

# Digital Communications

## Chapter 11 Multichannel and Multicarrier Systems

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# Motivation

In Chapter 9: *Communications Through Band-Limited Channels*, we have seen that

When channel is band-limited to  $[-W, W]$ , without extra care, the received signal at matched-filter output is

$$y_k = \sum_{n=-\infty}^{\infty} I_n x_{k-n} + z_k$$

where

- $I_k$  is the information symbol,
- $x_k$  is the overall discrete impulse response,
- $z_k$  is the additive noise

This gives an intersymbol interference (ISI) channel.

$$y_k = \sum_{n=-\infty}^{\infty} I_n x_{k-n} + z_k$$

With **Nyquist pulse**, it is possible to create an ISI-free channel

$$x_{k-n} = \delta_{k-n} \quad \text{and} \quad y_k = I_k + z_k.$$

However, due to

- mis-synchronization
- imperfect channel estimation, etc

ISI is sometimes inevitable.

$$y_k = \sum_{n=-\infty}^{\infty} I_n x_{k-n} + z_k$$

There is another simple solution to the ISI problem. The main idea is the following:

- Given the discrete channel impulse response  $x_k$ , we see

$$y_k = I_k * x_k + z_k$$

- By Fourier duality, taking discrete Fourier transform (DFT) at both sides gives

$$Y_k = \mathfrak{F}_k X_k + Z_k$$

This transforms “convolution” to “multiplication.”

- So, if we set, for example,  $\{\mathcal{I}_k\} \in \{-1, 1\}^*$  and transmit its IDFT  $\{I_k\}$ . Then, the ISI problem can be solved straightforwardly.
- Note that  $Y_k$  is only a function of  $\mathcal{I}_k$  and does not depend on  $\dots, \mathcal{I}_{k-2}, \mathcal{I}_{k-1}, \mathcal{I}_{k+1}, \mathcal{I}_{k+2}, \dots$
- This idea has been employed in many modern techniques such as **Orthogonal Frequency Division Multiplexing (OFDM)**.



## 11.2 Multicarrier communications:

11.2-3 Orthogonal frequency division multiplexing (OFDM)

11.2-4 Modulation and demodulation in an OFDM system

Let  $T$  be the symbol duration; then we know the set of waveforms

$$\left\{ \kappa e^{i2\pi \frac{k}{T}t} : t \in [0, T], k = 0, 1, \dots, Q-1 \right\}$$

is a set of orthonormal functions, where

$$\kappa = \sqrt{\frac{1}{T}}.$$

$$\begin{aligned} \left\langle \kappa e^{i2\pi \frac{k}{T}t}, \kappa e^{i2\pi \frac{j}{T}t} \right\rangle &= \int_0^T \kappa^2 e^{i2\pi \frac{k}{T}t} e^{-i2\pi \frac{j}{T}t} dt \\ &= \frac{1}{T} \int_0^T e^{i2\pi \frac{(k-j)}{T}t} dt \\ &= \delta_{k-j} \end{aligned}$$

Let

$$X_{k,n} = I_{k,n} + j Q_{k,n}$$

be the QAM symbol at the  $k$ th subcarrier and at the  $n$ th symbol period; then the multicarrier waveform is given by

$$s_\ell(t) = \kappa \sum_{n=-\infty}^{\infty} \left( \sum_{k=0}^{Q-1} X_{k,n} e^{j2\pi \frac{k}{T} t} \right) g(t - nT)$$

where  $g(t)$  is the pulse shaping function.

Hence,

$$s(t) = \mathbf{Re} \{ s_\ell(t) e^{j2\pi f_c t} \}$$

At the first glance, it seems to be a single-carrier  $f_c$  system; but, it is actually a multi-carrier system with single-carrier implementation.



$$\begin{aligned}
s(t) &= \mathbf{Re} \left\{ s_\ell(t) e^{i2\pi f_c t} \right\} \\
&= \mathbf{Re} \left\{ \kappa \sum_{n=-\infty}^{\infty} \left( \sum_{k=0}^{Q-1} X_{k,n} e^{i2\pi \frac{k}{T} t} \right) g(t - nT) e^{i2\pi f_c t} \right\} \\
&= \sum_{k=0}^{Q-1} \mathbf{Re} \left\{ \left( \sum_{n=-\infty}^{\infty} \kappa X_{k,n} g(t - nT) \right) e^{i2\pi f_k t} \right\}
\end{aligned}$$

where  $f_k = f_c + \frac{k}{T}$  is the  $k$ th carrier.

## 11.2-6 Spectral characteristics of multicarrier signals

Clearly,  $s_\ell(t)$  is a random process.

For simplicity, we may assume  $I_{k,n}$  and  $Q_{k,n}$  are i.i.d., zero mean, and variance  $\frac{1}{2}\sigma^2$ .

