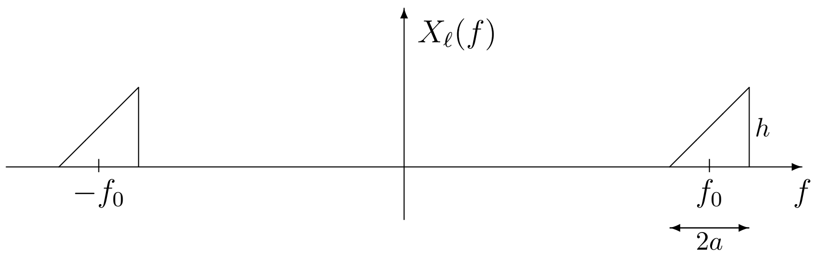
# 2015 Spring: The First Midterm of Digital Communications

The total points of this exam is 112.

1. For a given $x_\ell(t)$, define $x(t) = \mathbf{Re}\left\{x_\ell(t)e^{\imath 2\pi f_0 t}\right\}$ and $\hat{x}(t) = \mathbf{Im}\left\{x_\ell(t)e^{\imath 2\pi f_0 t}\right\}$. Assume the Fourier transform of $x_\ell(t)$ is the figure shown below, where the two triangles are identical, and the center frequency $f_0$ (respectively, $-f_0$) is in the middle of the base of the triangle on the right (respectively, left), and $X_\ell(f)$ is real-valued.




Answer the following questions.

(a) (6%) Is $x_\ell(t)$ a real-valued process? You should justify your answer by stating the reason why your answer is YES or NO.

(b) (6%) Derive the general relation between $X(f)$ and $X_\ell(f)$. Plot $X(f)$.

(c) (6%) Derive the general relation between $X(f) + \imath\hat{X}(f)$ and $X_\ell(f)$. Plot $X(f) + \imath\hat{X}(f)$.

(d) (6%) Compute the energies of $x(t)$ and $x_\ell(t)$, which are respectively denoted by $\mathcal{E}_x$ and $\mathcal{E}_{x_\ell}$.

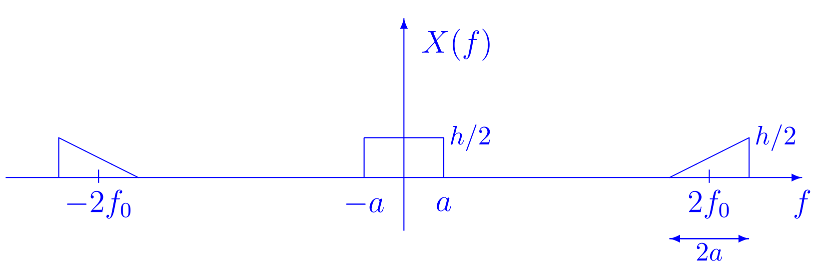
(e) (6%) In (d), you shall learn that $\mathcal{E}_{x_\ell}$ is actually smaller than twice of $\mathcal{E}_x$! From the derivation below, give the reason why $\mathcal{E}_{x_\ell} = 2\mathcal{E}_x$ is not true in this problem setting!

$$\begin{aligned}
\langle x(t), x(t)\rangle &= \langle X(f), X(f)\rangle \\
&= \left\langle \frac{1}{2}X_\ell(f-f_0) + \frac{1}{2}X_\ell^*(-f-f_0), \frac{1}{2}X_\ell(f-f_0) + \frac{1}{2}X_\ell^*(-f-f_0)\right\rangle \\
&= \frac{1}{4}\langle X_\ell(f-f_0), X_\ell(f-f_0)\rangle + \frac{1}{4}\langle X_\ell(f-f_0), X_\ell^*(-f-f_0)\rangle \\
&\quad + \frac{1}{4}\langle X_\ell^*(-f-f_0), X_\ell(f-f_0)\rangle + \frac{1}{4}\langle X_\ell^*(-f-f_0), X_\ell^*(-f-f_0)\rangle \\
&= \frac{1}{4}\langle x_\ell(t), y_\ell(t)\rangle + \frac{1}{4}(\langle x_\ell(t), x_\ell(t)\rangle)^* = \frac{1}{2}\langle x_\ell(t), x_\ell(t)\rangle.
\end{aligned}$$

