\ 5\pi/4, & (I_0, I_1) = (-1,-1) \\ \text{arbitrary}, & \text{otherwise} \end{cases}\end{aligned}$$

(d)

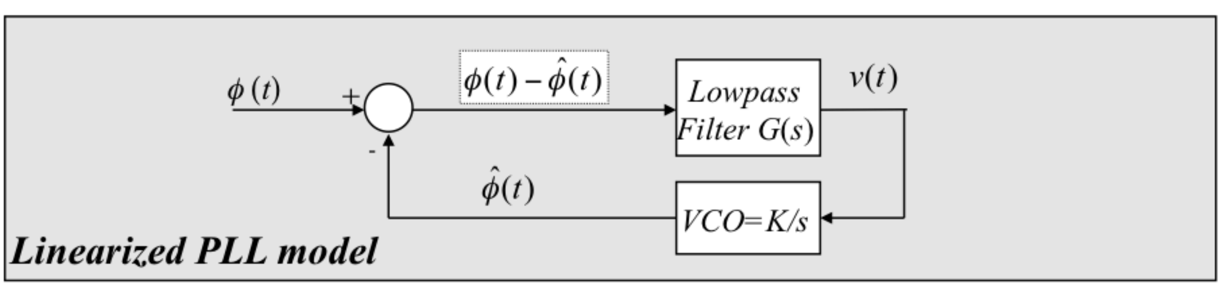
$$\begin{aligned}\mathbb{E}[\Lambda(\phi)] &= \mathbb{E}\left[\exp\left(\frac{2}{\sigma_\ell^2}\int_0^{T_0} \mathbf{Re}\left\{ r_\ell(t) s_\ell^*(t;\phi)\right\} dt\right)\right] \\ &= \mathbb{E}\left[\exp\left(\frac{2}{\sigma_\ell^2}\int_0^{2T} \mathbf{Re}\left\{ (1+\imath)e^{-\imath\phi}\sum_{n=0}^{1} I_n g(t-nT)\right\} dt\right)\right] \\ &= \mathbb{E}\left[\exp\left(\frac{2\sqrt{2}T\cos(\phi-\pi/4)}{\sigma_\ell^2}\sum_{n=0}^{1} I_n\right)\right] \\ &= \frac{1}{4}\exp\left(\frac{2\sqrt{2}T\cos(\phi-\pi/4)}{\sigma_\ell^2}\cdot 2\right) + \frac{1}{4}\exp\left(\frac{2\sqrt{2}T\cos(\phi-\pi/4)}{\sigma_\ell^2}(-2)\right) \\ &\quad + \frac{1}{2}\exp\left(\frac{2\sqrt{2}T\cos(\phi-\pi/4)}{\sigma_\ell^2}\cdot 0\right) \\ &= \frac{1}{2}\cosh\left(\frac{4\sqrt{2}T\cos(\phi-\pi/4)}{\sigma_\ell^2}\right) + \frac{1}{2}.\end{aligned}$$

Hence, $\mathbb{E}[\Lambda(\phi)]$ achieves its maximum when $\cos(\phi - \pi/4)$ achieves either $+1$ or -1. Consequently,

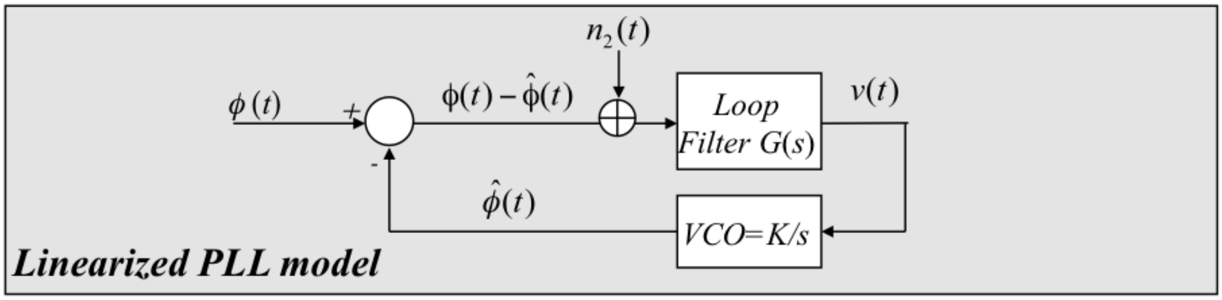
$$\hat{\phi} = \arg\max_{\phi} \mathbb{E}[\Lambda(\phi)] = \frac{\pi}{4} \text{ or } \frac{5\pi}{4}.$$