With  $\kappa = 1/\sqrt{T}$ , the autocorrelation function of  $s_\ell(t)$  is

$$\begin{aligned} R_{s_\ell}(t+\tau, t) &= \frac{1}{T} \mathbb{E} \left[ \left( \sum_{n=-\infty}^{\infty} \sum_{k=0}^{Q-1} X_{k,n} g(t+\tau-nT) e^{i2\pi \frac{k}{T}(t+\tau)} \right) \right. \\ &\quad \left. \left( \sum_{m=-\infty}^{\infty} \sum_{j=0}^{Q-1} X_{j,m}^* g^*(t-mT) e^{-i2\pi \frac{j}{T}t} \right) \right] \\ &= \frac{1}{T} \sigma^2 \sum_{k=0}^{Q-1} e^{i2\pi \frac{k}{T}\tau} \sum_{n=-\infty}^{\infty} g(t+\tau-nT) g^*(t-nT) \end{aligned}$$

- It is clear that

$$R_{s_\ell}(t + \tau, t) = R_{s_\ell}(t + \tau + mT, t + mT)$$

for any integer  $m$ ; hence  $s_\ell(t)$  is a **cyclostationary random process with period  $T$** .

- The average autocorrelation function is thus given by

$$\begin{aligned} \bar{R}_{s_\ell}(\tau) &= \frac{1}{T} \int_0^T R_{s_\ell}(t + \tau, t) dt \\ &= \frac{\sigma^2}{T^2} \sum_{k=0}^{Q-1} e^{i2\pi \frac{k}{T} \tau} \sum_{n=-\infty}^{\infty} \int_0^T g(t + \tau - nT) g^*(t - nT) dt \\ &= \frac{\sigma^2}{T^2} \sum_{k=0}^{Q-1} e^{i2\pi \frac{k}{T} \tau} \sum_{n=-\infty}^{\infty} \int_{-nT}^{-(n-1)T} g(u + \tau) g^*(u) du \\ &= \frac{\sigma^2}{T^2} \sum_{k=0}^{Q-1} e^{i2\pi \frac{k}{T} \tau} \int_{-\infty}^{\infty} g(t + \tau) g^*(t) dt \end{aligned}$$

# Power spectral density

The time-average power spectral density of  $s_\ell(t)$  is

$$\begin{aligned}\bar{S}_{s_\ell}(f) &= \int_{-\infty}^{\infty} \bar{R}_{s_\ell}(\tau) e^{-i2\pi f\tau} d\tau \\ &= \frac{\sigma^2}{T^2} \sum_{k=0}^{Q-1} \int_{-\infty}^{\infty} g^*(t) \left( \int_{-\infty}^{\infty} g(t+\tau) e^{-i2\pi(f-\frac{k}{T})\tau} d\tau \right) dt \\ &= \frac{\sigma^2}{T^2} \sum_{k=0}^{Q-1} \int_{-\infty}^{\infty} g^*(t) \left( \int_{-\infty}^{\infty} g(u) e^{-i2\pi(f-\frac{k}{T})u} du \right) e^{i2\pi(f-\frac{k}{T})t} dt \\ &= \frac{\sigma^2}{T^2} \sum_{k=0}^{Q-1} G\left(f - \frac{k}{T}\right) \left( \int_{-\infty}^{\infty} g(t) e^{-i2\pi(f-\frac{k}{T})t} dt \right)^* \\ &= \frac{\sigma^2}{T^2} \sum_{k=0}^{Q-1} \left| G\left(f - \frac{k}{T}\right) \right|^2.\end{aligned}$$

## Theorem 1

*The time-average power spectral density of  $s_\ell(t)$  is*

$$\bar{S}_{s_\ell}(f) = \frac{\sigma^2}{T^2} \sum_{k=0}^{Q-1} \left| G\left(f - \frac{k}{T}\right) \right|^2$$

*where  $Q$  is the number of subcarriers.*

# Example

Let  $g(t)$  be the rectangular pulse shape of height 1 and duration  $T$ ; then

$$G(f) = e^{-j\pi fT} T \operatorname{sinc}(fT).$$

Hence

$$\bar{S}_{s_\ell}(f) = \sigma^2 \sum_{k=0}^{Q-1} \left| \operatorname{sinc} \left( \left( f - \frac{k}{T} \right) T \right) \right|^2.$$

In particular,

$$\bar{S}_{s_\ell} \left( \frac{m}{T} \right) = \sigma^2 \sum_{k=0}^{Q-1} |\operatorname{sinc}(m-k)|^2 = \begin{cases} \sigma^2, & \text{if } 0 \leq m < Q \\ 0, & \text{otherwise.} \end{cases}$$