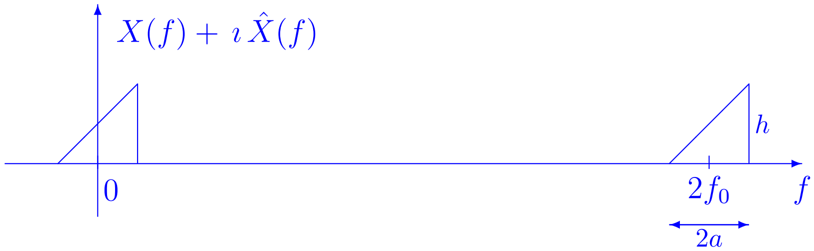
**Solutions.**

(a) $X_\ell(f)$ is not Hermitian symmetric; hence, $x_\ell(t)$ is not a real-valued signal.

(b) See Slide 2-13 for detail derivation, which gives $X(f) = \frac{1}{2}\left[X_\ell(f-f_0) + X_\ell^*(-f-f_0)\right]$.



(c) Since $x(t) + \imath\,\hat{x}(t) = x_\ell(t)e^{\imath 2\pi f_0 t}$, we have $X(f) + \imath\,\hat{X}(f) = X_\ell(f - f_0)$.




(d)

$$\mathcal{E}_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df = \frac{1}{2}ah^2 + 2\int_0^{2a} \left(\frac{h}{4a}\right)^2 f^2 df = \frac{5}{6}ah^2$$

and

$$\mathcal{E}_{x_\ell} = \int_{-\infty}^{\infty} |x_\ell(t)|^2 dt = \int_{-\infty}^{\infty} |X_\ell(f)|^2 df = 2\int_0^{2a} \left(\frac{h}{2a}\right)^2 f^2 df = \frac{4}{3}ah^2.$$

(e) From the derivation, it is clear that $\langle x(t), x(t)\rangle = \frac{1}{2}\langle x_\ell(t), x_\ell(t)\rangle$ if, and only if,

$$\langle X_\ell(f - f_0), X_\ell^*(-f - f_0)\rangle + \langle X_\ell^*(-f - f_0), X_\ell(f - f_0)\rangle = 0.$$

The setting in this problem clearly violates the above condition.

In our lecture, $x(t)$ must be a bandpass signal (See Slide 2-3) and $x_\ell(t)$ is a lowpass signal with bandwidth $W$ less than $f_0$. However, the $X_\ell(f)$ given in this problem has bandwidth $f_0 + a$, which is clearly larger than $f_0$. Hence, $X_\ell(f - f_0)$ overlaps with $X_\ell^*(-f - f_0)$, yielding $\langle X_\ell(f - f_0), X_\ell^*(-f - f_0)\rangle \neq 0$.

2. Relax the fundamental assumption to:

- The bandpass process $\boldsymbol{X}(t) = \mathbf{Re}\{\boldsymbol{X}_\ell(t)e^{\imath 2\pi f_c t}\}$ is cyclostationary with period $T$;
- The complex lowpass equivalent process $\boldsymbol{X}_\ell(t) = \boldsymbol{X}_i(t) + \imath\,\boldsymbol{X}_q(t)$ of bandpass process $\boldsymbol{X}(t)$ is cyclostationary with period $T$ in the sense that
  - $\boldsymbol{X}_i(t)$ and $\boldsymbol{X}_q(t)$ are cyclostationary with period $T$.
  - $\boldsymbol{X}_i(t)$ and $\boldsymbol{X}_q(t)$ are jointly cyclostationary with period $T$.

