13.1 Characterization of fading multipath channels
The multipath fading channels with additive noise

\[ \alpha_1(t)s(t - \tau_1) + \alpha_2(t)s(t - \tau_2) + \alpha_3(t)s(t - \tau_3) + n(t) \]
Time spread phenomenon of multipath channels
(Unpredictable) Time-variant factors

- Delay
- Number of spreads
- Size of the receive pulses
Transmitted signal

\[ s(t) = \text{Re} \left\{ s_\ell(t) e^{i2\pi f_c t} \right\} \]

Received signal in absence of additive noise

\[
\begin{align*}
    r(t) &= \int_{-\infty}^{\infty} c(\tau; t) s(t - \tau) d\tau \\
    &= \int_{-\infty}^{\infty} c(\tau; t) \text{Re} \left\{ s_\ell(t - \tau) e^{i2\pi f_c (t - \tau)} \right\} d\tau \\
    &= \text{Re} \left\{ \left( \int_{-\infty}^{\infty} c(\tau; t) e^{-i2\pi f_c \tau} s_\ell(t - \tau) d\tau \right) e^{i2\pi f_c t} \right\} \\
    &= \text{Re} \left\{ \left( s_\ell(t) \ast c(\tau; t) e^{-i2\pi f_c \tau} \right) e^{i2\pi f_c t} \right\}
\end{align*}
\]
Note that now it is not appropriate to write \( s_\ell(t) \ast c_\ell(t) \) because \( t \) and \( \tau \) are now specifically for time argument and convolution argument, respectively!

We should perhaps write \( s_\ell(t) \ast c_\ell(\tau) \) and \( s_\ell(t) \ast c_\ell(\tau; t) \), which respectively denote:

\[
s_\ell(t) \ast c_\ell(\tau) = \int_{-\infty}^{\infty} c_\ell(\tau)s_\ell(t - \tau)d\tau
\]

and

\[
s_\ell(t) \ast c_\ell(\tau; t) = \int_{-\infty}^{\infty} c_\ell(\tau; t)s_\ell(t - \tau)d\tau.
\]

From the previous slide, we know

\[
 c_\ell(\tau; t) = c(\tau; t)e^{-j2\pi fc\tau} \quad \text{and} \quad c(\tau; t) = |c_\ell(\tau; t)|.
\]
Rayleigh and Rician

The measurement suggests that $c_\ell(\tau; t)$ is a 2-D Gaussian random process in $t$ (not in $\tau$), which can be supported by the central limit theorem (CLT) because it is the “sum” effect of many paths.

- If zero mean, $|c_\ell(\tau; t)|$ is Rayleigh distributed. The channel is said to be a Rayleigh fading channel.

- If nonzero mean, $|c_\ell(\tau; t)|$ is Rician distributed. The channel is said to be a Rician fading channel.

When diversity technique is used, $c_\ell(\tau; t)$ can be well modeled by Nakagami $m$-distribution.

*Detail of these distributions can be found in Section 2.3.*
13.1-1 Channel correlation functions and power spectra
Assumption (WSS)

Assume $c_\ell(\tau; t)$ is WSS in $t$. 

$$R_{c_\ell}(\bar{\tau}, \tau; \Delta t) = \mathbb{E} \{ c_\ell(\bar{\tau}; t + \Delta t)c_\ell^*(\tau; t) \}$$

is only a function of time difference $\Delta t$.

Assumption (Uncorrelated scattering or US of a WSS channel)

For $\bar{\tau} \neq \tau$, assume $c_\ell(\bar{\tau}; t_1)$ and $c_\ell(\tau; t_2)$ are uncorrelated for any $t_1, t_2$.

- $\tau$ is the convolution argument and actually represents the delay for a certain path.

Assumption (Math definition of US)

$$R_{c_\ell}(\bar{\tau}, \tau; \Delta t) = R_{c_\ell}(\tau; \Delta t)\delta(\bar{\tau} - \tau)$$
Discussions

- It may appear “unnatural” to “define” the autocorrelation function of a channel impulse response using the Dirac delta function.

- However, we have already learned that $\tau$ is the convolution argument, and $\delta(\tau)$ is the impulse response of the identity channel. This somehow hints that there is a connection between “channel impulse response” and “Dirac delta function.”

- Recall that a WSS white (noise) process $z(\tau)$ is defined based on

$$R_z(\Delta \tau) = \mathbb{E}[z(\tau + \Delta \tau)z^*(\tau)] = \frac{N_0}{2} \delta(\Delta \tau).$$
Discussions (Continued)

- We can extensively view that the autocorrelation function of a 2-dimensional WSS white noise $z(\tau; t)$ is defined as

$$
\mathbb{E}[z(\tau + \Delta\tau; t_1)z^*(\tau; t_2)] = \frac{N_0(t_1, t_2)}{2}\delta(\Delta\tau).
$$

- US indicates that the accumulated power correlation from all other paths is essentially zero!

