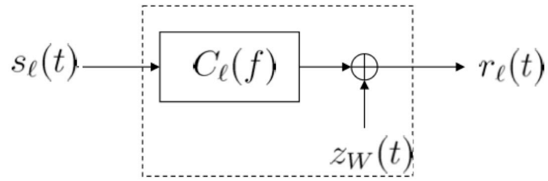
2015 Spring: The Second Midterm of Digital Communications

The total points of this exam is 110.

1.



The above figure shows an equivalent channel model with channel transfer function $C_\ell(f)$ bandlimited to W and random, and $z_W(t)$ bandlimited white noise. It can be formulated as:

$$r_\ell(t) = \int_{-\infty}^{\infty} s_\ell(f) C_\ell(f) e^{\imath 2\pi f t} df + z_W(t), \tag{1}$$

where $s_\ell(f)$ is the Fourier transform of $s_\ell(t)$.

(a) (6%) Since $C_\ell(f)$ is bandlimited to bandwidth W, its impulse response can be expressed using the samples $\{c_n = \frac{1}{W} c_\ell\left(\frac{n}{W}\right)\}_{n=-\infty}^{\infty}$, i.e.,

$$c_\ell(t) = W \sum_{n=-\infty}^{\infty} c_n \cdot \text{sinc}\left(W\left(t - \frac{n}{W}\right)\right).$$

Show that $C_\ell(f)$ can also be expressed using the samples $\{c_n\}_{n=-\infty}^{\infty}$.

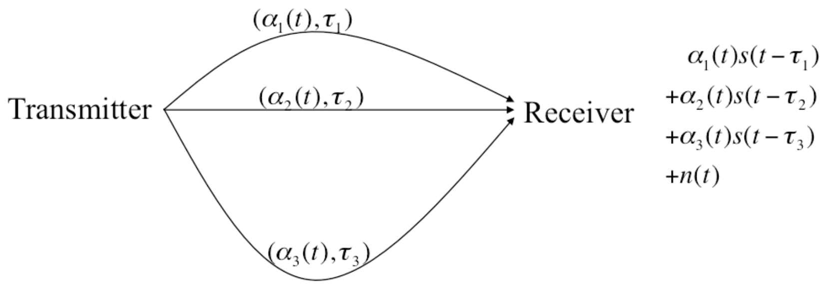
Hint: $\int_{-\infty}^{\infty} W \cdot \text{sinc}(Wt) e^{-\imath 2\pi f t} dt = \begin{cases} 1, & |f| < \frac{W}{2} \\ 0, & \text{otherwise} \end{cases}.$

(b) (6%) Suppose $c_n = 0$ with probability one for $n \le 0$ and $n \ge 4$ (i.e., only c_1, c_2 and c_3 are possibly non-zero). We can then derive:

$$\begin{aligned} r_\ell(t) &= \int_{-\infty}^{\infty} s_\ell(f) C_\ell(f) e^{\imath 2\pi f t} df + z_W(t) \\ &= \sum_{n=1}^{3} c_n \int_{-W/2}^{W/2} s_\ell(f) e^{\imath 2\pi f (t - n/W)} df + z_W(t) \\ &= \sum_{n=1}^{3} c_n \cdot s_\ell\left(t - \frac{n}{W}\right) + z_W(t). \end{aligned}$$

Draw the equivalent tapped-delay-line channel model diagram of the figure above.

(c) (6%) By borrowing the idea from (b), draw the equivalent tapped-delay-line channel model diagram of a general three-path fading channel illustrated below, where we assume $\tau_2 < \tau_1 < \tau_3$.



Hint: You may wish to replace the delay component $\boxed{\frac{1}{W}}$ *in subproblem (b) with the general delay component* $\boxed{\tau}$ *for some proper delay* τ.

Solutions.