Assume throughout this problem that $f_c T$ is an irrational number.

(a) (8%) Prove that if $\boldsymbol{X}(t)$ is zero-mean, then both $\boldsymbol{X}_i(t)$ and $\boldsymbol{X}_q(t)$ are zero-mean.
Hint: You may wish to first prove that

$$\begin{bmatrix} A_1 & -B_1 \\ A_0 & -B_0 \end{bmatrix} \begin{bmatrix} \mathbb{E}[\boldsymbol{X}_i(t)] \\ \mathbb{E}[\boldsymbol{X}_q(t)] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where $A_k = \cos(2\pi f_c(t + kT))$ and $B_k = \sin(2\pi f_c(t + kT))$.

(b) (8%) Prove that

$$\begin{cases} R_{\boldsymbol{X}_i}(t_1, t_2) = R_{\boldsymbol{X}_q}(t_1, t_2) \\ R_{\boldsymbol{X}_i, \boldsymbol{X}_q}(t_1, t_2) = -R_{\boldsymbol{X}_q, \boldsymbol{X}_i}(t_1, t_2) \end{cases}$$

Hint: Based on

$$\begin{aligned} R_{\boldsymbol{X}}(t_1,t_2) &= \mathbb{E}\left[\boldsymbol{X}(t_1)\boldsymbol{X}(t_2)\right] \\ &= \mathbb{E}\left[(\boldsymbol{X}_i(t_1)\cos(2\pi f_c(t_1)) - \boldsymbol{X}_q(t_1)\sin(2\pi f_c(t_1)))\right. \\ &\quad \left.(\boldsymbol{X}_i(t_2)\cos(2\pi f_c t_2) - \boldsymbol{X}_q(t_2)\sin(2\pi f_c t_2))\right] \\ &= \frac{R_{\boldsymbol{X}_i}(t_1,t_2)+R_{\boldsymbol{X}_q}(t_1,t_2)}{2}\cos(2\pi f_c(t_1-t_2)) \\ &\quad +\frac{R_{\boldsymbol{X}_i,\boldsymbol{X}_q}(t_1,t_2)-R_{\boldsymbol{X}_q,\boldsymbol{X}_i}(t_1,t_2)}{2}\sin(2\pi f_c(t_1-t_2)) \\ &\quad +\frac{R_{\boldsymbol{X}_i}(t_1,t_2)-R_{\boldsymbol{X}_q}(t_1,t_2)}{2}\cos(2\pi f_c(t_1+t_2)) \\ &\quad -\frac{R_{\boldsymbol{X}_i,\boldsymbol{X}_q}(t_1,t_2)+R_{\boldsymbol{X}_q,\boldsymbol{X}_i}(t_1,t_2)}{2}\sin(2\pi f_c(t_1+t_2)), \end{aligned}$$

you may wish to first prove that

$$\begin{bmatrix} A_0 & B_0 & 1 \\ A_1 & B_1 & 1 \\ A_2 & B_2 & 1 \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

where

$$\begin{cases} A_k = \cos(2\pi f_c(t_1+t_2)+4\pi f_c T k); \\ B_k = \sin(2\pi f_c(t_1+t_2)+4\pi f_c T k); \\ X = [R_{\boldsymbol{X}_i}(t_1,t_2)-R_{\boldsymbol{X}_q}(t_1,t_2)]/2; \\ Y = -[R_{\boldsymbol{X}_i,\boldsymbol{X}_q}(t_1,t_2)+R_{\boldsymbol{X}_q,\boldsymbol{X}_i}(t_1,t_2)/2; \\ Z = R_{\boldsymbol{X}_i}(t_1,t_2)+R_{\boldsymbol{X}_q}(t_1,t_2)\cos(2\pi f_c(t_1-t_2))/2 \\ \qquad +[R_{\boldsymbol{X}_i,\boldsymbol{X}_q}(t_1,t_2)-R_{\boldsymbol{X}_q,\boldsymbol{X}_i}(t_1,t_2)]\sin(2\pi f_c(t_1-t_2))/2 - R_{\boldsymbol{X}}(t_1,t_2) \end{cases}.$$

