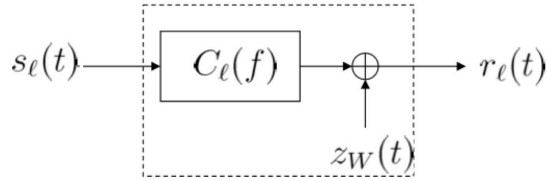


2015 Spring: The Second Midterm of Digital Communications

The total points of this exam is 110.

1.



The above figure shows an equivalent channel model with channel transfer function $C_\ell(f)$ bandlimited to W and random, and $z_W(t)$ bandlimited white noise. It can be formulated as:

$$r_\ell(t) = \int_{-\infty}^{\infty} s_\ell(f) C_\ell(f) e^{i2\pi ft} df + z_W(t), \quad (1)$$

where $s_\ell(f)$ is the Fourier transform of $s_\ell(t)$.

- (a) (6%) Since $C_\ell(f)$ is bandlimited to bandwidth W , its impulse response can be expressed using the samples $\{c_n = \frac{1}{W} c_\ell(\frac{n}{W})\}_{n=-\infty}^{\infty}$, i.e.,

$$c_\ell(t) = W \sum_{n=-\infty}^{\infty} c_n \cdot \text{sinc}\left(W\left(t - \frac{n}{W}\right)\right).$$

Show that $C_\ell(f)$ can also be expressed using the samples $\{c_n\}_{n=-\infty}^{\infty}$.

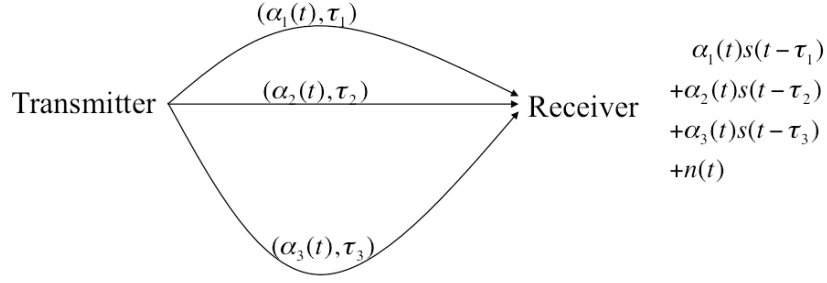
Hint: $\int_{-\infty}^{\infty} W \cdot \text{sinc}(Wt) e^{-i2\pi ft} dt = \begin{cases} 1, & |f| < \frac{W}{2} \\ 0, & \text{otherwise} \end{cases}$.

- (b) (6%) Suppose $c_n = 0$ with probability one for $n \leq 0$ and $n \geq 4$ (i.e., only c_1, c_2 and c_3 are possibly non-zero). We can then derive:

$$\begin{aligned} r_\ell(t) &= \int_{-\infty}^{\infty} s_\ell(f) C_\ell(f) e^{i2\pi ft} df + z_W(t) \\ &= \sum_{n=1}^3 c_n \int_{-W/2}^{W/2} s_\ell(f) e^{i2\pi f(t-n/W)} df + z_W(t) \\ &= \sum_{n=1}^3 c_n \cdot s_\ell\left(t - \frac{n}{W}\right) + z_W(t). \end{aligned}$$

Draw the equivalent tapped-delay-line channel model diagram of the figure above.

- (c) (6%) By borrowing the idea from (b), draw the equivalent tapped-delay-line channel model diagram of a general three-path fading channel illustrated below, where we assume $\tau_2 < \tau_1 < \tau_3$.



Hint: You may wish to replace the delay component $\boxed{\frac{1}{W}}$ in subproblem (b) with the general delay component $\boxed{\tau}$ for some proper delay τ .

2. (a) (6%) Give the PSDs of a broadband interference and a CW jammer.
 - (b) (6%) Give the definition of Jamming Margin.
 - (c) (6%) Explain what the pulsed interference is.
3. A 2-user CDMA system can be modeled as:

$$r_i = \underbrace{(2c_{m_1,i} - 1)(2b_{1,i} - 1)}_{\text{user 1}} 2\mathcal{E}_c + \underbrace{(2c_{m_2,i} - 1)(2b_{2,i} - 1)}_{\text{user 2}} 2\mathcal{E}_c + \nu_i, \quad i = 1, 2, \dots, n$$

where each $c_{m_1,i}, c_{m_2,i}, b_{1,i}, b_{2,i} \in \{0, 1\}$, $1 \leq m \leq M = 2$, and $\{\nu_i\}_{i=1}^n$ zero-mean Gaussian i.i.d. with $\mathbb{E}[\nu_i^2] = 2J_0\mathcal{E}_c$. In order to recover the information m_1 from the first user, the receiver will perform

$$y_i = (2b_{1,i} - 1) r_i = (2c_{m_1,i} - 1) 2\mathcal{E}_c + \underbrace{(2c_{m_2,i} - 1)(2b_{2,i} - 1)(2b_{1,i} - 1) 2\mathcal{E}_c + (2b_{1,i} - 1) \nu_i}_{=\nu'_i}. \quad (2)$$

- (a) (6%) Give one example of pseudo-random sequences $(b_{1,i}, \dots, b_{1,n})$ and $(b_{2,i}, \dots, b_{2,n})$ with $n \geq 7$ such that

$$\sum_{i=1}^n (2b_{1,i} - 1)(2b_{2,i} - 1) = -1. \quad (3)$$

Hint: Maximum-length shift register sequence or m-sequence with $m \geq 3$.