2. (Chapter 5)

(a) (8%) Fine the (Laplace-transform-based) close-loop system transfer function $H(s) = \dfrac{\hat{\phi}(s)}{\phi(s)}$ of the phase-lock loop below.



(b) (8%) When adding a noise $n_2(t)$ as shown in the figure below, prove that the system becomes $\hat{\phi}(t) = [\phi(t) + n_2(t)] \star h(t)$, where $h(t)$ is the close-loop system impulse response corresponding to $H(s)$ in (a).



Hint: $\dfrac{\hat{\phi}(s)}{\phi(s) + n_2(s)}$.

Solutions.

(a) See slide 5-27.

(b) See slide 5-36.

3. (Chapter 6)

(a) (6%) Section 6.6 shows that for M-ary orthogonal signals, the error probability P_e can be bounded above by:

$$P_e \;\le\; Q\left(\sqrt{2k\gamma_b} - \sqrt{2k\log(2)}\right) + \frac{Me^{-k\gamma_b/2}}{\sqrt{2}} Q\left(\frac{\sqrt{2k\log(2)} - \sqrt{k\gamma_b/2}}{\sqrt{1/2}}\right)$$

where $Q(\cdot)$ is the Q-function, $M = 2^k$, and γ_b is the signal to noise ratio per information bit. Using $Q(x) \le \frac{1}{2}e^{-x^2/2}$, prove that when $\gamma_b > \log(2)$, $\lim_{k\to\infty} P_e = 0$.

(b) (6%) In 1948, Shannon proved that

- if $R < C$, then P_e can be made arbitrarily small (by extending the code size);
- if $R > C$, then P_e is bounded away from zero,

where $C = \max_{P_X} I(X;Y)$ is the channel capacity, and R is the code rate. For AWGN channels,

$$C = W\log_2\left(1 + \frac{P}{N_0 W}\right) \quad \text{bit/second},$$

where the units of W, N_0 and P are Hz, Joule and Watt, respectively. Use this theorem to prove that for M-ary orthogonal FSK, if

$$\gamma_b < \lim_{k\to\infty} \frac{2^{2k/2^k} - 1}{2k/2^k} = \log(2),$$

then P_e is bounded away from zero.
Hint: $\gamma_b = \mathcal{E}_b/N_0$, $P = R\mathcal{E}_b$, $W = \frac{M}{2T}$ and $R = \frac{\log_2(M)}{T}$.

Solution.

(a)