(c) (8%) Prove that

$$R_{\boldsymbol{X}}(t_1,t_2) = \mathbf{Re}\left\{\frac{1}{2}R_{\boldsymbol{X}_\ell}(t_1,t_2)e^{\imath 2\pi f_c(t_1-t_2)}\right\}.$$

Hint: You may use directly the result in (b).

**Solutions.**

(a) By

$$\begin{aligned} \mathbb{E}[\boldsymbol{X}(t)] &= \mathbb{E}[\boldsymbol{X}_i(t)\cos(2\pi f_c t) - \boldsymbol{X}_q(t)\sin(2\pi f_c t)] \\ &= \mathbb{E}[\boldsymbol{X}_i(t)]\cos(2\pi f_c t) - \mathbb{E}[\boldsymbol{X}_q(t)]\sin(2\pi f_c t), \end{aligned}$$

we know that if $\boldsymbol{X}(t)$ is zero-mean, then $\mathbb{E}[\boldsymbol{X}_i(t)]\cos(2\pi f_c t) = \mathbb{E}[\boldsymbol{X}_q(t)]\sin(2\pi f_c t)$ must be true for every $t$. By cyclostationarity,

$$A_k\mathbb{E}[\boldsymbol{X}_i(t)] - B_k\mathbb{E}[\boldsymbol{X}_q(t)] = 0 \quad \text{for every integer } k,$$

where $A_k = \cos(2\pi f_c(t+kT))$ and $B_k = \sin(2\pi f_c(t+kT))$. Accordingly,

$$\Delta\begin{bmatrix} \mathbb{E}[\boldsymbol{X}_i(t)] \\ \mathbb{E}[\boldsymbol{X}_q(t)] \end{bmatrix} = \begin{bmatrix} A_1 & -B_1 \\ A_0 & -B_0 \end{bmatrix}\begin{bmatrix} \mathbb{E}[\boldsymbol{X}_i(t)] \\ \mathbb{E}[\boldsymbol{X}_q(t)] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The determinant of $\Delta$ is $B_1A_0 - B_0A_1 = \sin(2\pi f_c T) \neq 0$ because $f_cT$ is irrational. Therefore, $\mathbb{E}[\boldsymbol{X}_i(t)] = \mathbb{E}[\boldsymbol{X}_q(t)] = 0$.

(b) For a cyclostationary process, $R_{\boldsymbol{X}}(t_1,t_2) = R_{\boldsymbol{X}}(t_1+kT, t_2+kT)$. Thus,

$$\begin{aligned} R_{\boldsymbol{X}}(t_1,t_2) &= \frac{R_{\boldsymbol{X}_i}(t_1,t_2)+R_{\boldsymbol{X}_q}(t_1,t_2)}{2}\cos(2\pi f_c(t_1-t_2)) \\ &\quad+\frac{R_{\boldsymbol{X}_i,\boldsymbol{X}_q}(t_1,t_2)-R_{\boldsymbol{X}_q,\boldsymbol{X}_i}(t_1,t_2)}{2}\sin(2\pi f_c(t_1-t_2)) \\ &\quad+\frac{R_{\boldsymbol{X}_i}(t_1,t_2)-R_{\boldsymbol{X}_q}(t_1,t_2)}{2}\cos(2\pi f_c(t_1+t_2)+4\pi f_c T k) \\ &\quad-\frac{R_{\boldsymbol{X}_i,\boldsymbol{X}_q}(t_1,t_2)+R_{\boldsymbol{X}_q,\boldsymbol{X}_i}(t_1,t_2)}{2}\sin(2\pi f_c(t_1+t_2)+4\pi f_c T k). \end{aligned}$$