- (b) (8%) Suppose

$$(c_{m,1}, \dots, c_{m,n}) = \begin{cases} 000 \dots 000, & m = 1 \\ 111 \dots 111, & m = 2 \end{cases}$$

Under equal prior probability and the simplified treatment that $\{\nu'_i\}_{i=1}^n$ is zero-mean i.i.d. Gaussian distributed, i.e., (2) is simplified to

$$y_i = (2c_{m_1,i} - 1) 2\mathcal{E}_c + \nu'_i,$$

the optimal decision rule is given by:

$$\hat{m} = \arg \min_{1 \leq m \leq 2} \|\mathbf{y} - 2\mathcal{E}_c(2\mathbf{c}_m - 1)\|^2 = \arg \max_{1 \leq m \leq 2} \sum_{i=1}^n (2c_{m,i} - 1)y_i. \quad (4)$$

Prove that the error rate under decision rule (4) is equal to

$$P_{e,\text{sim}} = Q \left(\sqrt{n \frac{2\mathcal{E}_c}{2\mathcal{E}_c + J_0}} \right),$$

provided that the i.i.d. Gaussian $\{\nu'_i\}_{i=1}^n$ has mean

$$\mathbb{E}[(2c_{m_2,i} - 1)(2b_{2,i} - 1)(2b_{1,i} - 1)2\mathcal{E}_c + (2b_{1,i} - 1)\nu_i]$$

and variance

$$\text{Var}[(2c_{m_2,i} - 1)(2b_{2,i} - 1)(2b_{1,i} - 1)2\mathcal{E}_c + (2b_{1,i} - 1)\nu_i],$$

and $\{c_{m_2,i}\}_{i=1}^n$, $\{b_{1,i}\}_{i=1}^n$ and $\{b_{2,i}\}_{i=1}^n$ are uniform i.i.d. sequences and are independent to each other.

Hint: Under

$$\sum_{i=1}^n (2c_{m_2,i} - 1) y_i = (2m - 3) \sum_{i=1}^n y_i = (2m - 3) \left(n(2m_1 - 3)2\mathcal{E}_c + \sum_{i=1}^n \nu'_i \right),$$

derive $\Pr[\hat{m} = 1 | m_1 = 2]$ and $\Pr[\hat{m} = 2 | m_1 = 1]$. Note that $\Pr\{\mathcal{N}(m, \sigma^2) < r\} = Q\left(\frac{m-r}{\sigma}\right)$.

- (c) (8%) The simplified treatment in subproblem (b) is actually unreal since both $\{b_{1,i}\}_{i=1}^n$ and $\{b_{2,i}\}_{i=1}^n$ are known constant sequences to the receiver, and $\{c_{m_2,i}\}_{i=1}^n$ is constant although unknown to the receiver. Prove that the error rate under the same decision rule in subproblem (b) is actually equal to

$$P_e = \frac{1}{2}Q \left(\sqrt{2\alpha^2 \frac{\mathcal{E}_b}{J_0}} \right) + \frac{1}{2}Q \left(\sqrt{2(2-\alpha)^2 \frac{\mathcal{E}_b}{J_0}} \right),$$

where $\mathcal{E}_b = n\mathcal{E}_c$, provided that the two pseudo-random sequences satisfy

$$\sum_{i=1}^n (2b_{1,i} - 1)(2b_{2,i} - 1) = (1 - \alpha)n \text{ for some } 0 \leq \alpha \leq 2. \quad (5)$$

Hint: Under

$$\begin{aligned} \sum_{i=1}^n (2c_{m_2,i} - 1) y_i &= (2m - 3) \sum_{i=1}^n y_i \\ &= (2m - 3) \left(n(2m_1 - 3)2\mathcal{E}_c + (1 - \alpha)n(2m_2 - 3)2\mathcal{E}_c + \sum_{i=1}^n (2b_{1,i} - 1)\nu_i \right), \end{aligned}$$

derive $\Pr[\hat{m} = 1 | m_1 = 2]$ and $\Pr[\hat{m} = 2 | m_1 = 1]$.

4. (a) (8%) Suppose that random processes $\mathbf{p}_{\text{PN}}(t)$ and $\mathbf{z}(t)$ are independent to each other, and $\mathbf{z}(t)$ is a wide-sense stationary process. Show that the time-averaged autocorrelation function of the product random signal $\mathbf{p}_{\text{PN}}(t)\mathbf{z}(t)$ is given by

$$\bar{R}_{\mathbf{p} \times \mathbf{z}}(\tau) = R_{\mathbf{z}}(\tau) \bar{R}_{\mathbf{p}}(\tau)$$

where $R_{\mathbf{z}}(\tau)$ is the autocorrelation function of $\mathbf{z}(t)$, and $\bar{R}_{\mathbf{p}}(\tau)$ is the time-averaged autocorrelation function of $\mathbf{p}_{\text{PN}}(t)$.