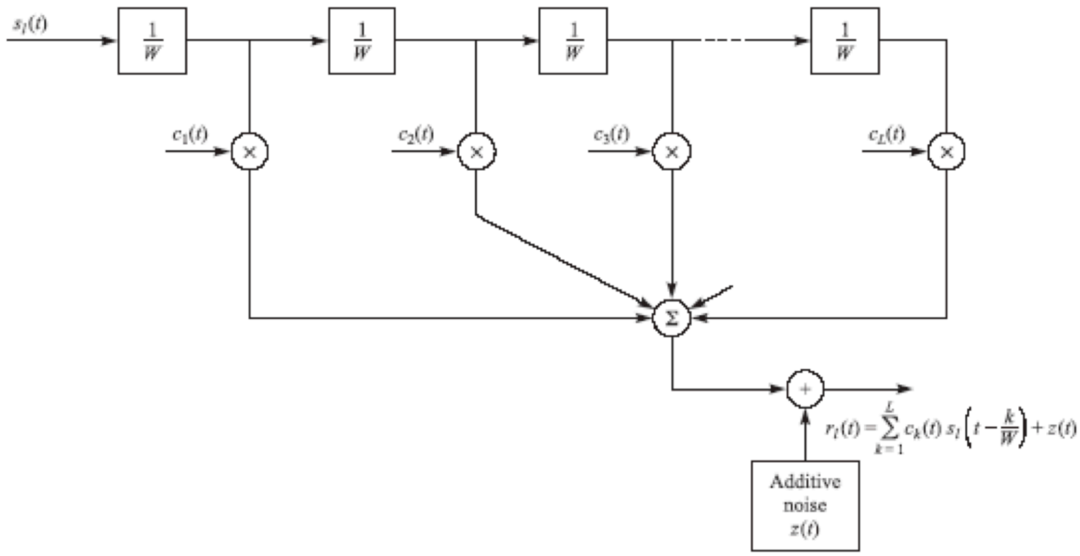
(a)

$$\begin{aligned}
C_\ell(f) &= \int_{-\infty}^{\infty} c_\ell(t) e^{-\imath 2\pi f t} dt \\
&= \int_{-\infty}^{\infty} \left(W \sum_{n=-\infty}^{\infty} c_n \cdot \operatorname{sinc}\left(W \left(t - \frac{n}{W} \right) \right) \right) e^{-\imath 2\pi f t} dt \\
&= \sum_{n=-\infty}^{\infty} c_n \int_{-\infty}^{\infty} W \cdot \operatorname{sinc}\left(W \left(t - \frac{n}{W} \right) \right) e^{-\imath 2\pi f t} dt \quad (\text{Let } u = t - \frac{n}{W}) \\
&= \sum_{n=-\infty}^{\infty} c_n \int_{-\infty}^{\infty} W \cdot \operatorname{sinc}(Wu) e^{-\imath 2\pi f (u + n/W)} du \\
&= \begin{cases} \displaystyle\sum_{n=-\infty}^{\infty} c_n e^{-\imath 2\pi f n/W}, & |f| \le W/2 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

(b)

$$\begin{aligned}
r_\ell(t) &= \int_{-\infty}^{\infty} s_\ell(f) C_\ell(f) e^{\imath 2\pi f t} df + z_W(t) \\
&= \sum_{n=1}^{3} c_n \int_{-W/2}^{W/2} s_\ell(f) e^{\imath 2\pi f (t - n/W)} df + z_W(t) \\
&= \sum_{n=1}^{3} c_n \cdot s_\ell\left(t - \frac{n}{W} \right) + z_W(t).
\end{aligned}$$

The equivalent tapped-delay-line channel model diagram is shown below with $L = 3$ and $(c_1(t), c_2(t), c_3(t)) = (c_1, c_2, c_3)$.



(c) The same as in subproblem (b) except $(c_1(t), c_2(t), c_3(t)) = (\alpha_2(t), \alpha_1(t), \alpha_3(t))$ and the three delay components become $\boxed{\tau_2}$, $\boxed{\tau_1 - \tau_2}$ and $\boxed{\tau_3 - \tau_1}$ (from left to right), respectively.

2. (a) (6%) Give the PSDs of a broadband interference and a CW jammer.

   (b) (6%) Give the definition of Jamming Margin.

   (c) (6%) ~~Explain what the pulsed interference is.~~ Sorry! This problem should be out of the scope of the final exam. So every one gets 6 points.

Solutions.

(a) $z(t)$ is a WSS broadband interference if PSD of $z(t)$ is

$$S_z(f) = 2J_0 \quad \text{for } |f| \le \frac{W}{2}.$$

Alternatively, $z(t)$ is a (WSS) CW (continuous wave) interference if PSD of $z(t)$ is

$$S_z(f) = J_{\text{av}}\delta(f)$$

(b) See Slide 12-37.

(c) See Slide 12-55.

