2015 Spring: The Second Midterm of Digital Communications

The total points of this exam is 130.

1. Suppose the complex-valued baseband output $\boldsymbol{r}_\ell$ is equal to the sum of channel input $\boldsymbol{s}_{m,\ell}$ and noise $\boldsymbol{n}_\ell$, and suppose there are M possible channel inputs to be transmitted (and hence $1 \le m \le M$). Denote by P_m the prior probability for input $\boldsymbol{s}_{m,\ell}$, and let $\boldsymbol{n}_\ell$ be Gaussian distributed with probability density function

$$\mathtt{f}(\boldsymbol{n}_\ell) = \left(\frac{1}{\pi\sigma^2}\right)^N \exp\left(-\frac{\|\boldsymbol{n}_\ell\|^2}{\sigma^2}\right)$$

where

$$\mathbb{E}[\boldsymbol{n}\boldsymbol{n}^\dagger] = \begin{bmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma^2 \end{bmatrix}.$$

Do the following problems.

(a) (15%) Prove that the MAP decision rule is given by

$$g_{\text{MAP}}(\boldsymbol{r}_\ell) = \arg\max_{1\le m\le M}\left[\sigma^2\log(P_m) - \|\boldsymbol{r}_\ell - \boldsymbol{s}_{m,\ell}\|^2\right]. \tag{1}$$

Hint: $g_{MAP}(\boldsymbol{r}_\ell) = \arg\max_{1\le m\le M}\left[P_m \cdot f(\boldsymbol{r}_\ell|\boldsymbol{s}_{m,\ell})\right]$, *where* $f(\boldsymbol{r}_\ell|\boldsymbol{s}_{m,\ell})$ *is the transition probability density function (pdf) of the channel.*

(b) (10%) If σ^2 is unknown to the receiver, how can we remove its impact in decision rule (1) through a proper setting of $\{P_m\}_{1\le m\le M}$? What will (1) be simplified to, under your choice of $\{P_m\}_{1\le m\le M}$?

(c) (10%) Now if the receiver, who does not know the value of noise variance σ^2, actually know its distribution, which is

$$\Pr[\sigma^2 = 2] = \Pr[\sigma^2 = 4] = \frac{1}{2},$$

determine the MAP decision rule for such statistical σ^2 under equal prior probability (i.e., $P_m = 1/M$ for $1 \le m \le M$).

Hint 1: $f(\boldsymbol{r}_\ell|\boldsymbol{s}_{m,\ell}) = \sum_{a\in\{2,4\}} f(\boldsymbol{r}_\ell|\boldsymbol{s}_{m,\ell},\sigma^2)\Pr(\sigma^2 = a) = \sum_{a\in\{2,4\}} \mathtt{f}(\boldsymbol{r}_\ell - \boldsymbol{s}_{m,\ell})\Pr(\sigma^2 = a)$.

Hint 2: *The below function is monotonically decreasing with respect to* λ *for* $\lambda > 0$*:*

$$\left(\frac{1}{2\pi}\right)^N \exp\left(-\frac{\lambda}{2}\right) + \left(\frac{1}{4\pi}\right)^N \exp\left(-\frac{\lambda}{4}\right).$$

2. For non-coherent transmission, we suppose the channel model becomes

$$\boldsymbol{r}_\ell = e^{\imath\phi}\boldsymbol{s}_{m,\ell} + \boldsymbol{n}_\ell,$$

where $\boldsymbol{n}_\ell$ has the same Gaussian distribution as in Problem 1. Denote by P_m the prior probability for input $\boldsymbol{s}_{m,\ell}$. Do the following problems.

(a) (15%) Given that ϕ is a two-value random variable with

$$\Pr[\phi = 0] = \Pr[\phi = \pi] = \frac{1}{2},$$

prove that the MAP decision rule is given by

$$g_{\text{MAP}}(\boldsymbol{r}_\ell) = \arg\max_{1 \le m \le M} P_m \exp\left(-\frac{\|\boldsymbol{s}_{m,\ell}\|^2}{\sigma^2}\right) \cosh\left(\frac{2|\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell}| \cos(\theta_m)}{\sigma^2}\right), \tag{2}$$