Hint: No statistical assumption on $\mathbf{p}_{\text{PN}}(t)$ is given except that its time-averaged PSD exists.

(b) (8%) From Chapter 3, we learn that the time-averaged PSD of

$$\mathbf{c}(t) = \sum_{n=-\infty}^{\infty} \mathbf{I}_n s(t - nT_b)$$

is equal to

$$\bar{S}_{\mathbf{c}}(f) = \frac{1}{T_b} S_{\mathbf{I}}(f) |S(f)|^2$$

where $S_{\mathbf{I}}(f) = \sum_{k=-\infty}^{\infty} R_{\mathbf{I}}(k) e^{-i2\pi k f T}$ is the PSD of $\{\mathbf{I}_n\}_{n=-\infty}^{\infty}$, $R_{\mathbf{I}}(k) = \mathbb{E}[\mathbf{I}_{m+k} \mathbf{I}_m^*]$ is the autocorrelation function of WSS $\{\mathbf{I}_n\}_{n=-\infty}^{\infty}$, and $S(f) = \int_{-\infty}^{\infty} s(t) e^{-i2\pi f t} dt$ is the spectrum of $s(t)$. Prove that

$$\bar{S}_{\mathbf{c}}(f) = \frac{2}{T_b} |G(f)|^2 (1 + \cos(2\pi f T_c))$$

if $\{\mathbf{I}_n\}_{n=-\infty}^{\infty}$ are uniform i.i.d. with $\mathbf{I}_n \in \{-1, 1\}$, and

$$s(t) = \begin{cases} g(t \bmod T_c) & 0 \leq t < T_b \\ 0 & \text{otherwise} \end{cases},$$

where $g(t) \neq 0$ only when $0 \leq t < T_c$, and $T_b/T_c = 2$.

Hint: Note that $s(t) = \sum_{n=0}^1 g(t - nT_c)$, and $G(f) = \int_0^{T_c} g(t) e^{-i2\pi f t} dt$.

(c) (8%) Continue from (b). After spreading $\mathbf{c}(t)$ with (deterministically chosen) PN sequence $(b_0, b_1) = (1, 0)$ (which accordingly satisfies the balanced property), we have

$$p_{\text{PN}}(t) \mathbf{c}(t) = \sum_{i=-\infty}^{\infty} (2b_i - 1) \mathbf{I}_{\lfloor i/2 \rfloor} p(t - iT_c) s(t - 2\lfloor i/2 \rfloor T_c),$$

where $p(t)$ is a rectangular pulse of height 1 and duration T_c , and $b_i = b_{i \bmod 2}$. Prove that

$$\bar{S}_{p \times \mathbf{c}}(f) = \frac{2}{T_b} |G(f)|^2 (1 - \cos(2\pi f T_c)).$$

Hint: In this derivation, we treat the PN sequence as a deterministic sequence rather than the textbook-assumed statistical sequence. Find $\tilde{g}(t)$ such that

$$p_{\text{PN}}(t) \mathbf{c}(t) = \sum_{k=-\infty}^{\infty} \mathbf{I}_k \tilde{g}(t - kT_b)$$

Then

$$\bar{S}_{p \times \mathbf{c}}(f) = \frac{1}{T_b} S_{\mathbf{I}}(f) |\tilde{G}(f)|^2.$$

5. For a lowpass signal of the form,

$$s_{\ell}(t) = \kappa \sum_{n=-\infty}^{\infty} \left(\sum_{k=0}^{Q-1} X_{k,n} e^{i2\pi \frac{k}{T} t} \right) g(t - nT)$$

answer the following questions.

- (a) (6%) Explain that why the signal can be regarded as a Q -carrier system with single-carrier implementation. What are the Q carrier frequencies in this system?
- (b) (8%) If $\{X_{k,n}\}_{-\infty < k,n < \infty}$ are zero-mean i.i.d. with variance σ^2 , show that

$$\bar{S}_{s_\ell}(f) = \frac{\sigma^2}{T^2} \sum_{k=0}^{Q-1} \left| G\left(f - \frac{k}{T}\right) \right|^2.$$

- (c) (6%) Why adding cyclic prefix (CP) to the signal $s_\ell(t)$ before its transmission?
Hint: It is relevant to the channel impulse response $c_\ell(t)$.
- (d) (8%) Show that with CP technique, the noiseless reception from the channel with impulse response $c_\ell(t)$ is given by

$$r_\ell(t) = \kappa \sum_{k=0}^{Q-1} C_\ell\left(\frac{k}{T}\right) X_k e^{i 2\pi \frac{km}{N}}$$

where $C_\ell(f)$ is the spectrum of $c_\ell(t)$.

Hint: $r_\ell(t) = \tilde{s}_\ell(t) \star c_\ell(t)$, where $\tilde{s}_\ell(t)$ is the periodic counterpart of $s_\ell(t)$.