# Example: $T = 1$ and $Q = 5$

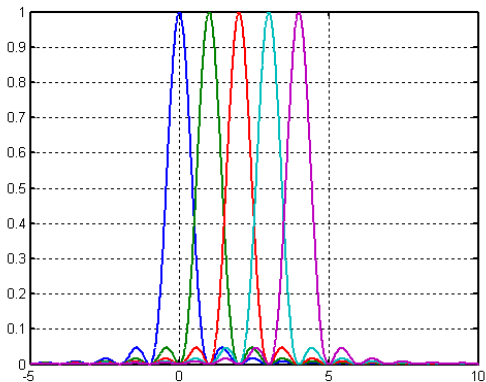


Figure:  $|G(f - \frac{k}{T})|^2$  for  $k = 0, 1, 2, 3, 4$



# Example: $T = 1$ and $Q = 5$

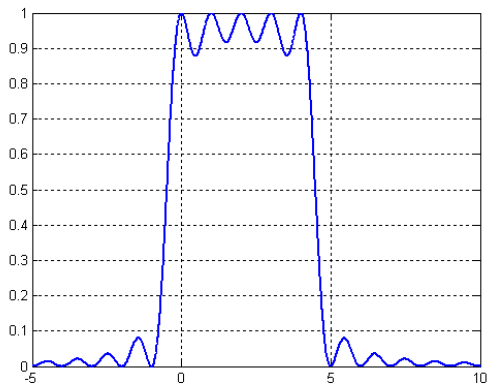


Figure:  $S_{s_l}(f)$

$$\bar{S}_{s_\ell}(f) = \sigma^2 \sum_{k=0}^{Q-1} \left| \text{sinc} \left( \left( f - \frac{k}{T} \right) T \right) \right|^2$$

- The PSD  $S_{s_\ell}(f)$  decays very slow at high frequencies at rate approximately

$$\bar{S}_{s_\ell}(f) \approx \frac{1}{f^2}.$$

- Out of band power leakage is severe and the resulting spectrum would not meet the FCC requirement.
- Can add a bandpass filter afterwards to remove the out-of-band signals, for example, using the root raised cosine filters.

## 11.2-5 An FFT algorithm implementation of an OFDM system

For simplicity, we again assume  $g(t)$  is the rectangular pulse shape of height 1 and duration  $T$  such that for  $0 \leq t < T$ ,

$$s_\ell(t) = \kappa \sum_{k=0}^{Q-1} X_{k,n} e^{i2\pi \frac{k}{T} t}$$

and zero, otherwise.

Then, we will introduce an efficient way to generate the following waveform:

$$s_\ell(t) = \kappa \sum_{k=0}^{Q-1} X_k e^{i2\pi \frac{k}{T} t}, \quad t \in [0, T)$$

# Generating $s_\ell(t)$ using iFFT + DAC

- Consider an  $N$ -point iFFT with  $N \geq Q$ . (Usually,  $N$  is equal to the power of two.)

- Set  $\hat{X}_k = \begin{cases} X_k, & \text{if } 0 \leq k < Q \\ 0, & \text{if } Q \leq k < N \end{cases}$

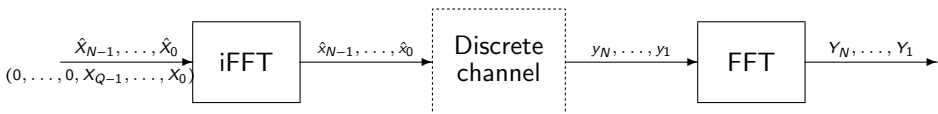
- The iFFT of  $\hat{X}_k$  is given by

$$\hat{x}_m = \frac{1}{N} \sum_{k=0}^{N-1} \hat{X}_k e^{i2\pi \frac{mk}{N}} = \frac{1}{N} \sum_{k=0}^{Q-1} X_k e^{i2\pi \frac{mk}{N}}$$

- Feeding  $N\hat{x}_m$  to a digital-analog-converter (DAC) at rate  $\frac{N}{T}$  gives

$$\hat{s}_\ell(t) = (\kappa N) \sum_{m=0}^{N-1} \hat{x}_m g_{\text{DAC}}\left(t - \frac{m}{N}T\right)$$

where  $g_{\text{DAC}}(t)$  is the rectangular pulse of height 1 and duration  $\frac{T}{N}$ .



Note that for  $n = 0, 1, \dots, N - 1$ ,

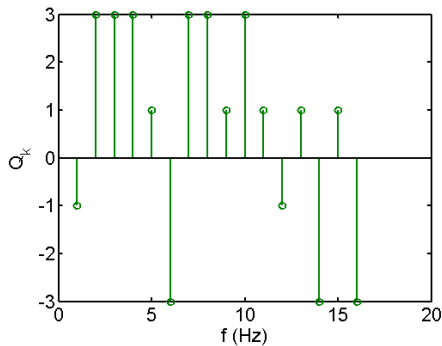
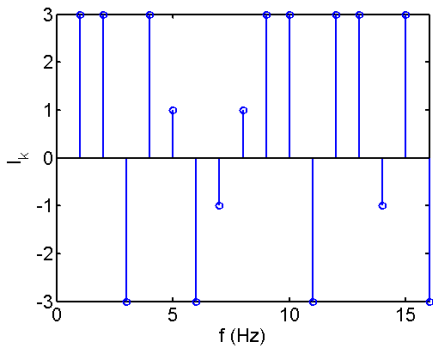
$$\begin{aligned}
 \hat{s}_\ell\left(\frac{n}{N}T\right) &= \kappa N \sum_{m=0}^{N-1} \hat{x}_m g_{\text{DAC}}\left(\frac{n}{N}T - \frac{m}{N}T\right) \\
 &= \kappa N \hat{x}_n = \kappa \sum_{k=0}^{Q-1} X_k e^{j2\pi \frac{kn}{N}} \\
 &= s_\ell\left(\frac{n}{N}T\right)
 \end{aligned}$$

We see

$$\hat{s}_\ell(t) = s_\ell(t) \text{ for } t = nT/N \text{ and } n = 0, 1, 2, \dots, N - 1.$$

The technique we had used is called **Zero Padding** in DSP.