where $\theta_m = \angle(\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell})$.
Hint 1: $f(\boldsymbol{r}_\ell | \boldsymbol{s}_{m,\ell}) = \sum_{a \in \{0,\pi\}} \mathtt{f}(\boldsymbol{r}_\ell - \boldsymbol{s}_{m,\ell}) \Pr(\phi = a)$.
Hint 2:

$$\begin{aligned} \left\|\boldsymbol{r}_\ell - \boldsymbol{s}_{m,\ell} e^{\imath\phi}\right\|^2 &= \|\boldsymbol{r}_\ell\|^2 + \|\boldsymbol{s}_{m,\ell} e^{\imath\pi}\|^2 - 2 \cdot \mathrm{Re}[\boldsymbol{r}_\ell^\dagger (\boldsymbol{s}_{m,\ell} e^{\imath\phi})] \\ &= \|\boldsymbol{r}_\ell\|^2 + \|\boldsymbol{s}_{m,\ell} e^{\imath\pi}\|^2 - 2 \cdot \mathrm{Re}[|\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell}| e^{\imath\theta_m} e^{\imath\phi}] \end{aligned}$$

(b) (10%) If σ^2 is unknown to the system, how can we remove its impact in decision rule (2) through a proper setting of $\{P_m\}_{1 \le m \le M}$ and $\{\|\boldsymbol{s}_{m,\ell}\|^2\}_{1 \le m \le M}$? What will (2) be simplified to, under your choice of $\{P_m\}_{1 \le m \le M}$ and $\{\|\boldsymbol{s}_{m,\ell}\|^2\}_{1 \le m \le M}$?
Hint: $\cosh(x)$ *is larger if* $|x|$ *is larger.*

3. For 2-ary orthogonal signaling with symbol energy $\mathcal{E}_b$, the lowpass equivalence has signal constellation:

$$\begin{aligned} \boldsymbol{s}_{1,\ell} &= \begin{pmatrix} \sqrt{2\mathcal{E}_b} & 0 \end{pmatrix}^\top \\ \boldsymbol{s}_{2,\ell} &= \begin{pmatrix} 0 & \sqrt{2\mathcal{E}_b} \end{pmatrix}^\top \end{aligned}.$$

Assume equal prior probability on $\boldsymbol{s}_{1,\ell}$ and $\boldsymbol{s}_{2,\ell}$.

(a) (15%) Based on the non-coherent channel model:

$$\boldsymbol{r}_\ell = e^{\imath\phi} \boldsymbol{s}_{m,\ell} + \boldsymbol{n}_\ell,$$

where ϕ is uniformly distributed over $[0, 2\pi)$ and $\mathbb{E}[\boldsymbol{n}_\ell \boldsymbol{n}_\ell^\dagger] = \begin{bmatrix} 2N_0 & 0 \\ 0 & 2N_0 \end{bmatrix}$, the optimal decision rule for equal prior probability is known to be

$$g_{\text{MAP}}(\boldsymbol{r}_\ell) = g_{\text{ML}}(\boldsymbol{r}_\ell) = \arg\max_{1 \le m \le 2} |\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell}| = \arg\max_{1 \le m \le 2} |\boldsymbol{s}_{m,\ell}^\dagger \boldsymbol{r}_\ell|. \tag{3}$$

Prove that the error probability under decision rule (3) is equal to $P_e = \frac{1}{2} e^{-\frac{\mathcal{E}_b}{2N_0}}$.
Hint 1: *Define* $R_m = |\boldsymbol{s}_{m,\ell}^\dagger \boldsymbol{r}_\ell|$, *where* $1 \le m \le 2$. *Then we known that if* $\boldsymbol{s}_{1,\ell}$ *is transmitted,* R_1 *is Ricean distributed with pdf*

$$f_{R_1}(r_1) = \frac{r_1}{\sigma} I_0\left(\frac{s r_1}{\sigma^2}\right) e^{-\frac{r_1^2 + s^2}{2\sigma^2}}, \qquad r_1 > 0$$

where $I_0(\cdot)$ *is the modified Bessel function of the first kind and order zero, and* R_2 *is Rayleigh distributed with pdf*

$$f_{R_2}(r_2) = \frac{r_2}{\sigma^2} e^{-\frac{r_2^2}{2\sigma^2}}, \qquad r_2 > 0$$

where $\sigma^2 = 2\mathcal{E}_b N_0$ and $s = 2\mathcal{E}_b$. Under equal prior, the error is then given by $P_e = \Pr\{R_2 \geq R_1\}$ subject to $\boldsymbol{s}_{1,\ell}$ transmitted.