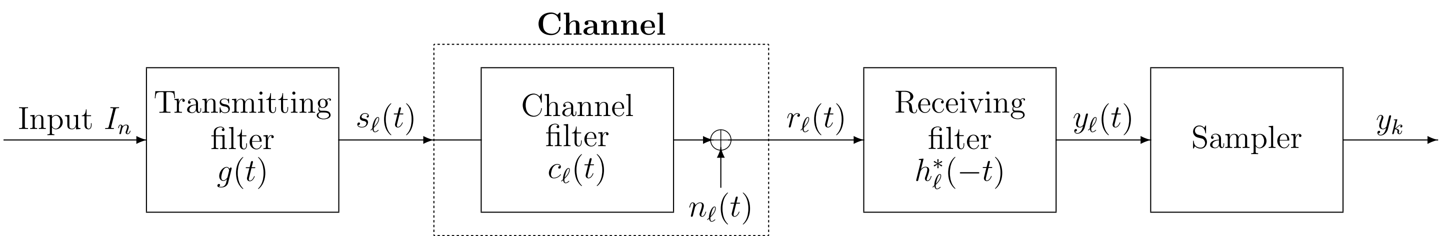
$$\begin{aligned}
P_e &\le Q\left(\sqrt{2k\gamma_b} - \sqrt{2k\log(2)}\right) + \frac{Me^{-k\gamma_b/2}}{\sqrt{2}} Q\left(\frac{\sqrt{2k\log(2)} - \sqrt{k\gamma_b/2}}{\sqrt{1/2}}\right) \\
&\le \begin{cases} \frac{1}{2} e^{-(\sqrt{2k\gamma_b} - \sqrt{2k\log(2)})^2/2} + \frac{2^k e^{-k\gamma_b/2}}{2\sqrt{2}} e^{-(\sqrt{4k\log(2)} - \sqrt{k\gamma_b})^2/2}, & \text{if } \log(2) < \gamma_b < 4\log(2) \\ \frac{1}{2} e^{-(\sqrt{2k\gamma_b} - \sqrt{2k\log(2)})^2/2} + \frac{2^k e^{-k\gamma_b/2}}{\sqrt{2}} \cdot 1, & \text{if } \gamma_b \ge 4\log(2) \end{cases} \\
&= \begin{cases} \frac{1}{2} e^{-k(\sqrt{\gamma_b} - \sqrt{\log(2)})^2} + \frac{1}{2\sqrt{2}} e^{-k(\sqrt{\gamma_b} - \sqrt{\log(2)})^2}, & \text{if } \log(2) < \gamma_b < 4\log(2) \\ \frac{1}{2} e^{-k(\sqrt{\gamma_b} - \sqrt{\log(2)})^2} + \frac{1}{\sqrt{2}} e^{-k(\gamma_b - 2\log(2))}, & \text{if } \gamma_b \ge 4\log(2) \end{cases}
\end{aligned}$$

(b) Since

$$R > C = W\log_2\left(1 + \frac{R\mathcal{E}_b}{N_0 W}\right) = W\log_2\left(1 + \tfrac{R}{W}\gamma_b\right) \Leftrightarrow \gamma_b < \frac{2^{R/W} - 1}{R/W},$$

and $R/W = \frac{2\log_2(M)}{M} = \frac{2k}{2^k}$, the desired result holds.

4. (Chapter 9)



(a) (8%) Suppose that T is the symbol period, $s_\ell(t) = \sum_{n=-\infty}^{\infty} I_n g(t - nT)$, $c_\ell(t) = \delta(t)$, $n_\ell(t) = 0$ (noise-free), $h_\ell(t) = g(t) \star c_\ell(t) = g(t)$, and

$$g(t) = \begin{cases} \frac{1}{\sqrt{u}}, & 0 \le t < u \\ 0, & \text{otherwise} \end{cases}$$

Assume that the sampler samples at time instant kT for integer k. Does the system have ISI if $u = \frac{T}{2}$? Answer the same question if $u = 2T$. You should justify your answers.

(b) (6%) Continue from (a). Draw the eye diagram for $u = \frac{T}{2}$ over the duration $[-T/2, 5T/2]$, where $I_n \in \{\pm 1\}$.

(c) (8%) Denote $x_\ell(t) = g(t) \star c_\ell(t) \star h_\ell^*(-t)$. Let $x_\ell(t)$ be a band-limited signal with band $[-W, W]$ and $x_\ell(0) = 1$. Prove that

$$x_k = x_\ell(kT) = \int_{-\infty}^{\infty} x_\ell(t)\delta(t - kT)\, dt = \delta_k$$

if and only if

$$\frac{1}{T}\sum_{m=-\infty}^{\infty} X_\ell\left(f - \frac{m}{T}\right) = 1 \qquad (1)$$

where $X_\ell(f) = \mathcal{F}\{x_\ell(t)\}$ and $\delta_k = \begin{cases} 1, & k=0 \\ 0, & k \neq 0 \end{cases}$ is the Kronecker delta function.