# Example: $Q = 16$ and $T = 1$



$$X_k = I_k + j Q_k \text{ for } 0 \leq k < Q = 16$$

# Example: $Q = 16$ and $T = 1$

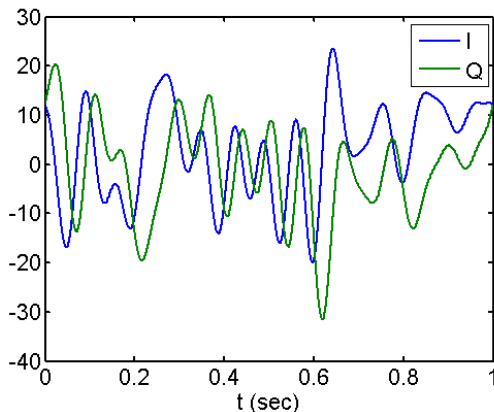
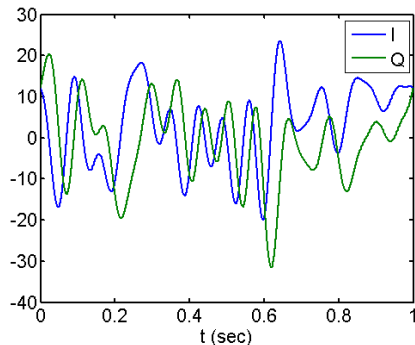


Figure:  $s_\ell(t) = I(t) + j Q(t) = \kappa \sum_{k=0}^{Q-1} X_k e^{j2\pi \frac{k}{T} t}$ ,  $t \in [0, T)$

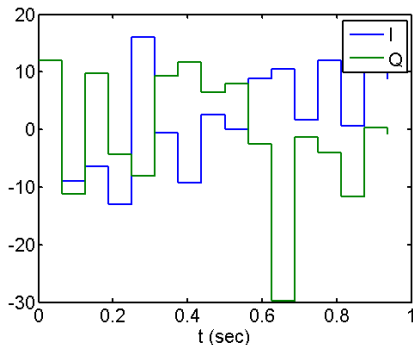


# Example: $Q = 16$ and $T = 1$ and $N = 16$

$s_\ell(t)$



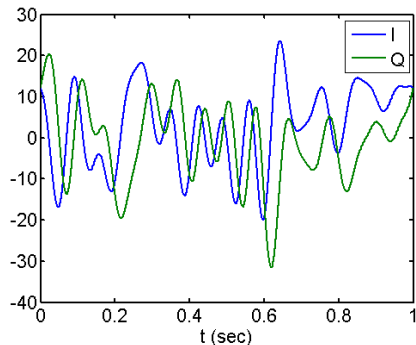
$\hat{s}_\ell(t)$



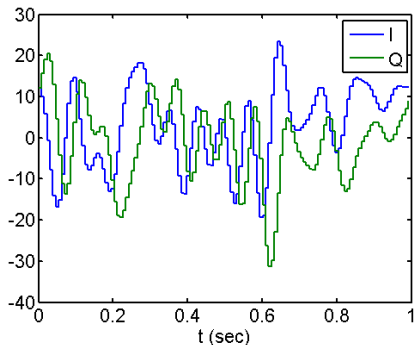
$$s_\ell(t) = \hat{s}_\ell(t) \text{ at } t = 0, \frac{1}{16}, \frac{2}{16}, \dots, \frac{15}{16} \text{ (sec)}$$

Example:  $Q = 16$  and  $T = 1$  and  $N = 128$

$s_\ell(t)$



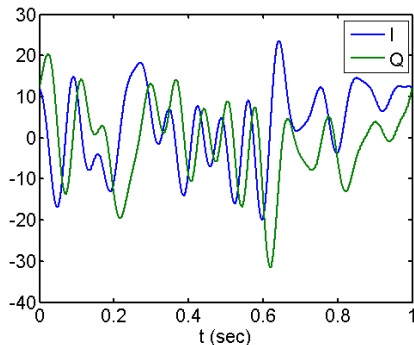
$\hat{s}_\ell(t)$



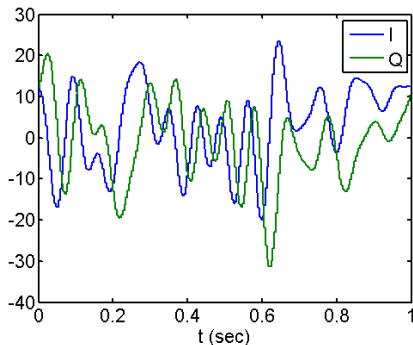
$$s_\ell(t) = \hat{s}_\ell(t) \text{ at } t = 0, \frac{1}{128}, \frac{2}{128}, \dots, \frac{127}{128} \text{ (sec)}$$

