2015 Spring: The Second Midterm of Digital Communications

The total points of this exam is 130.

1. Suppose the complex-valued baseband output $\boldsymbol{r}_\ell$ is equal to the sum of channel input $\boldsymbol{s}_{m,\ell}$ and noise $\boldsymbol{n}_\ell$, and suppose there are M possible channel inputs to be transmitted (and hence $1 \le m \le M$). Denote by P_m the prior probability for input $\boldsymbol{s}_{m,\ell}$, and let $\boldsymbol{n}_\ell$ be Gaussian distributed with probability density function

$$\mathtt{f}\left(\boldsymbol{n}_\ell\right) = \left(\frac{1}{\pi\sigma^2}\right)^N \exp\left(-\frac{\left\|\boldsymbol{n}_\ell\right\|^2}{\sigma^2}\right)$$

where

$$\mathbb{E}[\boldsymbol{n}\boldsymbol{n}^\dagger] = \begin{bmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma^2 \end{bmatrix}.$$

Do the following problems.

(a) (15%) Prove that the MAP decision rule is given by

$$g_{\mathrm{MAP}}(\boldsymbol{r}_\ell) = \arg\max_{1\le m\le M} \left[\sigma^2 \log(P_m) - \|\boldsymbol{r}_\ell - \boldsymbol{s}_{m,\ell}\|^2\right]. \tag{1}$$

Hint: $g_{MAP}(\boldsymbol{r}_\ell) = \arg\max_{1\le m\le M}\left[P_m \cdot f\left(\boldsymbol{r}_\ell|\boldsymbol{s}_{m,\ell}\right)\right]$, *where* $f\left(\boldsymbol{r}_\ell|\boldsymbol{s}_{m,\ell}\right)$ *is the transition probability density function (pdf) of the channel.*

(b) (10%) If σ^2 is unknown to the receiver, how can we remove its impact in decision rule (1) through a proper setting of $\{P_m\}_{1\le m\le M}$? What will (1) be simplified to, under your choice of $\{P_m\}_{1\le m\le M}$?

(c) (10%) Now if the receiver, who does not know the value of noise variance σ^2, actually know its distribution, which is

$$\Pr[\sigma^2 = 2] = \Pr[\sigma^2 = 4] = \frac{1}{2},$$

determine the MAP decision rule for such statistical σ^2 under equal prior probability (i.e., $P_m = 1/M$ for $1 \le m \le M$).

Hint 1: $f\left(\boldsymbol{r}_\ell|\boldsymbol{s}_{m,\ell}\right) = \sum_{a\in\{2,4\}} f\left(\boldsymbol{r}_\ell|\boldsymbol{s}_{m,\ell}, \sigma^2\right)\Pr(\sigma^2 = a) = \sum_{a\in\{2,4\}} \mathtt{f}(\boldsymbol{r}_\ell - \boldsymbol{s}_{m,\ell})\Pr(\sigma^2 = a)$.

Hint 2: *The below function is monotonically decreasing with respect to* λ *for* $\lambda > 0$*:*

$$\left(\frac{1}{2\pi}\right)^N \exp\left(-\frac{\lambda}{2}\right) + \left(\frac{1}{4\pi}\right)^N \exp\left(-\frac{\lambda}{4}\right).$$

Solutions.

(a)

$$\begin{aligned} g_{\text{MAP}}(\boldsymbol{r}_\ell) &= \arg\max_{1\le m\le M} \left[P_m \cdot f\left(\boldsymbol{r}_\ell | \boldsymbol{s}_{m,\ell}\right)\right] \\ &= \arg\max_{1\le m\le M} \left[P_m \cdot \mathtt{f}\left(\boldsymbol{r}_\ell - \boldsymbol{s}_{m,\ell}\right)\right] \\ &= \arg\max_{1\le m\le M} \left[P_m \left(\frac{1}{\pi\sigma^2}\right)^N \exp\left(-\frac{\|\boldsymbol{r}_\ell - \boldsymbol{s}_{m,\ell}\|^2}{\sigma^2}\right)\right] \\ &= \arg\max_{1\le m\le M} \left[\log(P_m) - \frac{\|\boldsymbol{r}_\ell - \boldsymbol{s}_{m,\ell}\|^2}{\sigma^2}\right] \\ &= \arg\max_{1\le m\le M} \left[\sigma^2\log(P_m) - \|\boldsymbol{r}_\ell - \boldsymbol{s}_{m,\ell}\|^2\right]. \end{aligned}$$