- Some researchers interpret “US” as “zero-correlation scattering.” So, from this, they don’t interpret it as

$$
\mathbb{E}[z(\tau + \Delta\tau; t_1)z^*(\tau; t_2)] = \mathbb{E}[z(\tau + \Delta\tau; t_1)]\mathbb{E}[z^*(\tau; t_2)] = 0,
$$

which requires “zero-mean” assumption but simply say

$$
\mathbb{E}[z(\tau + \Delta\tau; t_1)z^*(\tau; t_2)] = 0 \text{ when } \Delta\tau \neq 0.
$$
Multipath intensity profile of a US-WSS channel

The multipath intensity profile or delay power spectrum for a US-WSS multipath fading channel is given by:

\[ R_{c_{\ell}}(\tau) = R_{c_{\ell}}(\tau; \Delta t = 0). \]

It can be interpreted as the average signal power remained after delay \( \tau \).
The **multipath spread** or **delay spread** of a US-WSS multipath fading channel

- The **multipath spread** is the range of $\tau$ over which $R_{c\ell}(\tau)$ is essentially non-zero; it is usually denoted by $T_m$.

**FIGURE 13.1–2**
Multipath intensity profile.
Each $\tau$ corresponds to one path.

No Tx power will remain at Rx for paths with delay $\tau > T_m$. 
The transfer function of a channel impulse response $c_\ell(\tau; t)$ is the Fourier transform with respect to the convolutional argument $\tau$:

$$C_\ell(f; t) = \int_{-\infty}^{\infty} c_\ell(\tau; t) e^{-i 2\pi f \tau} \, d\tau$$

**Property**: If $c_\ell(\tau; t)$ is WSS, so is $C_\ell(f; t)$.

The autocorrelation function of WSS $C_\ell(f; t)$ is equal to:

$$R_{C_\ell}(\bar{f}, f; \Delta t) = \mathbb{E} \left\{ C_\ell(\bar{f}; t + \Delta t) C_\ell^*(f; t) \right\}$$
With an additional US assumption,

\[ \text{R}_{C_{\ell}}(\bar{f}, f; \Delta t) \]
\[ = \mathbb{E} \left\{ C_{\ell}(\bar{f}; t + \Delta t) C_{\ell}^*(f; t) \right\} \]
\[ = \mathbb{E} \left\{ \int_{-\infty}^{\infty} c_{\ell}(\bar{\tau}; t + \Delta t) e^{-i 2 \pi \bar{f} \bar{\tau}} d\bar{\tau} \int_{-\infty}^{\infty} c_{\ell}^*(\tau; t) e^{i 2 \pi f \tau} d\tau \right\} \]
\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{R}_{C_{\ell}}(\tau; \Delta t) \delta(\bar{\tau} - \tau) e^{i 2 \pi (f \tau - \bar{f} \bar{\tau})} d\tau d\bar{\tau} \]
\[ = \int_{-\infty}^{\infty} \text{R}_{C_{\ell}}(\tau; \Delta t) e^{-i 2 \pi (\bar{f} - f) \tau} d\tau \]
\[ = \text{R}_{C_{\ell}}(\Delta f; \Delta t) \text{, where } \Delta f = \bar{f} - f. \]

For a US-WSS multipath fading channel,

\[ \text{R}_{C_{\ell}}(\Delta f; \Delta t) = \mathbb{E} \left\{ C_{\ell}(f + \Delta f; t + \Delta t) C_{\ell}^*(f; t) \right\} \]

This is often called **spaced-frequency, spaced-time correlation function** of a US-WSS channel.
Coherent bandwidth

\[ R_{C_{\ell}} (\Delta f; \Delta t) = \int_{-\infty}^{\infty} R_{C_{\ell}} (\tau; \Delta t) e^{-\frac{i}{2} 2\pi (\Delta f) \tau} d\tau \]

For the case of \( \Delta t = 0 \), we have

\[ R_{C_{\ell}} (\Delta f) = \int_{-\infty}^{\infty} R_{C_{\ell}} (\tau) e^{-\frac{i}{2} 2\pi (\Delta f) \tau} d\tau \]

- Recall that \( R_{C_{\ell}} (\tau) = 0 \) outside \([0, T_m)\).
- In freq domain, \( (\Delta f)_c = \frac{1}{T_m} \) is correspondingly called coherent bandwidth.
FIGURE 13.1–3
Relationship between $R_C(\Delta f)$ and $R_c(\tau)$. 
Example.