Example:  $Q = 16$  and  $T = 1$  and  $N = 256$

$s_\ell(t)$



$\hat{s}_\ell(t)$

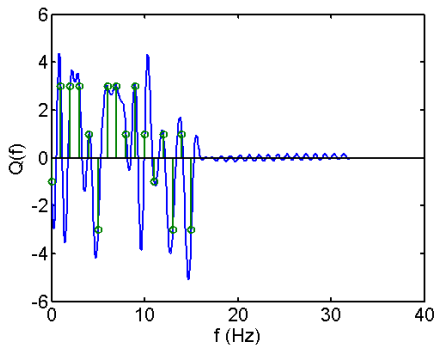
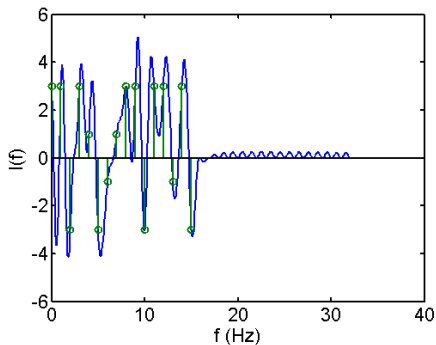


$$s_\ell(t) = \hat{s}_\ell(t) \text{ at } t = \frac{1}{256}, \frac{2}{256}, \dots, \frac{255}{256} \text{ (sec)}$$

# Example: $Q = 16$ and $T = 1$

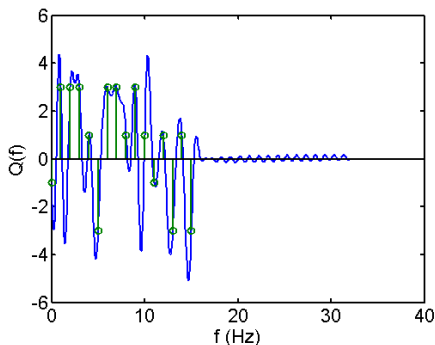
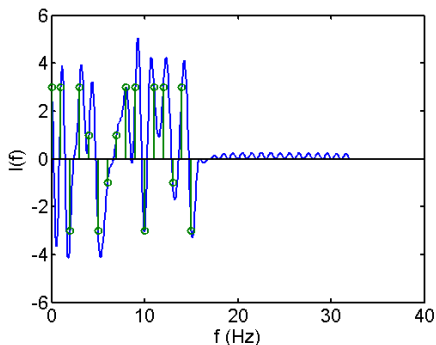
$$s_\ell(t) = \kappa \sum_{k=0}^{Q-1} X_k e^{i2\pi \frac{k}{T} t} g(t), \quad t \in [0, T)$$

$$S_\ell(f) = \mathcal{F}\{s_\ell(t)\} = I(f) + iQ(f) = \kappa \sum_{k=0}^{Q-1} X_k G(f - \frac{k}{T})$$



Out-of-band leakage due to rectangular pulse shape  $g(t)$

# Example: $Q = 16$ and $T = 1$



$$S_\ell\left(f = \frac{k}{T}\right) = X_k = I_{k+\nu} + j Q_k \text{ for } k = 0, 1, \dots, Q = 15 \text{ and } \kappa = \frac{1}{T}$$

# Transmission of multicarrier signal

$$\begin{aligned}
\hat{s}_\ell(t) &= (\kappa N) \sum_{n=-\infty}^{\infty} \left( \sum_{m=0}^{N-1} \hat{x}_{m,n} g_{\text{DAC}} \left( t - \frac{m}{N} T \right) \right) g(t - nT) \\
&= \kappa \sum_{n=-\infty}^{\infty} \left( \sum_{m=0}^{N-1} \left( \sum_{k=0}^{Q-1} X_{k,n} e^{i2\pi \frac{mk}{N}} \right) g_{\text{DAC}} \left( t - \frac{m}{N} T \right) \right) g(t - nT) \\
&= \kappa \sum_{n=-\infty}^{\infty} \left( \sum_{k=0}^{Q-1} X_{k,n} \left( \sum_{m=0}^{N-1} e^{i2\pi \frac{mk}{N}} g_{\text{DAC}} \left( t - \frac{m}{N} T \right) \right) \right) g(t - nT) \\
&= \kappa \sum_{n=-\infty}^{\infty} \left( \sum_{k=0}^{Q-1} X_{k,n} e^{i2\pi \frac{k}{T} \frac{\lfloor t(N/T) \rfloor}{(N/T)}} \right) g(t - nT) \\
s_\ell(t) &= \kappa \sum_{n=-\infty}^{\infty} \left( \sum_{k=0}^{Q-1} X_{k,n} e^{i2\pi \frac{k}{T} t} \right) g(t - nT)
\end{aligned}$$

The difference between ideal  $s_\ell(t)$  and physically realizable  $\hat{s}_\ell(t)$  is that the latter uses a “digitized” time scale.

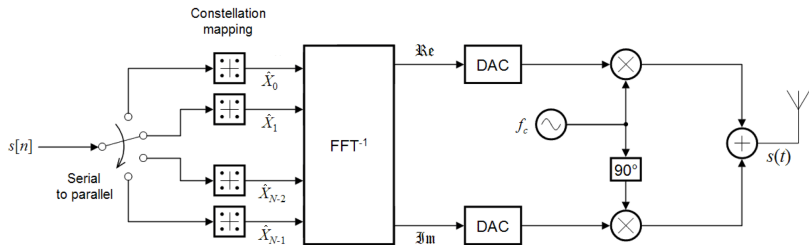
Denote  $a = \frac{\lfloor t(N/T) \rfloor}{t(N/T)}$ , which is approximately 1 when  $N$  large.