Thus, for any $t_1$ and $t_2$ fixed, we have $A_k X + B_k Y + Z = 0$ for every integer $k$, where

$$\begin{cases} A_k = \cos(2\pi f_c(t_1+t_2)+4\pi f_c T k); \\ B_k = \sin(2\pi f_c(t_1+t_2)+4\pi f_c T k); \\ X = [R_{\boldsymbol{X}_i}(t_1,t_2)-R_{\boldsymbol{X}_q}(t_1,t_2)]/2; \\ Y = -[R_{\boldsymbol{X}_i,\boldsymbol{X}_q}(t_1,t_2)+R_{\boldsymbol{X}_q,\boldsymbol{X}_i}(t_1,t_2)/2; \\ Z = R_{\boldsymbol{X}_i}(t_1,t_2)+R_{\boldsymbol{X}_q}(t_1,t_2)\cos(2\pi f_c(t_1-t_2))/2 \\ \qquad +[R_{\boldsymbol{X}_i,\boldsymbol{X}_q}(t_1,t_2)-R_{\boldsymbol{X}_q,\boldsymbol{X}_i}(t_1,t_2)]\sin(2\pi f_c(t_1-t_2))/2 - R_{\boldsymbol{X}}(t_1,t_2) \end{cases}.$$

As a result,

$$\mathbb{D}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} A_0 & B_0 & 1 \\ A_1 & B_1 & 1 \\ A_2 & B_2 & 1 \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

By observing that the determinant of $\mathbb{D}$ is equal to

$$2\sin(4\pi f_c T) - \sin(8\pi f_c T) = 2\sin(4\pi f_c T)[1-\cos(4\pi f_c T)] \neq 0$$

when $f_c T$ is not a rational number. We know that $X = Y = Z = 0$.

> Note for (a) and (b): Since we only require $\sin(2\pi f_c T) \neq 0$ and $2\sin(4\pi f_c T) - \sin(8\pi f_c T) \neq 0$, we do not really need the strong condition that $f_c T$ is an irrational number. For example, if $f_c T = n + 1/8$ for some integer $n$, the desired properties still hold. However, if $f_c T$ is an integer, then the situation becomes trick and no conclusive properties can be claimed.

(c) By (b), we have that

$$\begin{aligned} R_{\boldsymbol{X}_\ell}(t_1,t_2) &= \mathbb{E}[\boldsymbol{X}_\ell(t_1)\boldsymbol{X}_\ell^*(t_2)] \\ &= \mathbb{E}[(\boldsymbol{X}_i(t_1)+\imath\,\boldsymbol{X}_q(t_1))(\boldsymbol{X}_i(t_2)-\imath\,\boldsymbol{X}_q(t_2))] \\ &= R_{\boldsymbol{X}_i}(t_1,t_2)+R_{\boldsymbol{X}_q}(t_1,t_2)-\imath\,R_{\boldsymbol{X}_i,\boldsymbol{X}_q}(t_1,t_2)+\imath\,R_{\boldsymbol{X}_q,\boldsymbol{X}_i}(t_1,t_2) \\ &= 2R_{\boldsymbol{X}_i}(t_1,t_2)+\imath\,2R_{\boldsymbol{X}_q,\boldsymbol{X}_i}(t_1,t_2). \end{aligned}$$

Also by the derivation in (b),

$$R_{\boldsymbol{X}}(t_1,t_2) = R_{\boldsymbol{X}_i}(t_1,t_2)\cos(2\pi f_c(t_1-t_2)) - R_{\boldsymbol{X}_q,\boldsymbol{X}_i}(t_1,t_2)\sin(2\pi f_c(t_1-t_2)).$$