Hint: $\mathcal{F}\left\{ \sum_{k=-\infty}^{\infty} \delta(t-kT) \right\} = \frac{1}{T} \sum_{m=-\infty}^{\infty} \delta\left(f - \frac{m}{T}\right)$.

(d) (6%) Continue from (c). If we now shoot for controlled ISI as

$$x_k = \delta_k + \delta_{k-1} + \delta_{k-2},$$

what will be the new condition replacing (1)? No derivation is required. You may answer the question directly.

(e) (8%) In order to satisfy (1), a choice is to let $X_\ell(f) = X_{\text{rc}}(f)$, where

$$X_{rc}(f) = \begin{cases} T, & 0 \leq |f| \leq \frac{1-\beta}{2T} \\ \frac{T}{2}\left\{1 + \cos\left[\frac{\pi T}{\beta}\left(|f| - \frac{1-\beta}{2T}\right)\right]\right\}, & \frac{1-\beta}{2T} \leq |f| \leq \frac{1+\beta}{2T} \\ 0, & \text{otherwise} \end{cases}$$

where $0 \leq \beta \leq 1$ is the roll-off factor. So we have

$$\begin{cases} s_\ell(t) = \sum_{n=-\infty}^{\infty} I_n g(t-nT) \\ r_\ell(t) = c_\ell(t) \star s_\ell(t) + n_\ell(t) \\ y_\ell(t) = s_\ell(t) \star c_\ell(t) \star h_\ell^*(-t) + n_\ell(t) \star h_\ell^*(-t) = \sum_{n=-\infty}^{\infty} I_n x_{\text{rc}}(t-nT) + z_\ell(t) \\ y_k = y_\ell(kT) = I_k + z_k \end{cases}$$

where $I_n \in \{\pm d\}$, $z_\ell(t) = n_\ell(t) \star h_\ell^*(-t)$, $z_k = z_\ell(kT)$ and n_ℓ is an additive while noise with two-sided power spectrum σ_ℓ^2. Now for fixed transmission power

$$P_{av,\ell} = \frac{d^2||g(t)||^2}{T}$$

prove that

$$\frac{d}{\sqrt{\mathbb{E}[z_k^2]}} = \sqrt{\frac{P_{av,\ell}T/\sigma_\ell^2}{\|g(t)\|^2\|h_\ell(t)\|^2}}.$$

Hint: In your proof, you may directly use the fact that $S_{z_\ell}(f) = S_{n_\ell}(f)|H_\ell(f)|^2$.

(f) (8%) Continue from (e). If we wish to minimize the error rate

$$P_e = Q\left(\frac{d}{\sqrt{\mathbb{E}[z_k^2]}}\right)$$

without ISI by setting $g(t) \star c_\ell(t) \star h_\ell^*(-t) = x_{\text{rc}}(t)$, then the system design desires to maximize $\|g(t)\|^2\|h_\ell(t)\|^2$ subject to $g(t) \star c_\ell(t) \star h_\ell^*(-t) = x_{\text{rc}}(t)$, or equivalently, to maximize $\|G(f)\|^2\|H_\ell(f)\|^2$ subject to $G(f)C_\ell(f)H_\ell^*(f) = X_{\text{rc}}(f)$. If $C_\ell(f)$ is known to both transmitter and receiver, what are choices of $G(f)$ and $H_\ell^*(f)$ such that $\|G(f)\|^2\|H_\ell(f)\|^2$ is maximized.

Hint: Cauchy-Schwartz inequality and $G(f)H_\ell^*(f) = X_{\text{rc}}(f)/C_\ell(f)$.

Solutions.