3. A 2-user CDMA system can be modeled as:

$$r_i = \underbrace{(2c_{m_1,i} - 1)(2b_{1,i} - 1)\, 2\mathcal{E}_c}_{\text{user 1}} + \underbrace{(2c_{m_2,i} - 1)(2b_{2,i} - 1)\, 2\mathcal{E}_c}_{\text{user 2}} + \nu_i, \quad i = 1, 2, \ldots, n$$

where each $c_{m_1,i}$, $c_{m_2,i}$, $b_{1,i}$, $b_{2,i} \in \{0, 1\}$, $1 \le m \le M = 2$, and $\{\nu_i\}_{i=1}^n$ zero-mean Gaussian i.i.d. with $\mathbb{E}[\nu_i^2] = 2J_0\mathcal{E}_c$. In order to recover the information m_1 from the first user, the receiver will perform

$$y_i = (2b_{1,i} - 1)\, r_i = (2c_{m_1,i} - 1)\, 2\mathcal{E}_c + \underbrace{(2c_{m_2,i} - 1)(2b_{2,i} - 1)(2b_{1,i} - 1)\, 2\mathcal{E}_c + (2b_{1,i} - 1)\, \nu_i}_{=\nu_i'}. \quad (2)$$

(a) (6%) Give one example of pseudo-random sequences $(b_{1,i}, \ldots, b_{1,n})$ and $(b_{2,i}, \ldots, b_{2,n})$ with $n \geq 7$ such that

$$\sum_{i=1}^{n}(2b_{1,i}-1)(2b_{2,i}-1) = -1. \tag{3}$$

Hint: Maximum-length shift register sequence or m-sequence with $m \geq 3$.

(b) (8%) Suppose

$$(c_{m,1}, \ldots, c_{m,n}) = \begin{cases} 000\cdots 000, & m=1 \\ 111\cdots 111, & m=2 \end{cases}$$

Under equal prior probability and the simplified treatment that $\{\nu_i'\}_{i=1}^n$ is zero-mean i.i.d. Gaussian distributed, i.e., (2) is simplified to

$$y_i = (2c_{m_1,i}-1)\,2\mathcal{E}_c + \nu_i',$$

the optimal decision rule is given by:

$$\hat{m} = \arg\min_{1\leq m\leq 2} \|\boldsymbol{y} - 2\mathcal{E}_c(2\boldsymbol{c}_m - 1)\|^2 = \arg\max_{1\leq m\leq 2}\sum_{i=1}^{n}(2c_{m,i}-1)y_i. \tag{4}$$

Prove that the error rate under decision rule (4) is equal to

$$P_{e,\text{sim}} = Q\left(\sqrt{n\frac{2\mathcal{E}_c}{2\mathcal{E}_c + J_0}}\right),$$

provided that the i.i.d. Gaussian $\{\nu_i'\}_{i=1}^n$ has mean

$$\mathbb{E}\left[(2c_{m_2,i}-1)(2b_{2,i}-1)(2b_{1,i}-1)\,2\mathcal{E}_c + (2b_{1,i}-1)\,\nu_i\right]$$

and variance

$$\text{Var}\left[(2c_{m_2,i}-1)(2b_{2,i}-1)(2b_{1,i}-1)\,2\mathcal{E}_c + (2b_{1,i}-1)\,\nu_i\right],$$

and $\{c_{m_2,i}\}_{i=1}^n$, $\{b_{1,i}\}_{i=1}^n$ and $\{b_{2,i}\}_{i=1}^n$ are uniform i.i.d. sequences and are independent to each other.

Hint: Under

$$\sum_{i=1}^{n}(2c_{m,i}-1)\,y_i = (2m-3)\sum_{i=1}^{n}y_i = (2m-3)\left(n(2m_1-3)2\mathcal{E}_c + \sum_{i=1}^{n}\nu_i'\right),$$

derive $\Pr[\hat{m}=1|m_1=2]$ *and* $\Pr[\hat{m}=2|m_1=1]$. *Note that* $\Pr\{\mathcal{N}(m,\sigma^2) < r\} = Q\left(\frac{m-r}{\sigma}\right)$.

(c) (8%) The simplified treatment in subproblem (b) is actually unreal since both $\{b_{1,i}\}_{i=1}^n$ and $\{b_{2,i}\}_{i=1}^n$ are known constant sequences to the receiver, and $\{c_{m_2,i}\}_{i=1}^n$ is constant although unknown to the receiver. Prove that the error rate under the same decision rule in subproblem (b) is actually equal to

$$P_e = \frac{1}{2}Q\left(\sqrt{2\alpha^2\frac{\mathcal{E}_b}{J_0}}\right) + \frac{1}{2}Q\left(\sqrt{2(2-\alpha)^2\frac{\mathcal{E}_b}{J_0}}\right),$$

where $\mathcal{E}_b = n\mathcal{E}_c$, provided that the two pseudo-random sequences satisfy

$$\sum_{i=1}^{n}(2b_{1,i}-1)(2b_{2,i}-1) = (1-\alpha)n \text{ for some } 0 \le \alpha \le 2. \tag{5}$$

Hint: Under

$$\begin{aligned}\sum_{i=1}^{n}(2c_{m,i}-1)\,y_i &= (2m-3)\sum_{i=1}^{n} y_i \\ &= (2m-3)\left(n(2m_1-3)2\mathcal{E}_c + (1-\alpha)n\,(2m_2-3)\,2\mathcal{E}_c + \sum_{i=1}^{n}(2b_{1,i}-1)\,\nu_i\right),\end{aligned}$$

derive $\Pr[\hat{m}=1|m_1=2]$ *and* $\Pr[\hat{m}=2|m_1=1]$.*j*

Solutions.