Hence,

$$R_{\boldsymbol{X}}(t_1,t_2) = \mathbf{Re}\left\{\frac{1}{2}R_{\boldsymbol{X}_\ell}(t_1,t_2)e^{\imath 2\pi f_c(t_1-t_2)}\right\}.$$

3. (a) (8%) Suppose $\boldsymbol{Y}(t)$ is the output due to zero-mean input $\boldsymbol{X}(t)$ through an LTI filter with impulse response $h(t)$. Let $\{\varphi_k(t)\}_{k=1}^{\infty}$ be the eigenfunctions corresponding to $R_{\boldsymbol{X}}(t,s) = \mathbb{E}[\boldsymbol{X}(t)\boldsymbol{X}^*(s)]$ in the sense that for every $k$,

$$\int_0^T R_{\boldsymbol{X}}(t,s)_k\varphi(s)ds = \lambda_k\varphi_k(t).$$

Prove that $R_{\boldsymbol{X}}(t,s) = \sum_{k=1}^{\infty}\lambda_k\varphi_k(t)\varphi_k^*(s)$ implies $R_{\boldsymbol{Y}}(t,s) = \sum_{k=1}^{\infty}\lambda_k\psi_k(t)\psi_k^*(s)$, where $\psi_k(t) = \varphi_k(t) \star h(t)$.

(b) (8%) Let $\boldsymbol{Z}_k = \langle \boldsymbol{X}(t), \varphi_k(t)\rangle = \int_0^T \boldsymbol{X}(t)\varphi_k^*(t)dt$. Prove that $\boldsymbol{X}(t) = \sum_{k=1}^{\infty}\boldsymbol{Z}_k \cdot \varphi_k(t)$ implies $\boldsymbol{Y}(t) = \sum_{k=1}^{\infty}\boldsymbol{Z}_k \cdot \psi_k(t)$.

**Solutions.**

(a) It can be proved as follows.

$$\begin{aligned}
R_{\boldsymbol{Y}}(t,s) &= \mathbb{E}[\boldsymbol{Y}(t)\boldsymbol{Y}^*(s)] \\
&= \mathbb{E}\left[\left(\int_{-\infty}^{\infty} h(u)\boldsymbol{X}(t-u)du\right)\left(\int_{-\infty}^{\infty} h(v)\boldsymbol{X}(s-v)dv\right)^*\right] \\
&= \int_{-\infty}^{\infty} h(u)\left(\int_{-\infty}^{\infty} h^*(v)R_{\boldsymbol{X}}(t-u,s-v)dv\right)du \\
&= \int_{-\infty}^{\infty} h(u)\left(\int_{-\infty}^{\infty} h^*(v)\sum_{k=1}^{\infty}\lambda_k\varphi_k(t-u)\varphi_k^*(s-v)dv\right)du \\
&= \sum_{k=1}^{\infty}\lambda_k\left(\int_{-\infty}^{\infty} h(u)\varphi_k(t-u)du\right)\left(\int_{-\infty}^{\infty} h(v)\varphi_k(s-v)dv\right)^* \\
&= \sum_{k=1}^{\infty}\lambda_k\psi(t)\psi^*(s).
\end{aligned}$$

(b) The desired result can be proved through:

$$\begin{aligned}
\boldsymbol{Y}(t) &= h(t)\star\boldsymbol{X}(t) = \int_{-\infty}^{\infty} h(\tau)\boldsymbol{X}(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} h(\tau)\left(\sum_{k=1}^{\infty}\boldsymbol{Z}_k\cdot\varphi_k(t-\tau)\right)d\tau = \sum_{k=1}^{\infty}\boldsymbol{Z}_k\int_{-\infty}^{\infty} h(\tau)\varphi_k(t-\tau)d\tau \\
&= \sum_{k=1}^{\infty}\boldsymbol{Z}_k\psi_k(t).
\end{aligned}$$