Then the transmitted signal is given by

$$\begin{aligned}\hat{s}(t) &= \mathbf{Re} \left\{ \hat{s}_\ell(t) e^{i2\pi f_c t} \right\} \\ &= \kappa \sum_{n=-\infty}^{\infty} \mathbf{Re} \left\{ \left( \sum_{k=0}^{Q-1} X_{k,n} e^{i2\pi \frac{k}{T} \frac{\lfloor t(N/T) \rfloor}{(N/T)}} \right) e^{i2\pi f_c t} \right\} g(t - nT) \\ &= \kappa \sum_{n=-\infty}^{\infty} \sum_{k=0}^{Q-1} \left\{ I_{k,n} \cos \left[ 2\pi \left( f_c + a \frac{k}{T} \right) t \right] \right. \\ &\quad \left. - Q_{k,n} \sin \left[ 2\pi \left( f_c + a \frac{k}{T} \right) t \right] \right\} g(t - nT)\end{aligned}$$



# Transmission of multicarrier signal



# OFDM = Multicarrier + Cyclic prefix

- Why adding cyclic prefix?  
To combat the channel effect due to  $c_\ell(t)$ .
- We can virtually think that

$$s_\ell(t) = \begin{cases} \kappa \sum_{k=0}^{Q-1} X_k e^{i2\pi \frac{k}{T} t}, & t \in [0, T) \\ 0, & \text{otherwise} \end{cases}$$

or more physically

$$\hat{s}_\ell(t) = \begin{cases} \kappa \sum_{k=0}^{Q-1} X_k e^{i2\pi \frac{k}{T} t}, & t \in [0, T) \\ 0, & \text{otherwise} \end{cases}$$

- Virtually extend  $s_\ell(t)$  to make it periodic

$$\tilde{s}_\ell(t) = \sum_{n=-\infty}^{\infty} s_\ell(t - nT) = \kappa \sum_{k=0}^{Q-1} X_k e^{i2\pi \frac{k}{T} t} \text{ for } t \in \mathbb{R}$$

We will transmit  $\tilde{s}_\ell(t)$  (of duration  $P + T$ ) instead of  $s_\ell(t)$  (of duration  $T$ ) for OFDM, where  $P$  is the length of  $c_\ell(t)$ .

In other words, we essentially assume that

$$c_\ell(t) = 0 \text{ for } t < 0 \text{ and } t \geq P.$$

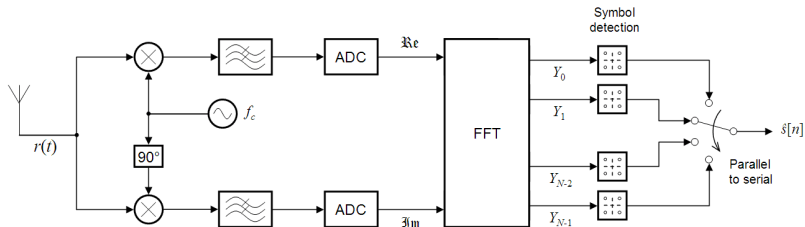
The extra periodic part  $P$  is called **cyclic prefix** in OFDM.

Usually,  $T$  should be made much larger than  $P$  in order to reduce the loss in transmission time and to save extra transmission power. For example,  $T = 3.2\mu\text{s}$  and  $P = 0.8\mu\text{s}$  for IEEE 802.11.

The necessity of adding CP will be clear in the analysis of Rx.

# Receiver for multicarrier signal

# Receiver for multicarrier signal



## Oversampling

While there are only  $Q$  tones transmitted, oversampling is required to avoid aliasing caused by out-of-band signals from other users.

Assuming the channel has a lowpass equivalent impulse response  $c_\ell(t)$ , the received noise-free received signal is

$$r_\ell(t) = \tilde{s}_\ell(t) \star c_\ell(t) = \int_0^P c_\ell(\tau) \tilde{s}_\ell(t - \tau) d\tau,$$

where  $\tilde{s}_\ell(t)$  periodic with period  $T$ .

Since all we need is  $r_\ell(t)$  for  $t \in [0, T)$ , it is clear from the above formula that we only need  $\tilde{s}_\ell(t)$  for  $t \in [-P, T)$ .

By this CP technique, the received signal simplifies to:

$$\begin{aligned} r_\ell(t) &= \tilde{s}_\ell(t) \star c_\ell(t) \\ &= \kappa \left( \sum_{k=0}^{Q-1} X_k e^{i2\pi \frac{k}{T} t} \right) \star c_\ell(t) \\ &= \kappa \sum_{k=0}^{Q-1} X_k \int_{-\infty}^{\infty} c_\ell(\tau) e^{i2\pi \frac{k}{T} (t-\tau)} d\tau \end{aligned}$$

$$\begin{aligned}
 r_\ell(t) &= \kappa \sum_{k=0}^{Q-1} X_k e^{i2\pi \frac{k}{T} t} \int_{-\infty}^{\infty} c_\ell(\tau) e^{-i2\pi \frac{k}{T} \tau} d\tau \\
 &= \kappa \sum_{k=0}^{Q-1} X_k e^{i2\pi \frac{k}{T} t} C_\ell \left( \frac{k}{T} \right).
 \end{aligned}$$

Note  $r_\ell(t)$  is actually periodic with period  $T$ .

