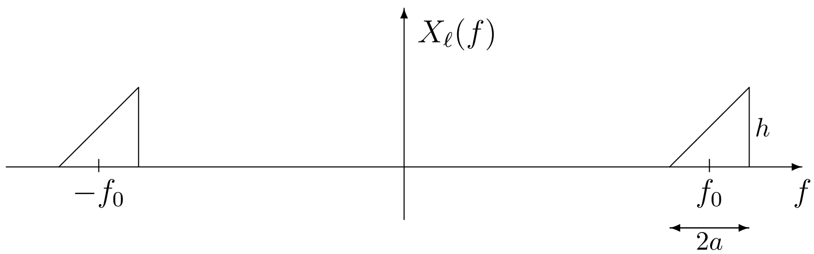
2015 Spring: The First Midterm of Digital Communications

The total points of this exam is 112.

1. For a given $x_\ell(t)$, define $x(t) = \mathbf{Re}\left\{x_\ell(t)e^{\imath 2\pi f_0 t}\right\}$ and $\hat{x}(t) = \mathbf{Im}\left\{x_\ell(t)e^{\imath 2\pi f_0 t}\right\}$. Assume the Fourier transform of $x_\ell(t)$ is the figure shown below, where the two triangles are identical, and the center frequency f_0 (respectively, $-f_0$) is in the middle of the base of the triangle on the right (respectively, left), and $X_\ell(f)$ is real-valued.



Answer the following questions.

(a) (6%) Is $x_\ell(t)$ a real-valued process? You should justify your answer by stating the reason why your answer is YES or NO.

(b) (6%) Derive the general relation between $X(f)$ and $X_\ell(f)$. Plot $X(f)$.

(c) (6%) Derive the general relation between $X(f) + \imath\hat{X}(f)$ and $X_\ell(f)$. Plot $X(f) + \imath\hat{X}(f)$.

(d) (6%) Compute the energies of $x(t)$ and $x_\ell(t)$, which are respectively denoted by $\mathcal{E}_x$ and $\mathcal{E}_{x_\ell}$.

(e) (6%) In (d), you shall learn that $\mathcal{E}_{x_\ell}$ is actually smaller than twice of $\mathcal{E}_x$! From the derivation below, give the reason why $\mathcal{E}_{x_\ell} = 2\mathcal{E}_x$ is not true in this problem setting!

$$\begin{aligned}
\langle x(t), x(t)\rangle &= \langle X(f), X(f)\rangle \\
&= \left\langle \frac{1}{2}X_\ell(f-f_0) + \frac{1}{2}X_\ell^*(-f-f_0), \frac{1}{2}X_\ell(f-f_0) + \frac{1}{2}X_\ell^*(-f-f_0)\right\rangle \\
&= \frac{1}{4}\langle X_\ell(f-f_0), X_\ell(f-f_0)\rangle + \frac{1}{4}\langle X_\ell(f-f_0), X_\ell^*(-f-f_0)\rangle \\
&\quad + \frac{1}{4}\langle X_\ell^*(-f-f_0), X_\ell(f-f_0)\rangle + \frac{1}{4}\langle X_\ell^*(-f-f_0), X_\ell^*(-f-f_0)\rangle \\
&= \frac{1}{4}\langle x_\ell(t), y_\ell(t)\rangle + \frac{1}{4}(\langle x_\ell(t), x_\ell(t)\rangle)^* = \frac{1}{2}\langle x_\ell(t), x_\ell(t)\rangle.
\end{aligned}$$

2. Relax the fundamental assumption to:

- The bandpass process $\boldsymbol{X}(t) = \mathbf{Re}\{\boldsymbol{X}_\ell(t)e^{\imath 2\pi f_c t}\}$ is cyclostationary with period T;
- The complex lowpass equivalent process $\boldsymbol{X}_\ell(t) = \boldsymbol{X}_i(t) + \imath\boldsymbol{X}_q(t)$ of bandpass process $\boldsymbol{X}(t)$ is cyclostationary with period T in the sense that
  - $\boldsymbol{X}_i(t)$ and $\boldsymbol{X}_q(t)$ are cyclostationary with period T.
  - $\boldsymbol{X}_i(t)$ and $\boldsymbol{X}_q(t)$ are jointly cyclostationary with period T.

Assume throughout this problem that f_cT is an irrational number.

(a) (8%) Prove that if $\boldsymbol{X}(t)$ is zero-mean, then both $\boldsymbol{X}_i(t)$ and $\boldsymbol{X}_q(t)$ are zero-mean.
Hint: You may wish to first prove that

$$\begin{bmatrix} A_1 & -B_1 \\ A_0 & -B_0 \end{bmatrix} \begin{bmatrix} \mathbb{E}[\boldsymbol{X}_i(t)] \\ \mathbb{E}[\boldsymbol{X}_q(t)] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where $A_k = \cos(2\pi f_c(t + kT))$ and $B_k = \sin(2\pi f_c(t + kT))$.