4. For $t \in [nT, (n+1)T)$, the CPFSK lowpass equivalent signal can be represented as

$$s_\ell(t) = \sqrt{\frac{2\mathcal{E}}{T}}e^{\imath\,\phi(t;\boldsymbol{I})}$$

with

$$\phi(t;\boldsymbol{I}) = \theta_n + 2\pi h\cdot I_n\cdot q(t-nT) = \theta_n + \pi h I_n\frac{t}{2T} - \pi h n I_n,$$

where

$$h = 2f_d T, \quad \theta_n = \pi h \sum_{k=-\infty}^{n-1} I_k, \text{ and } q(t) = \begin{cases} 0 & t < 0 \\ \frac{t}{2T} & 0 \le t < T \\ \frac{1}{2} & t \ge T \end{cases}.$$

(a) (6%) Plot the phase tree of the CPFSK signals from $t = 0$ to $t = 3T$, starting from $\phi(0; \boldsymbol{I}) = 0$.

(b) (6%) The CPFSK signal for $I_n \in \{\pm 1\}$ and $h = 1/2$ is particularly named minimum-shift keying (MSK). Explain why it is named so.

(c) (6%) Under $h = 1/2$, list all four (passband) frequencies corresponding to $I_n \in \{\pm 1, \pm 3\}$. Are the four passband signals corresponding to $I_n = -3, -1, +1, +3$ orthogonal to each other? Justify your answer.

**Solutions.**

(a) See Slide 3-77.

(b) See Slide 3-84.

(c) For $h = 1/2$, the passband signal formula is given by

$$s(t) = \cos\left[2\pi\left(f_c + \frac{I_n}{4T}\right)t - \frac{n\pi I_n}{2} + \theta_n\right]$$

Hence, the four passband frequencies are $f_c - \frac{3}{4T}$, $f_c - \frac{1}{4T}$, $f_c + \frac{1}{4T}$ and $f_c + \frac{3}{4T}$. From Slide 3-35, we know that as long as the frequency difference is a multiple of $1/(2T)$, the two bandpass signals are orthogonal. Hence, the four signals are orthogonal to each other.

Note that although the four passband signals are orthogonal to each other, we do not regard it as MSK modulation because the difference between the highest frequency shift and the lowest frequency shift (i.e., $\frac{3}{2T}$) is not equal to the minimum shift (i.e., $\frac{1}{2T}$) for orthogonality.

5. In our lecture, we learn that the autocorrelation function of $\boldsymbol{v}_\ell(t) = \sum_{n=-\infty}^{\infty} I_n g(t - nT)$ with wise-sense stationary $\{I_n\}$ is given by

$$R_{\boldsymbol{v}_\ell}(t_1, t_2) = \mathbb{E}\left[\boldsymbol{v}_\ell(t_1)\boldsymbol{v}_\ell^*(t_2)\right] = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \mathbb{E}\left[I_n I_m^*\right] g(t_1 - nT) g^*(t_2 - mT).$$

We then continue to derive its time-averaged PSD and obtain

$$\overline{S}_{\boldsymbol{v}_\ell}(f) = \frac{1}{T} S_{\boldsymbol{I}}(f) |G(f)|^2,$$

where $S_{\boldsymbol{I}}(f) = \sum_{k=-\infty}^{\infty} R_{\boldsymbol{I}}(k) e^{-\imath 2\pi k f T}$ and $R_{\boldsymbol{I}}(k) = \mathbb{E}[I_{n+k} I_n^*]$. Based on this result, answer the following questions.

(a) (6%) The OQPSK baseband signal can be represented as

$$s_{\text{OQPSK},\ell}(t) = \sum_{n=-\infty}^{\infty} J_{2n} g(t - 2nT) + \imath \sum_{n=-\infty}^{\infty} J_{2n+1} g(t - (2n+1)T).$$

Give the time-averaged PSD of the OQPSK baseband signal, if $\{J_n\}$ are an i.i.d. sequence with $\Pr[J_n = -1] = \Pr[J_n = 1] = 1/2$.