(a) From Slide 3-74, we can choose any two non-zero sequences as the answer. For example, $(b_{1,i},\ldots,b_{1,n}) = (0011101)$ and $(b_{2,i},\ldots,b_{2,n}) = (01000111)$.

(b) First, derive

$$\begin{aligned}\mathbb{E}[\nu_i'] &= \mathbb{E}\left[(2c_{m_2,i}-1)\,(2b_{2,i}-1)\,(2b_{1,i}-1)\,2\mathcal{E}_c + (2b_{1,i}-1)\,\nu_i\right] \\ &= \mathbb{E}\left[(2c_{m_2,i}-1)\right]\mathbb{E}\left[(2b_{2,i}-1)\right]\mathbb{E}\left[(2b_{1,i}-1)\right]2\mathcal{E}_c + \mathbb{E}\left[(2b_{1,i}-1)\right]\mathbb{E}\left[\nu_i\right] = 0\end{aligned}$$

and

$$\begin{aligned}\mathrm{Var}[\nu_i'] &= \mathbb{E}[(\nu_i')^2] \\ &= 4\mathcal{E}_c^2\mathbb{E}\left[(2c_{m_2,i}-1)^2\right]\mathbb{E}\left[(2b_{2,i}-1)^2\right]\mathbb{E}\left[(2b_{1,i}-1)^2\right] + \mathbb{E}\left[(2b_{1,i}-1)^2\right]\mathbb{E}\left[\nu_i^2\right] \\ &= 4\mathcal{E}_c^2 + \mathbb{E}\left[\nu_i^2\right] = 4\mathcal{E}_c^2 + 2J_0\mathcal{E}_c.\end{aligned}$$

Then

$$\begin{aligned}\Pr[\hat{m}=2|m_1=1] &= \Pr\left(-\left(-n2\mathcal{E}_c + \sum_{i=1}^{n}\nu_i'\right) < \left(-n2\mathcal{E}_c + \sum_{i=1}^{n}\nu_i'\right)\right) \\ &= \Pr\left(\sum_{i=1}^{n}\nu_i' > n2\mathcal{E}_c\right) \\ &= \Pr\left(\sum_{i=1}^{n}\nu_i' < -n2\mathcal{E}_c\right) \\ &= Q\left(\frac{n2\mathcal{E}_c}{\sqrt{n(4\mathcal{E}_c^2 + 2J_0\mathcal{E}_c)}}\right) \\ &= Q\left(\sqrt{n\frac{2\mathcal{E}_c}{2\mathcal{E}_c + J_0}}\right)\end{aligned}$$

and

$$\begin{aligned}
\Pr[\hat{m}=1|m_1=2] &= \Pr\left(-\left(n2\mathcal{E}_c+\sum_{i=1}^{n}\nu_i'\right) > \left(n2\mathcal{E}_c+\sum_{i=1}^{n}\nu_i'\right)\right) \\
&= \Pr\left(\sum_{i=1}^{n}\nu_i' < -n2\mathcal{E}_c\right) \\
&= Q\left(\sqrt{n\frac{2\mathcal{E}_c}{2\mathcal{E}_c+J_0}}\right).
\end{aligned}$$

Consequently,

$$P_e = Q\left(\sqrt{n\frac{2\mathcal{E}_c}{2\mathcal{E}_c+J_0}}\right).$$

Hence, by a similar simplification, a K-user CDMA system will have performance

$$P_e = Q\left(\sqrt{n\frac{2\mathcal{E}_c}{2\mathcal{E}_c(K-1)+J_0}}\right).$$

(c) If $m_1 = 1$ is transmitted, then

$$\begin{aligned}
\sum_{i=1}^{n}(2c_{m,i}-1)\,y_i &= \sum_{i=1}^{n}(2(m-1)-1)\,y_i = (2m-3)\sum_{i=1}^{n}y_i \\
&= (2m-3)\left(-n2\mathcal{E}_c+(1-\alpha)n\,(2m_2-3)\,2\mathcal{E}_c+\sum_{i=1}^{n}(2b_{1,i}-1)\,\nu_i\right),
\end{aligned}$$