Hint 2: *From elementary calculus,*

$$\int_0^\infty \left(\int_{r_1}^\infty \frac{r_2}{\sigma^2} e^{-\frac{r_2^2}{2\sigma^2}} \, dr_2 \right) f_{R_1}(r_1)\, dr_1 = \int_0^\infty \left(-e^{-\frac{r_2^2}{2\sigma^2}} \bigg|_{r_1}^\infty \right) f_{R_1}(r_1)\, dr_1$$

and with $s' = s/\sqrt{2}$ and $r' = r_1\sqrt{2}$,

$$\int_0^\infty \frac{r_1}{\sigma} I_0\left(\frac{s r_1}{\sigma^2}\right) e^{-\frac{2r_1^2+s^2}{2\sigma^2}} \, dr_1 = \int_0^\infty \frac{r'}{2\sigma} I_0\left(\frac{s' r'}{\sigma^2}\right) e^{-\frac{r'^2+2s'^2}{2\sigma^2}} \, dr'.$$

(b) (15%) If the system is actually out-of-sync such that

$$\boldsymbol{r}_\ell = e^{\imath\phi} \boldsymbol{s}_{m,\ell} + \boldsymbol{n}_\ell, \tag{4}$$

where ϕ is a constant, but the receiver adopts ignorantly the perfect-sync decision rule below:

$$g_{\text{MAP}}(\boldsymbol{r}_\ell) = g_{\text{ML}}(\boldsymbol{r}_\ell) = \arg\max_{1\leq m\leq 2} \text{Re}[\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell}] = \arg\max_{1\leq m\leq 2} \text{Re}[\boldsymbol{s}_{m,\ell}^\dagger \boldsymbol{r}_\ell]. \tag{5}$$

Prove that the error probability under system (4) and decision rule (5) becomes $P_e = Q\left(\cos(\phi)\sqrt{\frac{\mathcal{E}_b}{N_0}}\right)$.

Hint 1: *Define $R_m = Re[\boldsymbol{s}_{m,\ell}^\dagger \boldsymbol{r}_\ell]$, where $1 \leq m \leq 2$. Then we know that if $\boldsymbol{s}_{1,\ell}$ is transmitted, R_1 is Gaussian distributed with mean s and variance σ^2, and R_2 is Gaussian distributed with mean 0 and variance σ^2, where $\sigma^2 = 2\mathcal{E}_b N_0$ and $s = 2\mathcal{E}_b\cos(\phi)$. Under equal prior, the error is then given by $P_e = \Pr\{R_2 \geq R_1\}$ subject to $\boldsymbol{s}_{1,\ell}$ transmitted.*

Hint 2: $\Pr\{\mathcal{N}(m,\sigma^2) < r\} = Q\left(\frac{m-r}{\sigma}\right)$

4. (a) (10%) Suppose the delay spread and Doppler spread of a multipath fading channel are $T_m = 100$ ns and $B_d = 100$ Hz, respectively. Can we choose signal duration T to result in a slowly-fading, frequency non-selective channel, provided the transmission bandwidth $B = 1/T$? Justify your answer by giving the conditions for slowly fading and frequency non-selectivity.

(b) (15%) In Problem 3(a), we obtain that under the channel model

$$\boldsymbol{r}_\ell = e^{\imath\phi} \boldsymbol{s}_{m,\ell} + \boldsymbol{n}_\ell,$$

and uniformly distributed ϕ, the non-coherent decision rule in (3) gives

$$P_e = \frac{1}{2} e^{-\frac{\mathcal{E}_b}{2N_0}}.$$

Now suppose the channel additionally suffers a random fading α, modeled as

$$\boldsymbol{r}_\ell = \alpha e^{\imath\phi} \boldsymbol{s}_{m,\ell} + \boldsymbol{n}_\ell. \tag{6}$$

Prove that the resulting error probability is equal to $\frac{1}{\bar{\gamma}_b+2}$.