Give \( R_{c\ell}(\tau) = 10^7(10^{-7} - \tau) \) for \( 0 \leq \tau < 100 \) ns. Then,

\[
R_{c\ell}(\Delta f) = \frac{10^7}{4\pi^2(\Delta f)^2} \left( e^{-\frac{i}{2\pi}2\pi \cdot 10^{-7} \cdot \Delta f} - 1 \right) - \frac{1}{2\pi \Delta f}.
\]

\[
\frac{|R_{c\ell}(\Delta f)|}{|R_{c\ell}(0)|}
\]

\( R_{c\ell}(\tau) \)
Since the channel output due to input $s_\ell(t)$ is equal to:

$$r_\ell(f) = s_\ell(f) C_\ell(f; t),$$

where we abuse the notations to denote the Fourier transforms of $r_\ell(t)$ and $s_\ell(t)$ respectively as $r_\ell(f)$ and $s_\ell(f)$, we would say

$$r_\ell(f) = s_\ell(f) C_\ell(f; t)$$

will have weak (power) affection on

$$r_\ell(f + \Delta f) = s_\ell(f + \Delta f) C_\ell(f + \Delta f; t)$$

when $\Delta f > (\Delta f)_c$. 
If signal transmitted bandwidth \( B > (\Delta f)_c \), the channel is called **frequency selective**.

If signal transmitted bandwidth \( B < (\Delta f)_c \), the channel is called **frequency non-selective**.
For frequency selective channels, the signal shape is more severely distorted than that of frequency non-selective channels.

Criterion for frequency selectivity:

\[ B > (\Delta f)_c \iff \frac{1}{T} > \frac{1}{T_m} \iff T < T_m. \]
Doppler effect appears via the argument $\Delta t$. 

$$\mathbf{(\alpha_1(t), \tau_1(t))}$$
$$\mathbf{(\alpha_2(t), \tau_2(t))}$$
$$\mathbf{(\alpha_3(t), \tau_3(t))}$$
The Doppler power spectrum is

\[ S_{C_\ell}(\lambda) = \int_{-\infty}^{\infty} R_{C_\ell}(\Delta f = 0; \Delta t) e^{-\frac{i 2\pi \lambda (\Delta t)}{b}} d(\Delta t), \]

where \( \lambda \) is referred to as the Doppler frequency.

- \( B_d \) = Doppler spread is the range such that \( S_{C_\ell}(\lambda) \) is essentially zero.
- \( (\Delta t)_c = \frac{1}{B_d} \) is called the coherent time.
- If symbol period \( T > (\Delta t)_c \), the channel is classified as Fast Fading.
  - I.e., channel statistics changes within one symbol!
- If symbol period \( T < (\Delta t)_c \), the channel is classified as Slow Fading.
**FIGURE 13.1–4**

Relationship between $R_C(\Delta t)$ and $S_C(\lambda)$. 

$$(\Delta t)_c \approx \frac{1}{B_d}$$

- Spaced-time correlation function
- Doppler power spectrum
Summary:

\( R_{c\ell}(\tau; \Delta t) \) Channel autocorrelation function

1-D FT:

\[
R_{c\ell}(\Delta f; \Delta t) = \mathcal{F}_\tau \{ R_{c\ell}(\tau; \Delta t) \}
\]

\[
S(\tau; \lambda) = \mathcal{F}_{\Delta t} \{ R_{c\ell}(\tau; \Delta t) \}
\]

2D FT:

\[
??\ = \mathcal{F}_{\tau, \Delta t} \{ R_{c\ell}(\tau; \Delta t) \}
\]

\( R_{c\ell}(\Delta f; \Delta t) \)

1-D FT:

\[
??\ = \mathcal{F}_{\Delta f} \{ R_{c\ell}(\Delta f; \Delta t) \}
\]

\[
S_{c\ell}(\Delta f; \lambda) = \mathcal{F}_{\Delta t} \{ R_{c\ell}(\Delta f; \Delta t) \}
\]

2D FT:

\[
S(-\tau; \lambda) = \mathcal{F}_{\Delta f, \Delta t} \{ R_{c\ell}(\Delta f; \Delta t) \}
\]

Spaced-freq spaced-time correlation func

Scattering function

Doppler power spectrum \((\Delta f = 0)\)
The scattering function can be used to identify “delay spread” and “Doppler spread” at the same time.