and

$$\begin{aligned}
\Pr[\hat{m}=2|m_1=1] &= \Pr\left[-\left(-n2\mathcal{E}_c+(1-\alpha)n\,(2m_2-3)\,2\mathcal{E}_c+\sum_{i=1}^{n}(2b_{1,i}-1)\,\nu_i\right)\right.\\
&\qquad\left.<\left(-n2\mathcal{E}_c+(1-\alpha)n\,(2m_2-3)\,2\mathcal{E}_c+\sum_{i=1}^{n}(2b_{1,i}-1)\,\nu_i\right)\right]\\
&= \Pr\left[\sum_{i=1}^{n}(2b_{1,i}-1)\,\nu_i>(1-(1-\alpha)(2m_2-3))n2\mathcal{E}_c\right]\\
&= \Pr\left[\sum_{i=1}^{n}\nu_i>(1-(1-\alpha)(2m_2-3))n2\mathcal{E}_c\right] \qquad (6)\\
&= \Pr\left[\sum_{i=1}^{n}\nu_i<-(1-(1-\alpha)(2m_2-3))n2\mathcal{E}_c\right]\\
&= Q\left(\frac{(1-(1-\alpha)(2m_2-3))n2\mathcal{E}_c}{\sqrt{n2J_0\mathcal{E}_c}}\right)\\
&= Q\left(\sqrt{2(1-(1-\alpha)(2m_2-3))^2\frac{n\mathcal{E}_c}{J_0}}\right)\\
&= Q\left(\sqrt{2(1-(1-\alpha)(2m_2-3))^2\frac{\mathcal{E}_b}{J_0}}\right),
\end{aligned}$$

where (6) holds because the receiver knows $(b_{1,1},\ldots,b_{1,n})$ and hence $\sum_{i=1}^{n}(2b_{1,i}-1)\,\nu_i$ has the same distribution as $\sum_{i=1}^{n}\nu_i$. Similarly, if $m_2=1$ is transmitted, then

$$\sum_{i=1}^{n}(2c_{m,i}-1)\,y_i = (2m-3)\left(n2\mathcal{E}_c+(1-\alpha)n\,(2m_2-3)\,2\mathcal{E}_c+\sum_{i=1}^{n}(2b_{1,i}-1)\,\nu_i\right),$$

and

$$\begin{aligned}
\Pr[\hat{m}=1|m_1=2] &= \Pr\left[-\left(n2\mathcal{E}_c+(1-\alpha)n\,(2m_2-3)\,2\mathcal{E}_c+\sum_{i=1}^{n}(2b_{1,i}-1)\,\nu_i\right)\right.\\
&\qquad\left.>\left(n2\mathcal{E}_c+(1-\alpha)n\,(2m_2-3)\,2\mathcal{E}_c+\sum_{i=1}^{n}(2b_{1,i}-1)\,\nu_i\right)\right]\\
&= \Pr\left[\sum_{i=1}^{n}(2b_{1,i}-1)\,\nu_i<-(1+(1-\alpha)(2m_2-3))n2\mathcal{E}_c\right]\\
&= \Pr\left[\sum_{i=1}^{n}\nu_i<-(1+(1-\alpha)(2m_2-3))n2\mathcal{E}_c\right]\\
&= Q\left(\frac{(1+(1-\alpha)(2m_2-3))n2\mathcal{E}_c}{\sqrt{n2J_0\mathcal{E}_c}}\right)\\
&= Q\left(\sqrt{2(1+(1-\alpha)(2m_2-3))^2\frac{n\mathcal{E}_c}{J_0}}\right)\\
&= Q\left(\sqrt{2(1+(1-\alpha)(2m_2-3))^2\frac{\mathcal{E}_b}{J_0}}\right).
\end{aligned}$$

Consequently, under equal prior probability,

$$\begin{aligned} P_e &= \frac{1}{2}\Pr[\hat{m}=2|m_1=1]+\frac{1}{2}\Pr[\hat{m}=1|m_1=2] \\ &= \frac{1}{2}Q\left(\sqrt{2(1-(1-\alpha)(2m_2-3))^2\frac{\mathcal{E}_b}{J_0}}\right)+\frac{1}{2}Q\left(\sqrt{2(1+(1-\alpha)(2m_2-3))^2\frac{\mathcal{E}_b}{J_0}}\right) \\ &= \frac{1}{2}Q\left(\sqrt{2\alpha^2\frac{\mathcal{E}_b}{J_0}}\right)+\frac{1}{2}Q\left(\sqrt{2(2-\alpha)^2\frac{\mathcal{E}_b}{J_0}}\right). \end{aligned}$$