(b) (8%) Prove that

$$\begin{cases} R_{\boldsymbol{X}_i}(t_1, t_2) = R_{\boldsymbol{X}_q}(t_1, t_2) \\ R_{\boldsymbol{X}_i,\boldsymbol{X}_q}(t_1, t_2) = -R_{\boldsymbol{X}_q,\boldsymbol{X}_i}(t_1, t_2) \end{cases}$$

Hint: Based on

$$\begin{aligned} R_{\boldsymbol{X}}(t_1, t_2) &= \mathbb{E}\left[\boldsymbol{X}(t_1)\boldsymbol{X}(t_2)\right] \\ &= \mathbb{E}\left[(\boldsymbol{X}_i(t_1)\cos(2\pi f_c(t_1)) - \boldsymbol{X}_q(t_1)\sin(2\pi f_c(t_1)))\right. \\ &\quad \left.(\boldsymbol{X}_i(t_2)\cos(2\pi f_c t_2) - \boldsymbol{X}_q(t_2)\sin(2\pi f_c t_2))\right] \\ &= \frac{R_{\boldsymbol{X}_i}(t_1, t_2) + R_{\boldsymbol{X}_q}(t_1, t_2)}{2}\cos(2\pi f_c(t_1 - t_2)) \\ &\quad + \frac{R_{\boldsymbol{X}_i,\boldsymbol{X}_q}(t_1, t_2) - R_{\boldsymbol{X}_q,\boldsymbol{X}_i}(t_1, t_2)}{2}\sin(2\pi f_c(t_1 - t_2)) \\ &\quad + \frac{R_{\boldsymbol{X}_i}(t_1, t_2) - R_{\boldsymbol{X}_q}(t_1, t_2)}{2}\cos(2\pi f_c(t_1 + t_2)) \\ &\quad - \frac{R_{\boldsymbol{X}_i,\boldsymbol{X}_q}(t_1, t_2) + R_{\boldsymbol{X}_q,\boldsymbol{X}_i}(t_1, t_2)}{2}\sin(2\pi f_c(t_1 + t_2)), \end{aligned}$$

you may wish to first prove that

$$\begin{bmatrix} A_0 & B_0 & 1 \\ A_1 & B_1 & 1 \\ A_2 & B_2 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

where

$$\begin{cases} A_k = \cos(2\pi f_c(t_1 + t_2) + 4\pi f_c T k); \\ B_k = \sin(2\pi f_c(t_1 + t_2) + 4\pi f_c T k); \\ X = [R_{\boldsymbol{X}_i}(t_1, t_2) - R_{\boldsymbol{X}_q}(t_1, t_2)]/2; \\ Y = -[R_{\boldsymbol{X}_i,\boldsymbol{X}_q}(t_1, t_2) + R_{\boldsymbol{X}_q,\boldsymbol{X}_i}(t_1, t_2)/2; \\ Z = R_{\boldsymbol{X}_i}(t_1, t_2) + R_{\boldsymbol{X}_q}(t_1, t_2)\cos(2\pi f_c(t_1 - t_2))/2 \\ \qquad +[R_{\boldsymbol{X}_i,\boldsymbol{X}_q}(t_1, t_2) - R_{\boldsymbol{X}_q,\boldsymbol{X}_i}(t_1, t_2)]\sin(2\pi f_c(t_1 - t_2))/2 - R_{\boldsymbol{X}}(t_1, t_2) \end{cases}$$

(c) (8%) Prove that

$$R_{\boldsymbol{X}}(t_1, t_2) = \mathbf{Re}\left\{\frac{1}{2}R_{\boldsymbol{X}_\ell}(t_1, t_2)e^{\imath 2\pi f_c(t_1 - t_2)}\right\}.$$

Hint: You may use directly the result in (b).

3. (a) (8%) Suppose $\boldsymbol{Y}(t)$ is the output due to zero-mean input $\boldsymbol{X}(t)$ through an LTI filter with impulse response $h(t)$. Let $\{\varphi_k(t)\}_{k=1}^{\infty}$ be the eigenfunctions corresponding to $R_{\boldsymbol{X}}(t,s) = \mathbb{E}[\boldsymbol{X}(t)\boldsymbol{X}^*(s)]$ in the sense that for every k,

$$\int_0^T R_{\boldsymbol{X}}(t,s)_k\varphi(s)ds = \lambda_k\varphi_k(t).$$

Prove that $R_{\boldsymbol{X}}(t,s) = \sum_{k=1}^{\infty}\lambda_k\varphi_k(t)\varphi_k^*(s)$ implies $R_{\boldsymbol{Y}}(t,s) = \sum_{k=1}^{\infty}\lambda_k\psi_k(t)\psi_k^*(s)$, where $\psi_k(t) = \varphi_k(t) \star h(t)$.