(b) Setting $P_m = 1/M$ yields

$$\begin{aligned} g_{\text{MAP}}(\boldsymbol{r}_\ell) &= \arg\max_{1\le m\le M} \left[\sigma^2\log(P_m) - \|\boldsymbol{r}_\ell - \boldsymbol{s}_{m,\ell}\|^2\right] \\ &= \arg\max_{1\le m\le M} \left[\sigma^2\log(1/M) - \|\boldsymbol{r}_\ell - \boldsymbol{s}_{m,\ell}\|^2\right] \\ &= \arg\min_{1\le m\le M} \|\boldsymbol{r}_\ell - \boldsymbol{s}_{m,\ell}\|^2. \end{aligned}$$

(c) Under equal prior,

$$\begin{aligned} g_{\text{MAP}}(\boldsymbol{r}_\ell) &= \arg\max_{1\le m\le M} f\left(\boldsymbol{r}_\ell | \boldsymbol{s}_{m,\ell}\right) \\ &= \arg\max_{1\le m\le M} \Bigg[\underbrace{\frac{1}{2}\left(\frac{1}{2\pi}\right)^N \exp\left(-\frac{\|\boldsymbol{r}_\ell - \boldsymbol{s}_{m,\ell}\|^2}{2}\right)}_{\sigma^2=2} \\ &\qquad + \underbrace{\frac{1}{2}\left(\frac{1}{4\pi}\right)^N \exp\left(-\frac{\|\boldsymbol{r}_\ell - \boldsymbol{s}_{m,\ell}\|^2}{4}\right)}_{\sigma^2=4}\Bigg] \\ &= \arg\min_{1\le m\le M} \|\boldsymbol{r}_\ell - \boldsymbol{s}_{m,\ell}\|^2, \end{aligned}$$

where the last step follows because

$$\left(\frac{1}{2\pi}\right)^N \exp\left(-\frac{\lambda}{2}\right) + \left(\frac{1}{4\pi}\right)^N \exp\left(-\frac{\lambda}{4}\right).$$

is strict decreasing with respect to λ for $\lambda > 0$.

From this, you shall know that "equal prior" is really an advantageous system setting.

2. For non-coherent transmission, we suppose the channel model becomes

$$\boldsymbol{r}_\ell = e^{\imath\phi}\boldsymbol{s}_{m,\ell} + \boldsymbol{n}_\ell,$$

where $\boldsymbol{n}_\ell$ has the same Gaussian distribution as in Problem 1. Denote by P_m the prior probability for input $\boldsymbol{s}_{m,\ell}$. Do the following problems.

(a) (15%) Given that ϕ is a two-value random variable with

$$\Pr[\phi = 0] = \Pr[\phi = \pi] = \frac{1}{2},$$

prove that the MAP decision rule is given by

$$g_{\text{MAP}}(\boldsymbol{r}_\ell) = \arg\max_{1\le m\le M} P_m \exp\left(-\frac{\|\boldsymbol{s}_{m,\ell}\|^2}{\sigma^2}\right)\cosh\left(\frac{2|\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell}|\cos(\theta_m)}{\sigma^2}\right), \tag{2}$$

where $\theta_m = \angle(\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell})$.

Hint 1: $f(\boldsymbol{r}_\ell|\boldsymbol{s}_{m,\ell}) = \sum_{a\in\{0,\pi\}} \mathtt{f}(\boldsymbol{r}_\ell - \boldsymbol{s}_{m,\ell})\Pr(\phi = a)$.

Hint 2:

$$\begin{aligned}\left\|\boldsymbol{r}_\ell - \boldsymbol{s}_{m,\ell}e^{\imath\phi}\right\|^2 &= \|\boldsymbol{r}_\ell\|^2 + \|\boldsymbol{s}_{m,\ell}e^{\imath\pi}\|^2 - 2\cdot\text{Re}[\boldsymbol{r}_\ell^\dagger(\boldsymbol{s}_{m,\ell}e^{\imath\phi})] \\ &= \|\boldsymbol{r}_\ell\|^2 + \|\boldsymbol{s}_{m,\ell}e^{\imath\pi}\|^2 - 2\cdot\text{Re}[|\boldsymbol{r}_\ell^\dagger\boldsymbol{s}_{m,\ell}|e^{\imath\theta_m}e^{\imath\phi}]\end{aligned}$$

(b) (10%) If σ^2 is unknown to the system, how can we remove its impact in decision rule (2) through a proper setting of $\{P_m\}_{1\le m\le M}$ and $\{\|\boldsymbol{s}_{m,\ell}\|^2\}_{1\le m\le M}$? What will (2) be simplified to, under your choice of $\{P_m\}_{1\le m\le M}$ and $\{\|\boldsymbol{s}_{m,\ell}\|^2\}_{1\le m\le M}$?