4. (a) (8%) Suppose that random processes $\boldsymbol{p}_{\text{PN}}(t)$ and $\boldsymbol{z}(t)$ are independent to each other, and $\boldsymbol{z}(t)$ is a wide-sense stationary process. Show that the time-averaged autocorrelation function of the product random signal $\boldsymbol{p}_{\text{PN}}(t)z(t)$ is given by

$$\bar{R}_{\boldsymbol{p}\times\boldsymbol{z}}(\tau)=R_{\boldsymbol{z}}(\tau)\bar{R}_{\boldsymbol{p}}(\tau)$$

where $R_{\boldsymbol{z}}(\tau)$ is the autocorrelation function of $\boldsymbol{z}(t)$, and $\bar{R}_{\boldsymbol{p}}(\tau)$ is the time-averaged autocorrelation function of $\boldsymbol{p}_{\text{PN}}(t)$.

Hint: No statistical assumption on $\boldsymbol{p}_{\text{PN}}(t)$ is given except that its time-averaged PSD exists.

(b) (8%) From Chapter 3, we learn that the time-averaged PSD of

$$\boldsymbol{c}(t)=\sum_{n=-\infty}^{\infty}\boldsymbol{I}_n s(t-nT_b)$$

is equal to

$$\bar{S}_{\boldsymbol{c}}(f)=\frac{1}{T_b}S_{\boldsymbol{I}}(f)|S(f)|^2$$

where $S_{\boldsymbol{I}}(f)=\sum_{k=-\infty}^{\infty}R_{\boldsymbol{I}}(k)e^{-\imath 2\pi kfT}$ is the PSD of $\{\boldsymbol{I}_n\}_{n=-\infty}^{\infty}$, $R_{\boldsymbol{I}}(k)=\mathbb{E}\left[\boldsymbol{I}_{m+k}\boldsymbol{I}_m^*\right]$ is the autocorrelation function of WSS $\{\boldsymbol{I}_n\}_{n=-\infty}^{\infty}$, and $S(f)=\int_{-\infty}^{\infty}s(t)e^{-\imath 2\pi ft}dt$ is the spectrum of $s(t)$. Prove that

$$\bar{S}_{\boldsymbol{c}}(f)=\frac{2}{T_b}|G(f)|^2(1+\cos(2\pi fT_c))$$

if $\{\boldsymbol{I}_n\}_{n=-\infty}^{\infty}$ are uniform i.i.d. with $\boldsymbol{I}_n\in\{-1,1\}$, and

$$s(t)=\begin{cases} g(t \bmod T_c) & 0\le t<T_b \\ 0 & \text{otherwise}\end{cases},$$

where $g(t)\neq 0$ only when $0\le t<T_c$, and $T_b/T_c=2$.

Hint: Note that $s(t)=\sum_{n=0}^{1}g(t-nT_c)$, and $G(f)=\int_0^{T_c}g(t)e^{-\imath 2\pi ft}dt$.

(c) (8%) Continue from (b). After spreading $\boldsymbol{c}(t)$ with (deterministically chosen) PN sequence $(b_0,b_1)=(1,0)$ (which accordingly satisfies the balanced property), we have

$$p_{\text{PN}}(t)\boldsymbol{c}(t) = \sum_{i=-\infty}^{\infty}(2b_i-1)\boldsymbol{I}_{\lfloor i/2\rfloor}p(t-iT_c)s(t-2\lfloor i/2\rfloor T_c),$$

where $p(t)$ is a rectangular pulse of height 1 and duration T_c, and $b_i=b_{i \bmod 2}$. Prove that

$$\bar{S}_{p\times\boldsymbol{c}}(f)=\frac{2}{T_b}|G(f)|^2(1-\cos(2\pi fT_c)).$$

Hint: In this derivation, we treat the PN sequence as a deterministic sequence rather than the textbook-assumed statistical sequence. Find $\tilde{g}(t)$ *such that*

$$p_{PN}(t)\boldsymbol{c}(t) = \sum_{k=-\infty}^{\infty} \boldsymbol{I}_k \tilde{g}(t - kT_b)$$

Then

$$\bar{S}_{p\times\boldsymbol{c}}(f) = \frac{1}{T_b} S_{\boldsymbol{I}}(f)|\tilde{G}(f)|^2.$$

Solutions.