(b) (8%) Let $\boldsymbol{Z}_k = \langle \boldsymbol{X}(t), \varphi_k(t)\rangle = \int_0^T \boldsymbol{X}(t)\varphi_k^*(t)dt$. Prove that $\boldsymbol{X}(t) = \sum_{k=1}^{\infty}\boldsymbol{Z}_k \cdot \varphi_k(t)$ implies $\boldsymbol{Y}(t) = \sum_{k=1}^{\infty}\boldsymbol{Z}_k \cdot \psi_k(t)$.

4. For $t \in [nT, (n+1)T)$, the CPFSK lowpass equivalent signal can be represented as

$$s_\ell(t) = \sqrt{\frac{2\mathcal{E}}{T}}e^{\imath\,\phi(t;\boldsymbol{I})}$$

with

$$\phi(t;\boldsymbol{I}) = \theta_n + 2\pi h \cdot I_n \cdot q(t-nT) = \theta_n + \pi h I_n \frac{t}{2T} - \pi h n I_n,$$

where

$$h = 2f_d T, \quad \theta_n = \pi h \sum_{k=-\infty}^{n-1} I_k, \text{ and } q(t) = \begin{cases} 0 & t < 0 \\ \frac{t}{2T} & 0 \le t < T \\ \frac{1}{2} & t \ge T \end{cases}.$$

(a) (6%) Plot the phase tree of the CPFSK signals from $t = 0$ to $t = 3T$, starting from $\phi(0;\boldsymbol{I}) = 0$.

(b) (6%) The CPFSK signal for $I_n \in \{\pm 1\}$ and $h = 1/2$ is particularly named minimum-shift keying (MSK). Explain why it is named so.

(c) (6%) Under $h = 1/2$, list all four (passband) frequencies corresponding to $I_n \in \{\pm 1, \pm 3\}$. Are the four passband signals corresponding to $I_n = -3, -1, +1, +3$ orthogonal to each other? Justify your answer.

5. In our lecture, we learn that the autocorrelation function of $\boldsymbol{v}_\ell(t) = \sum_{n=-\infty}^{\infty} I_n g(t-nT)$ with wise-sense stationary $\{I_n\}$ is given by

$$R_{\boldsymbol{v}_\ell}(t_1,t_2) = \mathbb{E}\left[\boldsymbol{v}_\ell(t_1)\boldsymbol{v}_\ell^*(t_2)\right] = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\mathbb{E}\left[I_n I_m^*\right] g(t_1-nT)g^*(t_2-mT).$$

We then continue to derive its time-averaged PSD and obtain

$$\overline{S}_{\boldsymbol{v}_\ell}(f) = \frac{1}{T}S_{\boldsymbol{I}}(f)|G(f)|^2,$$

where $S_{\boldsymbol{I}}(f) = \sum_{k=-\infty}^{\infty} R_{\boldsymbol{I}}(k)e^{-\imath\,2\pi kfT}$ and $R_{\boldsymbol{I}}(k) = \mathbb{E}[I_{n+k}I_n^*]$. Based on this result, answer the following questions.

(a) (6%) The OQPSK baseband signal can be represented as

$$s_{\text{OQPSK},\ell}(t) = \sum_{n=-\infty}^{\infty} J_{2n} g(t - 2nT) + \imath \sum_{n=-\infty}^{\infty} J_{2n+1} g(t - (2n+1)T).$$

Give the time-averaged PSD of the OQPSK baseband signal, if $\{J_n\}$ are an i.i.d. sequence with $\Pr[J_n = -1] = \Pr[J_n = 1] = 1/2$.

Hint: $s_\ell(t) = \sum_{k=-\infty}^{\infty} c_k J_k g(t - kT)$, where $c_k = 1$ for k even and $c_k = \imath$ for k odd. You may use directly the above quoted result from our lecture.

(b) (6%) Re-do problem (a) for $J_n = \prod_{k=-\infty}^{n} \tilde{J}_k$ with $\{\tilde{J}_k\}$ i.i.d. and $\Pr[\tilde{J}_k = -1] = \Pr[\tilde{J}_k = 1] = 1/2$.

(c) (6%) Continue from (b). Define $g(t) = \sin\left(\pi \frac{t}{2T}\right) [u_{-1}(t) - u_{-1}(t - 2T)]$ (so as to realize the MSK modulation). The Fourier transform of $g(t)$ is given by

$$G(f) = \frac{4T \cos(2\pi f T)}{\pi(1 - 16 f^2 T^2)} e^{-\imath 2\pi f T}.$$

Please determine the time-averaged PSD of the signal in (b).