(a) It can be derived that $y_\ell(t) = \sum_{n=-\infty}^{\infty} I_n x_\ell(t - nT)$, where

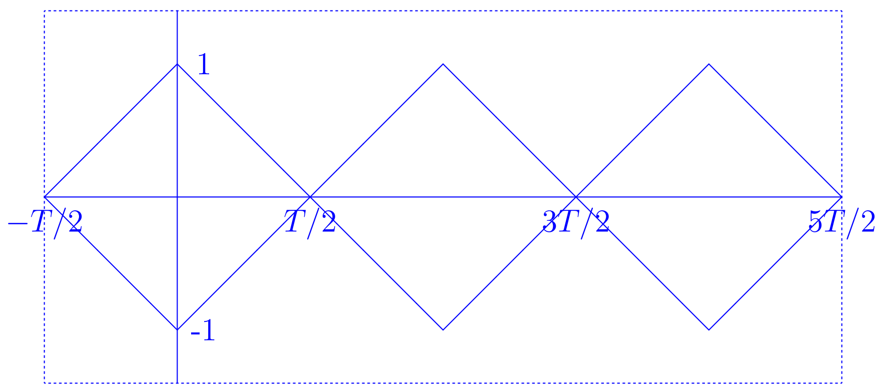
$$x_\ell(t) = h_\ell(t) \star h_\ell^*(-t) = \int_{-\infty}^{\infty} h_\ell(\tau) h_\ell^*(-(t-\tau)) d\tau = \int_{-\infty}^{\infty} g(\tau) g(\tau - t) d\tau = \begin{cases} \frac{u-|t|}{u}, & |t| \le u \\ 0, & \text{otherwise} \end{cases}$$

Thus

$$y_\ell(kT) = \sum_{n=-\infty}^{\infty} I_n x_\ell(kT - nT) = \begin{cases} I_k, & u = \frac{T}{2} \\ \frac{1}{2} I_{k-1} + I_k + \frac{1}{2} I_{k+1}, & u = 2T \end{cases}$$

Thus, there is no ISI when $u = \frac{T}{2}$, and ISI occurs when $u = 2T$.

(b)



(c) See Slide 9-25.

(d)

$$\frac{1}{T} \sum_{m=-\infty}^{\infty} X_\ell\left(f - \frac{m}{T}\right) = 1 + e^{-\imath 2\pi f T} + e^{-\imath 4\pi f T}.$$

(e)

$$\mathbb{E}[z_k^2] = \int_{-\infty}^{\infty} S_{z_\ell}(f)\, df = \int_{-\infty}^{\infty} S_{n_\ell}(f) |H_\ell(f)|^2\, df = \sigma_\ell^2 \int_{-\infty}^{\infty} |H_\ell(f)|^2\, df = \sigma_\ell^2 \|h_\ell(t)\|^2$$

So

$$\sqrt{\frac{d^2}{\mathbb{E}[z_k^2]}} = \sqrt{\frac{P_{av,\ell} T / \sigma_\ell^2}{\|g(t)\|^2 \|h_\ell(t)\|^2}}$$

(f) Cauchy-Schwartz inequality gives that

$$\|G(f)\|^2 \|H_\ell(f)\|^2 = \|G(f)\|^2 \|H_\ell^*(f)\|^2 \ge \left| \int_{-\infty}^{\infty} G(f) H_\ell^*(f) df \right|^2 = \left| \int_{-\infty}^{\infty} \frac{X_{\rm rc}(f)}{C_\ell(f)} df \right|^2$$

with equality holding when $H_\ell^*(f) = aG(f)$. So one choice is to let $a = 1$ and

$$G^2(f) = \frac{X_{\rm rc}(f)}{C_\ell(f)}, \quad \text{or equaivalently} \quad G(f) = \left| \frac{X_{\rm rc}(f)}{C_\ell(f)} \right|^{1/2} e^{\imath \angle (X_{\rm rc}(f)/C_\ell(f))/2}.$$