Hint: *By combining α into signal, we can view the system as one that transmits*

$$\begin{array}{rl} \tilde{\boldsymbol{s}}_{1,\ell} = & \begin{pmatrix} \alpha\sqrt{2\mathcal{E}_b} & 0 \end{pmatrix}^\top = \begin{pmatrix} \sqrt{2\alpha^2\mathcal{E}_b} & 0 \end{pmatrix}^\top \\ \tilde{\boldsymbol{s}}_{2,\ell} = & \begin{pmatrix} 0 & \sqrt{2\alpha^2\mathcal{E}_b} \end{pmatrix}^\top = \begin{pmatrix} 0 & \sqrt{2\alpha^2\mathcal{E}_b} \end{pmatrix}^\top \end{array}.$$

over the channel

$$\boldsymbol{r}_\ell = e^{\imath\phi}\tilde{\boldsymbol{s}}_{m,\ell} + \boldsymbol{n}_\ell.$$

Given that α is Rayleigh distributed, we obtain that $\gamma_b = \gamma_b(\alpha) = \alpha^2\mathcal{E}_b/N_0$ is χ^2-distributed with two degrees of freedom. In other words, the pdf of γ_b is given by $f(\gamma_b) = \frac{1}{\bar{\gamma}_b}e^{-\gamma_b/\bar{\gamma}_b}$, where $\bar{\gamma}_b = \mathbb{E}[\gamma_b]$.

(c) (15%) Now in order to improve the system performance in (b), we introduce the diversity technique:

$$\boldsymbol{r}_{k,\ell} = \alpha_k e^{\imath\phi}\boldsymbol{s}_{m,\ell} + \boldsymbol{n}_{k,\ell}, \quad k = 1, 2, \ldots, L.$$

Suppose $\{\alpha_k\}_{k=1}^L$ is perfectly estimated at the receiver; hence we can use **maximal ratio combiner** (even in a noncoherent transmission system). As a result,

$$\boldsymbol{r}_\ell = \sum_{k=1}^{L}\alpha_k\boldsymbol{r}_{k,\ell} = \left(\sum_{k=1}^{L}\alpha_k^2\right)e^{\imath\phi}\boldsymbol{s}_{m,\ell} + \left(\sum_{k=1}^{L}\alpha_k\boldsymbol{n}_{k,\ell}\right). \tag{7}$$

Since $\{\alpha_k\}_{k=1}^L$ are known, we can equivalently transfer the system in (7) (by dividing α) to:

$$\tilde{\boldsymbol{r}}_\ell = \boldsymbol{r}_\ell/\alpha = \alpha e^{\imath\phi}\boldsymbol{s}_{m,\ell} + \tilde{\boldsymbol{n}}_\ell, \tag{8}$$

where $\alpha = \sqrt{\sum_{k=1}^L \alpha_k^2}$ and $\tilde{\boldsymbol{n}}_\ell = \left(\sum_{k=1}^L \alpha_k\boldsymbol{n}_{k,\ell}\right)\Big/\sqrt{\sum_{k=1}^L \alpha_k^2}$. Since $\mathbb{E}[\tilde{\boldsymbol{n}}_\ell\tilde{\boldsymbol{n}}_\ell^\dagger] = \mathbb{E}[\boldsymbol{n}_{k,\ell}\boldsymbol{n}_{k,\ell}^\dagger]$ for independent and identically distributed $\{\boldsymbol{n}_{k,\ell}\}_{k=1}^L$, (8) is equivalently to (6) except that $\gamma_b(\alpha) = \alpha^2\mathcal{E}_b/N_0$ becomes χ^2-distributed with $2L$ degrees of freedom, i.e.,

$$f(\gamma_b) = \frac{1}{(L-1)!\bar{\gamma}_c^L}\gamma_b^{L-1}e^{-\gamma_b/\bar{\gamma}_c}$$

where $\bar{\gamma}_c = \mathbb{E}[\alpha_k^2]\mathcal{E}_b/N_0$. Prove that the resulting error probability is equal to $\frac{2^{L-1}}{(\bar{\gamma}_c+2)^L}$.

Hint:

$$\int_0^\infty \gamma_b^{L-1}e^{-\gamma_b/\bar{\gamma}_c}d\gamma_b = (L-1)!\bar{\gamma}_c^L$$

or equivalently,

$$\int_0^\infty \gamma_b^{L-1}e^{-\gamma_b(1/2+1/\bar{\gamma}_c)}d\gamma_b = (L-1)!\frac{1}{(1/2+1/\bar{\gamma}_c)^L}.$$