Sample  $r_\ell(t)$  at rate  $\frac{\tilde{N}}{T}$ , where  $\tilde{N}$  is not necessarily equal to  $N$ .

$$r_m = r_\ell \left( \frac{m}{\tilde{N}} T \right) = \kappa \sum_{k=0}^{Q-1} C_\ell \left( \frac{k}{T} \right) X_k e^{i2\pi \frac{km}{\tilde{N}}}$$

\* The extra receptions for  $m = -1, -2, \dots, -\frac{P}{T} \tilde{N}$  due to CP are unused.

When using the physical  $\hat{s}_\ell(t)$  instead of ideal  $s_\ell(t)$ ,

$$\begin{aligned}\hat{r}_m &= \hat{r}_\ell\left(\frac{m}{\tilde{N}}T\right) = \kappa \sum_{k=0}^{Q-1} C_\ell\left(\frac{k}{T}\right) X_k e^{i2\pi\frac{k}{T}\left(\frac{\lfloor m(N/\tilde{N}) \rfloor}{m(N/\tilde{N})}\right)\frac{m}{\tilde{N}}T} \\ &= \kappa \sum_{k=0}^{Q-1} C_\ell\left(\frac{k}{T}\right) X_k e^{i2\pi k \frac{\lfloor m(N/\tilde{N}) \rfloor}{N}} \text{ for } 0 \leq m \leq \tilde{N} - 1\end{aligned}$$

So, if  $N = \tilde{N}$  or  $N$  is a multiple of  $\tilde{N}$  (i.e., the sampling rate at Tx is higher), then  $\hat{r}_m = r_m$ .

However, if  $\tilde{N}$  is a multiple of  $N$ , say,  $\tilde{N} = uN$ , then

$$\hat{r}_m = \kappa \sum_{k=0}^{Q-1} C_\ell\left(\frac{k}{T}\right) X_k e^{i2\pi k \frac{\lfloor m/u \rfloor}{N}} = r_{u\lfloor m/u \rfloor}.$$

In other words, we only have  $N$  different samples at Rx since Tx only transmits  $N$  samples.



The FFT/iFFT duality we adopt here is:

$$\left\{ \begin{array}{l} \text{FFT} \quad \hat{X}_k = \sum_{m=0}^{N-1} \hat{x}_m e^{-j2\pi \frac{mk}{N}} \\ \text{iFFT} \quad \hat{x}_m = \frac{1}{N} \sum_{k=0}^{N-1} \hat{X}_k e^{j2\pi \frac{mk}{N}} \end{array} \right.$$

# Channel equalization

Given the received signal vector  $\mathbf{r} = [r_0, \dots, r_{\tilde{N}-1}]$ , the receiver applies FFT to  $\mathbf{r}$  (Implicitly,  $N$  is a multiple of  $\tilde{N}$  with  $\tilde{N} > Q$ .)

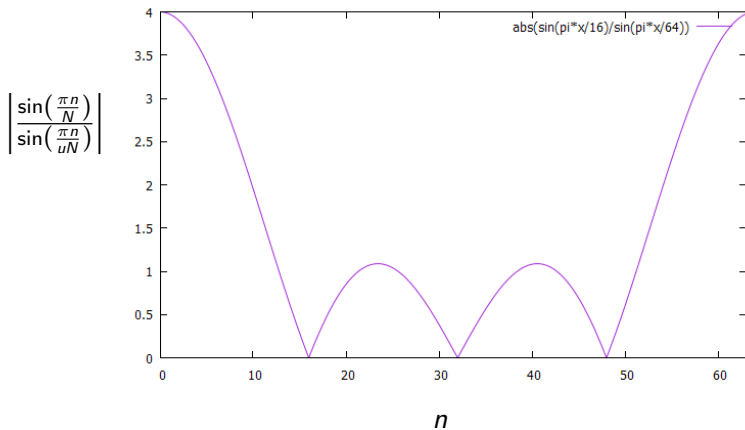
$$\begin{aligned} R_n &= \sum_{m=0}^{\tilde{N}-1} r_m e^{-j2\pi \frac{mn}{\tilde{N}}} \\ &= \sum_{m=0}^{\tilde{N}-1} \left( \kappa \sum_{k=0}^{Q-1} C_\ell \left( \frac{k}{T} \right) X_k e^{j2\pi \frac{km}{\tilde{N}}} \right) e^{-j2\pi \frac{mn}{\tilde{N}}} \\ &= \kappa \sum_{k=0}^{Q-1} C_\ell \left( \frac{k}{T} \right) X_k \sum_{m=0}^{\tilde{N}-1} e^{-j2\pi \frac{m(n-k)}{\tilde{N}}} \\ &= \kappa \sum_{k=0}^{Q-1} C_\ell \left( \frac{k}{T} \right) X_k \cdot \tilde{N} \delta_{n-k} \\ &= \begin{cases} \kappa \tilde{N} C_\ell \left( \frac{n}{T} \right) X_n, & 0 \leq n < Q \\ 0, & Q \leq n < \tilde{N} \end{cases} \end{aligned}$$