Hint: $\cosh(x)$ *is larger if* $|x|$ *is larger.*

Solutions.

(a)

$$
\begin{aligned}
g_{\text{MAP}}(\boldsymbol{r}_\ell) &= \arg\max_{1\le m\le M} \left[P_m \cdot f\left(\boldsymbol{r}_\ell | \boldsymbol{s}_{m,\ell}\right)\right] \\
&= \arg\max_{1\le m\le M} \left(P_m \left[\frac{1}{2}\cdot\left(\frac{1}{\pi\sigma^2}\right)^N \exp\left(-\frac{\|\boldsymbol{r}_\ell - \boldsymbol{s}_{m,\ell}\|^2}{\sigma^2}\right)\right.\right. \\
&\qquad \left.\left. + \frac{1}{2}\cdot\left(\frac{1}{\pi\sigma^2}\right)^N \exp\left(-\frac{\|\boldsymbol{r}_\ell - \boldsymbol{s}_{m,\ell}e^{\imath\pi}\|^2}{\sigma^2}\right)\right]\right) \\
&= \arg\max_{1\le m\le M} P_m \left[\exp\left(-\frac{\|\boldsymbol{r}_\ell\|^2 + \|\boldsymbol{s}_{m,\ell}\|^2 - 2\text{Re}[\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell}]}{\sigma^2}\right)\right. \\
&\qquad \left. + \exp\left(-\frac{\|\boldsymbol{r}_\ell\|^2 + \|\boldsymbol{s}_{m,\ell}e^{\imath\pi}\|^2 - 2\text{Re}[\boldsymbol{r}_\ell^\dagger (\boldsymbol{s}_{m,\ell}e^{\imath\pi})]}{\sigma^2}\right)\right] \\
&= \arg\max_{1\le m\le M} P_m \exp\left(-\frac{\|\boldsymbol{s}_{m,\ell}\|^2}{\sigma^2}\right)\left[\exp\left(\frac{2\cdot\text{Re}[|\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell}|e^{\imath\theta_m}]}{\sigma^2}\right)\right. \\
&\qquad \left. + \exp\left(\frac{2\cdot\text{Re}[|\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell}|e^{\imath\theta_m}e^{\imath\pi}]}{\sigma^2}\right)\right] \\
&= \arg\max_{1\le m\le M} P_m \exp\left(-\frac{\|\boldsymbol{s}_{m,\ell}\|^2}{\sigma^2}\right)\left[\exp\left(\frac{2|\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell}|\cos(\theta_m)}{\sigma^2}\right)\right. \\
&\qquad \left. + \exp\left(\frac{2|\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell}|\cos(\theta_m+\pi)}{\sigma^2}\right)\right] \\
&= \arg\max_{1\le m\le M} P_m \exp\left(-\frac{\|\boldsymbol{s}_{m,\ell}\|^2}{\sigma^2}\right)\left[\exp\left(\frac{2|\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell}|\cos(\theta_m)}{\sigma^2}\right)\right. \\
&\qquad \left. + \exp\left(-\frac{2|\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell}|\cos(\theta_m)}{\sigma^2}\right)\right] \\
&= \arg\max_{1\le m\le M} P_m \exp\left(-\frac{\|\boldsymbol{s}_{m,\ell}\|^2}{\sigma^2}\right)\cosh\left(\frac{2|\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell}|\cos(\theta_m)}{\sigma^2}\right),
\end{aligned}
$$

where $\theta_m = \angle(\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell})$.