(a)

$$\begin{aligned} R_{\boldsymbol{p}\times\boldsymbol{z}}(t_1, t_2) &= \mathbb{E}[\boldsymbol{p}_{\text{PN}}(t_1)\boldsymbol{z}(t_1)\boldsymbol{p}^*_{\text{PN}}(t_2)\boldsymbol{z}^*(t_2)] \\ &= \mathbb{E}[\boldsymbol{p}_{\text{PN}}(t_1)\boldsymbol{p}^*_{\text{PN}}(t_2)]\mathbb{E}[\boldsymbol{z}(t_1)\boldsymbol{z}^*(t_2)] \\ &= R_{\boldsymbol{p}}(t_1, t_2)R_{\boldsymbol{z}}(t_1, t_2). \end{aligned}$$

Hence,

$$\begin{aligned} \bar{R}_{\boldsymbol{p}\times\boldsymbol{z}}(\tau) &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} R_{\boldsymbol{p}\times\boldsymbol{z}}(t+\tau, t)dt \\ &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} R_{\boldsymbol{p}}(t+\tau, t)R_{\boldsymbol{z}}(t+\tau, t)dt \\ &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} R_{\boldsymbol{p}}(t+\tau, t)R_{\boldsymbol{z}}(\tau)dt \\ &= R_{\boldsymbol{z}}(\tau) \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} R_{\boldsymbol{p}}(t+\tau, t)dt \\ &= R_{\boldsymbol{z}}(\tau)\bar{R}_{\boldsymbol{p}}(\tau). \end{aligned}$$

(b) For uniform i.i.d. $\{\boldsymbol{I}_n\}_{n=-\infty}^{\infty}$ with $\boldsymbol{I}_n \in \{\pm 1\}$,

$$R_{\boldsymbol{I}}(k) = \mathbb{E}\left[\boldsymbol{I}_{m+k}\boldsymbol{I}^*_m\right] = \begin{cases} 0, & k \neq 0 \\ 1, & k = 0 \end{cases}$$

Thus, $S_{\boldsymbol{I}}(f) = \sum_{k=-\infty}^{\infty} R_{\boldsymbol{I}}(k) e^{-\imath 2\pi k f T_b} = 1$. We then derive

$$\begin{aligned} S(f) &= \int_{-\infty}^{\infty} s(t) e^{-\imath 2\pi f t} dt \\ &= \int_{-\infty}^{\infty} \sum_{n=0}^{1} g(t - nT_c) e^{-\imath 2\pi f t} dt \\ &= \sum_{n=0}^{1} \int_{-\infty}^{\infty} g(t - nT_c) e^{-\imath 2\pi f t} dt \\ &= G(f) \sum_{n=0}^{1} e^{-\imath 2\pi f n T_c} \\ &= G(f) \left(1 + e^{-\imath 2\pi f T_c}\right) \\ &= G(f) e^{-\imath \pi f T_c} \left(e^{\imath \pi f T_c} + e^{-\imath \pi f T_c}\right) \\ &= G(f) 2 \cos(\pi f T_c) e^{-\imath \pi f T_c}. \end{aligned}$$

Consequently,

$$\begin{aligned} \bar{S}_{\boldsymbol{c}}(f) &= \frac{1}{T_b} S_{\boldsymbol{I}}(f) |S(f)|^2 \\ &= \frac{4}{T_b} |G(f)|^2 \cos^2(\pi f T_c) \\ &= \frac{2}{T_b} |G(f)|^2 \left[1 + \cos(2\pi f T_c)\right]. \end{aligned}$$

(c)

$$\begin{aligned} p_{\text{PN}}(t)\boldsymbol{c}(t) &= \sum_{i=-\infty}^{\infty} (2b_i - 1) \boldsymbol{I}_{\lfloor i/2 \rfloor} p(t - iT_c) s(t - 2\lfloor i/2 \rfloor T_c) \\ &= \sum_{i=-\infty}^{\infty} (2b_i - 1) \boldsymbol{I}_{\lfloor i/2 \rfloor} p(t - iT_c) \sum_{n=0}^{1} g(t - 2\lfloor i/2 \rfloor T_c - nT_c) \\ &= \sum_{k=-\infty}^{\infty} (2b_{2k} - 1) \boldsymbol{I}_k p(t - 2kT_c) \sum_{n=0}^{1} g(t - 2kT_c - nT_c) \\ &\quad + \sum_{k=-\infty}^{\infty} (2b_{2k+1} - 1) \boldsymbol{I}_k p(t - (2k+1)T_c) \sum_{n=0}^{1} g(t - 2kT_c - nT_c) \\ &= \sum_{k=-\infty}^{\infty} \boldsymbol{I}_k \left(g(t - 2kT_c) + g(t - (2k+1)T_c)\right) \\ &\quad \times \left[(2b_{2k} - 1) p(t - 2kT_c) + (2b_{2k+1} - 1) p(t - (2k+1)T_c)\right] \\ &= \sum_{k=-\infty}^{\infty} \boldsymbol{I}_k \tilde{g}(t - kT_b) \end{aligned}$$