# When oversampling occurs

When  $\tilde{N} = uN$ ,

$$\begin{aligned}
 R_n &= \sum_{m=0}^{\tilde{N}-1} \hat{r}_m e^{-i2\pi \frac{mn}{\tilde{N}}} \\
 &= \sum_{m=0}^{\tilde{N}-1} \left( \kappa \sum_{k=0}^{Q-1} C_\ell \left( \frac{k}{T} \right) X_k e^{i2\pi \frac{km}{\tilde{N}} \frac{\lfloor m/u \rfloor}{m/u}} \right) e^{-i2\pi \frac{mn}{\tilde{N}}} \quad (m = ui + j) \\
 &= \kappa \sum_{k=0}^{Q-1} C_\ell \left( \frac{k}{T} \right) X_k \left( \sum_{j=0}^{u-1} e^{-i2\pi \frac{nj}{\tilde{N}}} \sum_{i=0}^{N-1} e^{-i2\pi \frac{i(n-k)}{N}} \right) \\
 &= \begin{cases} \kappa \left( \sum_{j=0}^{u-1} e^{-i2\pi \frac{nj}{uN}} \right) N C_\ell \left( \frac{n \bmod N}{T} \right) X_{n \bmod N}, & 0 \leq n \bmod N < Q \\ 0, & Q \leq n \bmod N < N \end{cases} \\
 &= \begin{cases} \kappa \frac{e^{-i2\pi \frac{(u-1)n}{uN}} \sin\left(\frac{\pi n}{N}\right)}{\sin\left(\frac{\pi n}{uN}\right)} N C_\ell \left( \frac{n \bmod N}{T} \right) X_{n \bmod N}, & 0 \leq n \bmod N < Q \\ 0, & Q \leq n \bmod N < N \end{cases}
 \end{aligned}$$

# Example. $N = 16$ and $\tilde{N} = 64$



# Channel equalization

With noise present, we have

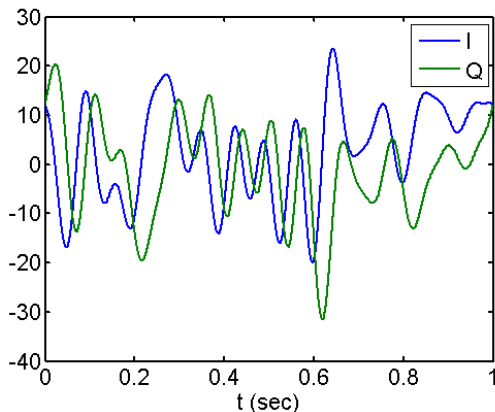
$$R_k = \kappa \tilde{N} C_\ell \left( \frac{k}{T} \right) X_k + Z_k$$

- Only one-tap equalization (i.e.,  $\kappa \tilde{N} C_\ell \left( \frac{k}{T} \right)$ ) is needed.

# Disadvantages of OFDM

While OFDM allows for simple equalization, it also introduces other problems such as:

High peak-to-average power ratio (PAPR) at  $s_\ell(t)$



# What you learn from Chapter 11



- DFT transforms convolution (ISI at time domain) to multiplication (no-ISI at freq domain).
- OFDM: Single carrier implementation of a multi-carrier system **with the help of DFT**.

$$\begin{aligned} s(t) &= \mathbf{Re} \left\{ s_\ell(t) e^{i2\pi f_c t} \right\} \\ &= \mathbf{Re} \left\{ \kappa \sum_{n=-\infty}^{\infty} \underbrace{\left( \sum_{k=0}^{Q-1} X_{k,n} e^{i2\pi \frac{k}{T} t} \right)}_{DFT} g(t - nT) e^{i2\pi f_c t} \right\} \\ &= \sum_{k=0}^{Q-1} \mathbf{Re} \left\{ \left( \sum_{n=-\infty}^{\infty} \kappa X_{k,n} g(t - nT) \right) e^{i2\pi f_k t} \right\} \end{aligned}$$

- Time-averaged autocorrelation function and PSD of OFDM
- OFDM realization: Ideal  $s(t)$  versus DAC versions  $\hat{s}(t)$ .
  - $Q$  carriers,  $N$ -point DFT at TX,  $\tilde{N}$ -point DFT at RX with  $\min(N, \tilde{N}) > Q$ .
  - When  $N$  and  $\tilde{N}$  are powers of 2, FFT implementations can be used.
- CP helps removing ISI at RX such that  $r_\ell(t)$  only affects by  $c_\ell(t)$  but not the previous OFDM symbol.

$$\begin{aligned}
 r_\ell(t) &= \tilde{s}_\ell(t) \star c_\ell(t) \\
 &= \kappa \left( \sum_{k=0}^{Q-1} X_k e^{i2\pi \frac{k}{T} t} \right) \star c_\ell(t) \\
 &= \kappa \sum_{k=0}^{Q-1} X_k \int_{-\infty}^{\infty} c_\ell(\tau) e^{i2\pi \frac{k}{T} (t-\tau)} d\tau
 \end{aligned}$$

- The analysis of  $R_n$  based on CP technique: Undersampling and oversampling.