(b) Setting $P_m = 1/M$ and constant $\|\boldsymbol{s}_{m,\ell}\|^2$ yields

$$
\begin{aligned}
g_{\text{MAP}}(\boldsymbol{r}_\ell) &= \arg\max_{1\le m\le M} P_m \exp\left(-\frac{\|\boldsymbol{s}_{m,\ell}\|^2}{\sigma^2}\right)\cosh\left(\frac{2|\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell}|\cos(\theta_m)}{\sigma^2}\right) \\
&= \arg\max_{1\le m\le M} \cosh\left(\frac{2|\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell}|\cos(\theta_m)}{\sigma^2}\right) \\
&= \arg\max_{1\le m\le M} |\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell}|\cdot|\cos(\theta_m)| \text{ or equivalently } \arg\max_{1\le m\le M} |\text{Re}[\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell}]|.
\end{aligned}
$$

3. For 2-ary orthogonal signaling with symbol energy $\mathcal{E}_b$, the lowpass equivalence has signal constellation:

$$
\begin{aligned}
\boldsymbol{s}_{1,\ell} &= \begin{pmatrix} \sqrt{2\mathcal{E}_b} & 0 \end{pmatrix}^\top \\
\boldsymbol{s}_{2,\ell} &= \begin{pmatrix} 0 & \sqrt{2\mathcal{E}_b} \end{pmatrix}^\top
\end{aligned}.
$$

Assume equal prior probability on $\boldsymbol{s}_{1,\ell}$ and $\boldsymbol{s}_{2,\ell}$.

(a) (15%) Based on the non-coherent channel model:

$$\boldsymbol{r}_\ell = e^{\imath\phi}\boldsymbol{s}_{m,\ell} + \boldsymbol{n}_\ell,$$

where ϕ is uniformly distributed over $[0, 2\pi)$ and $\mathbb{E}[\boldsymbol{n}_\ell \boldsymbol{n}_\ell^\dagger] = \begin{bmatrix} 2N_0 & 0 \\ 0 & 2N_0 \end{bmatrix}$, the optimal decision rule for equal prior probability is known to be

$$g_{\text{MAP}}(\boldsymbol{r}_\ell) = g_{\text{ML}}(\boldsymbol{r}_\ell) = \arg\max_{1\le m\le 2} |\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell}| = \arg\max_{1\le m\le 2} |\boldsymbol{s}_{m,\ell}^\dagger \boldsymbol{r}_\ell|. \tag{3}$$

Prove that the error probability under decision rule (3) is equal to $P_e = \frac{1}{2}e^{-\frac{\mathcal{E}_b}{2N_0}}$.

Hint 1: *Define $R_m = |\boldsymbol{s}_{m,\ell}^\dagger \boldsymbol{r}_\ell|$, where $1 \le m \le 2$. Then we known that if $\boldsymbol{s}_{1,\ell}$ is transmitted, R_1 is Ricean distributed with pdf*

$$f_{R_1}(r_1) \;=\; \frac{r_1}{\sigma} I_0\left(\frac{s r_1}{\sigma^2}\right) e^{-\frac{r_1^2+s^2}{2\sigma^2}}, \qquad r_1 > 0$$

where $I_0(\cdot)$ is the modified Bessel function of the first kind and order zero, and R_2 is Rayleigh distributed with pdf

$$f_{R_2}(r_2) \;=\; \frac{r_2}{\sigma^2} e^{-\frac{r_2^2}{2\sigma^2}}, \qquad r_2 > 0$$

where $\sigma^2 = 2\mathcal{E}_b N_0$ and $s = 2\mathcal{E}_b$. Under equal prior, the error is then given by $P_e = \Pr\{R_2 \ge R_1\}$ subject to $\boldsymbol{s}_{1,\ell}$ transmitted.

Hint 2: *From elementary calculus,*

$$\int_0^\infty \left(\int_{r_1}^\infty \frac{r_2}{\sigma^2} e^{-\frac{r_2^2}{2\sigma^2}}\, dr_2\right) f_{R_1}(r_1)\, dr_1 = \int_0^\infty \left(-e^{-\frac{r_2^2}{2\sigma^2}}\Big|_{r_1}^{\infty}\right) f_{R_1}(r_1)\, dr_1$$

and with $s' = s/\sqrt{2}$ and $r' = r_1\sqrt{2}$,

$$\int_0^\infty \frac{r_1}{\sigma} I_0\left(\frac{s r_1}{\sigma^2}\right) e^{-\frac{2r_1^2+s^2}{2\sigma^2}}\, dr_1 = \int_0^\infty \frac{r'}{2\sigma} I_0\left(\frac{s' r'}{\sigma^2}\right) e^{-\frac{r'^2+2s'^2}{2\sigma^2}}\, dr'.$$

(b) (15%) If the system is actually out-of-sync such that

$$\boldsymbol{r}_\ell = e^{\imath\phi}\boldsymbol{s}_{m,\ell} + \boldsymbol{n}_\ell, \tag{4}$$

where ϕ is a constant, but the receiver adopts ignorantly the perfect-sync decision rule below:

$$g_{\text{MAP}}(\boldsymbol{r}_\ell) = g_{\text{ML}}(\boldsymbol{r}_\ell) = \arg\max_{1\le m\le 2} \text{Re}[\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell}] = \arg\max_{1\le m\le 2} \text{Re}[\boldsymbol{s}_{m,\ell}^\dagger \boldsymbol{r}_\ell]. \tag{5}$$

Prove that the error probability under system (4) and decision rule (5) becomes $P_e = Q\left(\cos(\phi)\sqrt{\frac{\mathcal{E}_b}{N_0}}\right)$.