Hint: $s_\ell(t) = \sum_{k=-\infty}^{\infty} c_k J_k g(t - kT)$, where $c_k = 1$ for $k$ even and $c_k = \imath$ for $k$ odd. You may use directly the above quoted result from our lecture.

(b) (6%) Re-do problem (a) for $J_n = \prod_{k=-\infty}^{n} \tilde{J}_k$ with $\{\tilde{J}_k\}$ i.i.d. and $\Pr[\tilde{J}_k = -1] = \Pr[\tilde{J}_k = 1] = 1/2$.

(c) (6%) Continue from (b). Define $g(t) = \sin\left(\pi \frac{t}{2T}\right)[u_{-1}(t) - u_{-1}(t - 2T)]$ (so as to realize the MSK modulation). The Fourier transform of $g(t)$ is given by

$$G(f) = \frac{4T\cos(2\pi f T)}{\pi(1 - 16f^2T^2)} e^{-\imath 2\pi f T}.$$

Please determine the time-averaged PSD of the signal in (b).

**Solutions.**

(a)

$$\begin{aligned}
R_{\text{OQPSK},\ell}(t_1, t_2) &= \mathbb{E}\left[s_{\text{OQPSK},\ell}(t_1) s^*_{\text{OQPSK},\ell}(t_2)\right] \\
&= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_n c_m^* \mathbb{E}[J_n J_m] g(t_1 - nT) g(t_2 - mT) \\
&= \sum_{n=-\infty}^{\infty} |c_n|^2 \mathbb{E}[J_n^2] g(t_1 - nT) g(t_2 - nT) \\
&= \sum_{n=-\infty}^{\infty} g(t_1 - nT) g(t_2 - nT)
\end{aligned}$$

Since for i.i.d. $\{I_n\}$ with $\Pr[I_n = -1] = \Pr[I_n = 1] = 1/2$,

$$\begin{aligned}
R_{\boldsymbol{v}_\ell}(t_1, t_2) &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \mathbb{E}\left[I_n I_m^*\right] g(t_1 - nT) g^*(t_2 - mT) \\
&= \sum_{n=-\infty}^{\infty} g(t_1 - nT) g^*(t_2 - nT),
\end{aligned}$$

we have that the time-averaged PSD of the given OQPSK signal is equal to:

$$\overline{S}_{\text{OQPSK},\ell}(f) = \overline{S}_{\boldsymbol{v}_\ell}(f) = \frac{1}{T}|G(f)|^2.$$

(b)

$$\begin{aligned}
R_{\text{MSK},\ell}(t_1, t_2) &= \mathbb{E}\left[s_\ell(t_1) s_\ell^*(t_2)\right] \\
&= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_n a_m^* \mathbb{E}[J_n J_m] g(t_1 - nT) g(t_2 - mT) \\
&= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_n a_m^* \mathbb{E}\left[\prod_{k_1=-\infty}^{n} \tilde{J}_{k_1} \prod_{k_2=-\infty}^{m} \tilde{J}_{k_2}\right] g(t_1 - nT) g(t_2 - mT) \\
&= \sum_{n=-\infty}^{\infty} g(t_1 - nT) g(t_2 - nT).
\end{aligned}$$

Hence, its time-averaged PSD is also equal to

$$\overline{S}_{\mathrm{MSK},\ell}(f) = \overline{S}_{\boldsymbol{v}_\ell}(f) = \frac{1}{T}|G(f)|^2.$$

(c)

$$\overline{S}_{\mathrm{MSK},\ell}(f) \;=\; \frac{1}{T}|G(f)|^2 = \frac{1}{T}\cdot\frac{16T^2\cos^2(2\pi fT)}{\pi^2(1-16f^2T^2)^2} = \frac{16T\cos^2(2\pi fT)}{\pi^2(1-16f^2T^2)^2}.$$