where

$$\begin{aligned}
\tilde{g}(t) &= [g(t)+g(t-T_c)]\,[(2b_0-1)p(t)+(2b_1-1)p(t-T_c)] \\
&= (2b_0-1)g(t)p(t)+(2b_1-1)g(t-T_c)p(t-T_c) \\
&= (2b_0-1)g(t)+(2b_1-1)g(t-T_c).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\tilde{G}(f) &= (2b_0-1)G(f)+(2b_1-1)G(f)e^{-\imath 2\pi f T_c} \\
&= G(f)\left[(2b_0-1)+(2b_1-1)e^{-\imath 2\pi f T_c}\right] \\
&= G(f)e^{-\imath\pi f T_c}\left[(2b_0-1)e^{\imath\pi f T_c}+(2b_1-1)e^{-\imath\pi f T_c}\right] \\
&= G(f)e^{-\imath\pi f T_c}\left[e^{\imath\pi f T_c}-e^{-\imath\pi f T_c}\right] \\
&= G(f)e^{-\imath\pi f T_c}\imath 2\sin(\pi f T_c)
\end{aligned}$$

and

$$\begin{aligned}
\bar{S}_{p\times\boldsymbol{c}}(f) &= \frac{1}{T_b}S_{\boldsymbol{I}}(f)|\tilde{G}(f)|^2 \\
&= \frac{4}{T_b}|G(f)|^2\sin^2(\pi f T_c) \\
&= \frac{2}{T_b}|G(f)|^2\left[1-\cos(2\pi f T_c)\right]
\end{aligned}$$

Take $g(t)$ be a rectangular pulse of hight $1/\sqrt{T_c}$ and duration T_c. Then $|G(f)|^2 = T_c^2\text{sinc}^2(fT_c)$. This gives $\bar{S}_{\boldsymbol{c}}(f) = T_c\text{sinc}^2(fT_c)\,[1+\cos(2\pi f T_c)]$ and $\bar{S}_{p\times\boldsymbol{c}}(f) = T_c\text{sinc}^2(fT_c)\,[1-\cos(2\pi f T_c)]$, which indicates that $\bar{S}_{\boldsymbol{c}}(f)$ has null-to-null bandwidth $1/T_b$ and $\bar{S}_{p\times\boldsymbol{c}}(f)$ has null-to-null bandwidth $1/T_c$ if the null point at $f=0$ is excluded in the bandwidth consideration. The maximum heights of $\bar{S}_{\boldsymbol{c}}(f)$ and $\bar{S}_{p\times\boldsymbol{c}}(f)$ are respectively $2T_c$ and T_c. So the "spectrum spreading" phenomenon introduced in our lecture (in terms of statistical PN sequences) still holds if a proper PN sequence is adopted.

5. For a lowpass signal of the form,

$$s_\ell(t) = \kappa \sum_{n=-\infty}^{\infty}\left(\sum_{k=0}^{Q-1} X_{k,n} e^{\imath 2\pi \frac{k}{T} t}\right) g(t-nT)$$

answer the following questions.

(a) (6%) Explain that why the signal can be regarded as a Q-carrier system with single-carrier implementation. What are the Q carrier frequencies in this system?

(b) (8%) If $\{X_{k,n}\}_{-\infty<k,n<\infty}$ are zero-mean i.i.d. with variance σ^2, show that

$$\bar{S}_{s_\ell}(f) = \frac{\sigma^2}{T^2}\sum_{k=0}^{Q-1}\left|G\left(f-\frac{k}{T}\right)\right|^2.$$

(c) (6%) Why adding cyclic prefix (CP) to the signal $s_\ell(t)$ before its transmission?
Hint: It is relevant to the channel impulse response $c_\ell(t)$.

(d) (8%) Show that with CP technique, the noiseless reception from the channel with impulse response $c_\ell(t)$ is given by

$$r_\ell(t) \;=\; \kappa \sum_{k=0}^{Q-1} C_\ell\left(\frac{k}{T}\right) X_k e^{\imath 2\pi \frac{km}{\tilde{N}}}$$

where $C_\ell(f)$ is the spectrum of $c_\ell(t)$.
Hint: $r_\ell(t) = \tilde{s}_\ell(t) \star c_\ell(t)$, where $\tilde{s}_\ell(t)$ is the periodic counterpart of $s_\ell(t)$.

Solutions.

(a) See Slide 11-9.

(b) See Slide 11-11 to Slide 11-13.

(c) See Slide 11-13.

(d) See Slide 11-38 to Slide 11-39.