Hint 1: *Define $R_m = Re[\boldsymbol{s}_{m,\ell}^\dagger \boldsymbol{r}_\ell]$, where $1 \le m \le 2$. Then we know that if $\boldsymbol{s}_{1,\ell}$ is transmitted, R_1 is Gaussian distributed with mean s and variance σ^2, and R_2 is Gaussian distributed with mean 0 and variance σ^2, where $\sigma^2 = 2\mathcal{E}_b N_0$ and $s = 2\mathcal{E}_b\cos(\phi)$. Under equal prior, the error is then given by $P_e = \Pr\{R_2 \ge R_1\}$ subject to $\boldsymbol{s}_{1,\ell}$ transmitted.*

Hint 2: $\Pr\{\mathcal{N}(m, \sigma^2) < r\} = Q\left(\frac{m-r}{\sigma}\right)$

Solutions.

(a) Given $\boldsymbol{s}_{1,\ell}$ transmitted, the reception is $\boldsymbol{r}_\ell = e^{\imath\phi}\boldsymbol{s}_{1,\ell} + \boldsymbol{n}_\ell$.

As a reminder, the noncoherent ML computes

$$\begin{aligned} R_1 = \left|\boldsymbol{s}_{1,\ell}^\dagger \boldsymbol{r}_\ell\right| &= \left|\boldsymbol{s}_{1,\ell}^\dagger (e^{\imath\phi}\boldsymbol{s}_{1,\ell} + \boldsymbol{n}_\ell)\right| = \left|2\mathcal{E}_b e^{\imath\phi} + \boldsymbol{s}_{1,\ell}^\dagger \boldsymbol{n}_\ell\right| \\ R_2 = \left|\boldsymbol{s}_{2,\ell}^\dagger \boldsymbol{r}_\ell\right| &= \left|\boldsymbol{s}_{2,\ell}^\dagger (e^{\imath\phi}\boldsymbol{s}_{1,\ell} + \boldsymbol{n}_\ell)\right| = \left|\boldsymbol{s}_{2,\ell}^\dagger \boldsymbol{n}_\ell\right| \end{aligned}$$

Note that since $\boldsymbol{s}_{1,\ell}^\dagger \boldsymbol{n}_\ell$ and $\boldsymbol{s}_{2,\ell}^\dagger \boldsymbol{n}_\ell$ are independent, R_1 and R_2 are independent variables. Recall that $\boldsymbol{n}_\ell$ is a complex Gaussian random vector with covariance matrix

$$\mathbb{E}[\boldsymbol{n}_\ell \boldsymbol{n}_\ell^\dagger] = 2N_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};$$

This implies $\boldsymbol{s}_{m,\ell}^\dagger \boldsymbol{n}_\ell$ is a circular symmetric complex Gaussian random variable with

$$\mathbb{E}\left[\boldsymbol{s}_{m,\ell}^\dagger \boldsymbol{n}_\ell \boldsymbol{n}_\ell^\dagger \boldsymbol{s}_{m,\ell}\right] = 2N_0 \cdot 2\mathcal{E}_b = 4\mathcal{E}_b N_0$$

Thus, as given by hint, R_1 is Ricean distributed with density

$$f_{R_1}(r_1) = \frac{r_1}{\sigma} I_0\left(\frac{s r_1}{\sigma^2}\right) e^{-\frac{r_1^2+s^2}{2\sigma^2}}, \qquad r_1 > 0$$

and R_2 is Rayleigh distributed with density

$$f_{R_2}(r_2) = \frac{r_2}{\sigma^2} e^{-\frac{r_2^2}{2\sigma^2}}, \qquad r_2 > 0,$$

where $\sigma^2 = 2\mathcal{E}_b N_0$ and $s = 2\mathcal{E}_b$. As a result,

$$\begin{aligned} P_e &= \Pr\{R_2 \geq R_1\} \\ &= \int_0^\infty \Pr\{R_2 \geq r_1 | R_1 = r_1\} f_{R_1}(r_1)\, dr_1 \\ &= \int_0^\infty \Pr\{R_2 \geq r_1\} f_{R_1}(r_1)\, dr_1 \quad \text{(since } R_1 \text{ and } R_2 \text{ are independent)} \\ &= \int_0^\infty \left[\int_{r_1}^\infty f_{R_2}(r_2)\, dr_2\right] f_{R_1}(r_1)\, dr_1 \\ &= \int_0^\infty \left[\int_{r_1}^\infty \frac{r_2}{\sigma^2} e^{-\frac{r_2^2}{2\sigma^2}}\, dr_2\right] f_{R_1}(r_1)\, dr_1 \\ &= \int_0^\infty \left[-e^{-\frac{r_2^2}{2\sigma^2}}\Big|_{r_1}^\infty\right] f_{R_1}(r_1)\, dr_1 \\ &= \int_0^\infty e^{-\frac{r_1^2}{2\sigma^2}} f_{R_1}(r_1)\, dr_1 \\ &= \int_0^\infty e^{-\frac{r_1^2}{2\sigma^2}} \left(\frac{r_1}{\sigma} I_0\left(\frac{s r_1}{\sigma^2}\right) e^{-\frac{r_1^2+s^2}{2\sigma^2}}\right) dr_1 \\ &= \int_0^\infty \frac{r_1}{\sigma} I_0\left(\frac{s r_1}{\sigma^2}\right) e^{-\frac{2r_1^2+s^2}{2\sigma^2}}\, dr_1. \end{aligned}$$

Setting $s' = s/\sqrt{2}$ and $r' = r_1\sqrt{2}$ gives

$$\begin{aligned}
P_e &= \int_0^\infty \frac{r_1}{\sigma} I_0\left(\frac{sr_1}{\sigma^2}\right) e^{-\frac{2r_1^2+s^2}{2\sigma^2}}\, dr_1 \\
&= \int_0^\infty \frac{r'}{2\sigma} I_0\left(\frac{s'r'}{\sigma^2}\right) e^{-\frac{r'^2+2s'^2}{2\sigma^2}}\, dr' \\
&= \frac{1}{2} e^{-\frac{s'^2}{2\sigma^2}} \int_0^\infty \frac{r'}{\sigma} I_0\left(\frac{s'r'}{\sigma^2}\right) e^{-\frac{r'^2+s'^2}{2\sigma^2}}\, dr' \\
&= \frac{1}{2} e^{-\frac{s^2}{4\sigma^2}} \\
&= \frac{1}{2} e^{-\frac{\mathcal{E}_b}{2N_0}}.
\end{aligned}$$

(b) Given $\boldsymbol{s}_{m,1}$ transmitted, the reception is $\boldsymbol{r}_\ell = e^{\imath\phi}\boldsymbol{s}_{1,\ell} + \boldsymbol{n}_\ell$. The coherent ML computes

$$\begin{aligned}
R_1 = \mathrm{Re}[\boldsymbol{s}_{1,\ell}^\dagger \boldsymbol{r}_\ell] &= \mathrm{Re}[\boldsymbol{s}_{1,\ell}^\dagger (e^{\imath\phi}\boldsymbol{s}_{1,\ell} + \boldsymbol{n}_\ell)] = 2\mathcal{E}_b\cos(\phi) + \mathrm{Re}[\boldsymbol{s}_{1,\ell}^\dagger \boldsymbol{n}_\ell] \\
R_2 = \mathrm{Re}[\boldsymbol{s}_{2,\ell}^\dagger \boldsymbol{r}_\ell] &= \mathrm{Re}[\boldsymbol{s}_{2,\ell}^\dagger (e^{\imath\phi}\boldsymbol{s}_{1,\ell} + \boldsymbol{n}_\ell)] = \mathrm{Re}[\boldsymbol{s}_{2,\ell}^\dagger \boldsymbol{n}_\ell]
\end{aligned}$$

Note that since $\boldsymbol{s}_{1,\ell}^\dagger \boldsymbol{n}_\ell$ and $\boldsymbol{s}_{2,\ell}^\dagger \boldsymbol{n}_\ell$ are independent, R_1 and R_2 are independent Gaussian variables; hence, $R_1 - R_2$ is Gaussian distributed with mean $s = 2\mathcal{E}_b\cos(\phi)$ and variance $2\sigma^2$. As a result,

$$\begin{aligned}
P_e &= \Pr\{R_2 \ge R_1\} \\
&= \Pr\{R_1 - R_2 \le 0\} \\
&= Q\left(\frac{s-0}{\sqrt{2\sigma^2}}\right) \\
&= Q\left(\frac{2\mathcal{E}_b\cos(\phi)}{\sqrt{4\mathcal{E}_b N_0}}\right) \\
&= Q\left(\cos(\phi)\sqrt{\frac{\mathcal{E}_b}{N_0}}\right).
\end{aligned}$$

From this, you shall learn that the out-of-sync coherent detection may perform worse than the non-coherent detection and the performance loss can be quantitized by $10\log_{10}(\cos^2(\phi))$ dB.

4. (a) (10%) Suppose the delay spread and Doppler spread of a multipath fading channel are $T_m = 100$ ns and $B_d = 100$ Hz, respectively. Can we choose signal duration T to result in a slowly-fading, frequency non-selective channel, provided the transmission bandwidth $B = 1/T$? Justify your answer by giving the conditions for slowly fading and frequency non-selectivity.

(b) (15%) In Problem 3(a), we obtain that under the channel model

$$\boldsymbol{r}_\ell = e^{\imath\phi}\boldsymbol{s}_{m,\ell} + \boldsymbol{n}_\ell,$$

and uniformly distributed ϕ, the non-coherent decision rule in (3) gives

$$P_e = \frac{1}{2} e^{-\frac{\mathcal{E}_b}{2N_0}}.$$

Now suppose the channel additionally suffers a random fading α, modeled as

$$\boldsymbol{r}_\ell = \alpha e^{\imath\phi}\boldsymbol{s}_{m,\ell} + \boldsymbol{n}_\ell. \tag{6}$$

Prove that the resulting error probability is equal to $\frac{1}{\bar{\gamma}_b+2}$.

Hint: *By combining α into signal, we can view the system as one that transmits*

$$\begin{aligned}\tilde{\boldsymbol{s}}_{1,\ell} &= \begin{pmatrix} \alpha\sqrt{2\mathcal{E}_b} & 0 \end{pmatrix}^\top = \begin{pmatrix} \sqrt{2\alpha^2\mathcal{E}_b} & 0 \end{pmatrix}^\top \\ \tilde{\boldsymbol{s}}_{2,\ell} &= \begin{pmatrix} 0 & \sqrt{2\alpha^2\mathcal{E}_b} \end{pmatrix}^\top = \begin{pmatrix} 0 & \sqrt{2\alpha^2\mathcal{E}_b} \end{pmatrix}^\top\end{aligned}.$$

over the channel

$$\boldsymbol{r}_\ell = e^{\imath\phi}\tilde{\boldsymbol{s}}_{m,\ell} + \boldsymbol{n}_\ell.$$

Given that α is Rayleigh distributed, we obtain that $\gamma_b = \gamma_b(\alpha) = \alpha^2\mathcal{E}_b/N_0$ is χ^2-distributed with two degrees of freedom. In other words, the pdf of γ_b is given by $f(\gamma_b) = \frac{1}{\bar{\gamma}_b}e^{-\gamma_b/\bar{\gamma}_b}$, where $\bar{\gamma}_b = \mathbb{E}[\gamma_b]$.

(c) (15%) Now in order to improve the system performance in (b), we introduce the diversity technique:

$$\boldsymbol{r}_{k,\ell} = \alpha_k e^{\imath\phi}\boldsymbol{s}_{m,\ell} + \boldsymbol{n}_{k,\ell}, \quad k = 1, 2, \ldots, L.$$

Suppose $\{\alpha_k\}_{k=1}^L$ is perfectly estimated at the receiver; hence we can use **maximal ratio combiner** (even in a noncoherent transmission system). As a result,

$$\boldsymbol{r}_\ell = \sum_{k=1}^L \alpha_k \boldsymbol{r}_{k,\ell} = \left(\sum_{k=1}^L \alpha_k^2\right) e^{\imath\phi}\boldsymbol{s}_{m,\ell} + \left(\sum_{k=1}^L \alpha_k \boldsymbol{n}_{k,\ell}\right). \tag{7}$$

Since $\{\alpha_k\}_{k=1}^L$ are known, we can equivalently transfer the system in (7) (by dividing α) to:

$$\tilde{\boldsymbol{r}}_\ell = \boldsymbol{r}_\ell/\alpha = \alpha e^{\imath\phi}\boldsymbol{s}_{m,\ell} + \tilde{\boldsymbol{n}}_\ell, \tag{8}$$

where $\alpha = \sqrt{\sum_{k=1}^L \alpha_k^2}$ and $\tilde{\boldsymbol{n}}_\ell = \left(\sum_{k=1}^L \alpha_k \boldsymbol{n}_{k,\ell}\right)\Big/\sqrt{\sum_{k=1}^L \alpha_k^2}$. Since $\mathbb{E}[\tilde{\boldsymbol{n}}_\ell\tilde{\boldsymbol{n}}_\ell^\dagger] = \mathbb{E}[\boldsymbol{n}_{k,\ell}\boldsymbol{n}_{k,\ell}^\dagger]$ for independent and identically distributed $\{\boldsymbol{n}_{k,\ell}\}_{k=1}^L$, (8) is equivalently to (6) except that $\gamma_b(\alpha) = \alpha^2\mathcal{E}_b/N_0$ becomes χ^2-distributed with $2L$ degrees of freedom, i.e.,

$$f(\gamma_b) = \frac{1}{(L-1)!\bar{\gamma}_c^L}\gamma_b^{L-1}e^{-\gamma_b/\bar{\gamma}_c}$$

where $\bar{\gamma}_c = \mathbb{E}[\alpha_k^2]\mathcal{E}_b/N_0$. Prove that the resulting error probability is equal to $\frac{2^{L-1}}{(\bar{\gamma}_c+2)^L}$.

Hint:

$$\int_0^\infty \gamma_b^{L-1}e^{-\gamma_b/\bar{\gamma}_c}d\gamma_b = (L-1)!\bar{\gamma}_c^L$$

or equivalently,

$$\int_0^\infty \gamma_b^{L-1}e^{-\gamma_b(1/2+1/\bar{\gamma}_c)}d\gamma_b = (L-1)!\frac{1}{(1/2+1/\bar{\gamma}_c)^L}.$$

Solutions.

(a) We require $T < (\Delta t)_c$ and $B < (\Delta f)_c$ respectively for slowly fading and frequency non-selectivity. By $B = 1/T$, $(\Delta f)_c = 1/T_m$, $(\Delta t)_c = 1/B_d$, we know T should satisfy

$$T_m = 100 \times 10^{-9} < T < 1/B_d = 1/100.$$

Such T apparently exists.

(b)

$$
\begin{aligned}
P_e &= \int_0^\infty \frac{1}{2} e^{-\gamma_b/2} f(\gamma_b) d\gamma_b \\
&= \int_0^\infty \frac{1}{2} e^{-\gamma_b/2} \left(\frac{1}{\bar{\gamma}_b} e^{-\gamma_b/\bar{\gamma}_b} \right) d\gamma_b \\
&= \frac{1}{2\bar{\gamma}_b} \int_0^\infty e^{-\gamma_b(1/2+1/\bar{\gamma}_b)} d\gamma_b \\
&= \frac{1}{2\bar{\gamma}_b} \left(-\frac{1}{(1/2+1/\bar{\gamma}_b)} e^{-\gamma_b(1/2+1/\bar{\gamma}_b)} \Big|_0^\infty \right) \\
&= \frac{1}{2\bar{\gamma}_b} \frac{1}{(1/2+1/\bar{\gamma}_b)} \\
&= \frac{1}{\bar{\gamma}_b + 2}.
\end{aligned}
$$

(c)

$$
\begin{aligned}
P_e &= \int_0^\infty \frac{1}{2} e^{-\gamma_b/2} f(\gamma_b) d\gamma_b \\
&= \int_0^\infty \frac{1}{2} e^{-\gamma_b/2} \left(\frac{1}{(L-1)!\bar{\gamma}_c^L} \gamma_b^{L-1} e^{-\gamma_b/\bar{\gamma}_c} \right) d\gamma_b \\
&= \frac{1}{2(L-1)!\bar{\gamma}_c^L} \int_0^\infty \gamma_b^{L-1} e^{-\gamma_b(1/2+1/\bar{\gamma}_c)} d\gamma_b \\
&= \frac{1}{2(L-1)!\bar{\gamma}_c^L} \cdot (L-1)! \frac{1}{(1/2+1/\bar{\gamma}_c)^L} \\
&= \frac{2^{L-1}}{(\bar{\gamma}_c + 2)^L}.
\end{aligned}
$$