**FIGURE 13.1-5**
Relationships among the channel correlation functions and power spectra. [From Green (1962), with permission.]
Example. Medium-range tropospheric scatter channel

Scattering function of a medium-range tropospheric scatter channel. The taps delay increment is 0.1 μs.

\[ B_d = \text{Doppler spread, varies with paths} \]
\[ = (\text{often}) 3\text{dB bandwidth} \approx 1\text{Hz} \sim 30\text{Hz} \]

\[ T_m = \text{multipath spread} \approx 0.7\mu s \]
The **median delay spread** is the 50% value, meaning that 50% of all channels has a delay spread that is lower than the median value. Clearly, the median value is not so interesting for designing a wireless link, because you want to guarantee that the link works for at least 90% or 99% of all channels. Therefore the second column gives the measured **maximum delay spread** values. The reason to use maximum delay spread instead of a 90% or 99% value is that many papers only mention the maximum value. From the papers that do present cumulative distribution functions of their measured delay spreads, it can be deduced that the 99% value is only a few percent smaller than the maximum delay spread.
Measured delay spreads in frequency range of 800M to 1.5 GHz (surveyed by Richard van Nee)

<table>
<thead>
<tr>
<th>Median Delay Spread [ns]</th>
<th>Maximum Delay Spread [ns]</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>50</td>
<td>Office building</td>
</tr>
<tr>
<td>30</td>
<td>56</td>
<td>Office building</td>
</tr>
<tr>
<td>27</td>
<td>43</td>
<td>Office building</td>
</tr>
<tr>
<td>11</td>
<td>58</td>
<td>Office building</td>
</tr>
<tr>
<td>35</td>
<td>80</td>
<td>Office building</td>
</tr>
<tr>
<td>40</td>
<td>90</td>
<td>Shopping mall</td>
</tr>
<tr>
<td>80</td>
<td>120</td>
<td>Airport</td>
</tr>
<tr>
<td>120</td>
<td>180</td>
<td>Factory</td>
</tr>
<tr>
<td>50</td>
<td>129</td>
<td>Warehouse</td>
</tr>
<tr>
<td>120</td>
<td>300</td>
<td>Factory</td>
</tr>
</tbody>
</table>
Measured delay spreads in frequency range of 1.8 GHz to 2.4 GHz (surveyed by Richard van Nee)

<table>
<thead>
<tr>
<th>Median Delay Spread [ns]</th>
<th>Maximum Delay Spread [ns]</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>120</td>
<td>Large building (New York stock exchange)</td>
</tr>
<tr>
<td>40</td>
<td>95</td>
<td>Office building</td>
</tr>
<tr>
<td>40</td>
<td>150</td>
<td>Office building</td>
</tr>
<tr>
<td>60</td>
<td>200</td>
<td>Shopping center</td>
</tr>
<tr>
<td>106</td>
<td>270</td>
<td>Laboratory</td>
</tr>
<tr>
<td>19</td>
<td>30</td>
<td>Office building: single room only</td>
</tr>
<tr>
<td>20</td>
<td>65</td>
<td>Office building</td>
</tr>
<tr>
<td>30</td>
<td>75</td>
<td>Canteen</td>
</tr>
<tr>
<td>105</td>
<td>170</td>
<td>Shopping center</td>
</tr>
<tr>
<td>30</td>
<td>56</td>
<td>Office building</td>
</tr>
<tr>
<td>25</td>
<td>30</td>
<td>Office building: single room only</td>
</tr>
</tbody>
</table>
Measured delay spreads in frequency range of 4 GHz to 6 GHz (surveyed by Richard van Nee)

<table>
<thead>
<tr>
<th>Median Delay Spread [ns]</th>
<th>Maximum Delay Spread [ns]</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>120</td>
<td>Large building (New York stock exchange)</td>
</tr>
<tr>
<td>50</td>
<td>60</td>
<td>Office building</td>
</tr>
<tr>
<td>35</td>
<td>55</td>
<td>Meeting room (5mx5m) with metal walls</td>
</tr>
<tr>
<td>10</td>
<td>35</td>
<td>Single room with stone walls</td>
</tr>
<tr>
<td>40</td>
<td>130</td>
<td>Office building</td>
</tr>
<tr>
<td>40</td>
<td>120</td>
<td>Indoor sports arena</td>
</tr>
<tr>
<td>65</td>
<td>125</td>
<td>Factory</td>
</tr>
<tr>
<td>25</td>
<td>65</td>
<td>Office building</td>
</tr>
<tr>
<td>20</td>
<td>30</td>
<td>Office building: single room only</td>
</tr>
</tbody>
</table>

Conclusion by Richard van Nee: Measurements done at different frequencies show the multipath channel characteristics are almost the same from 1 to 5 GHz.
Jakes’ model

A widely used model for Doppler power spectrum is the so-called Jakes’ model (Jakes, 1974)

\[ R_{C_\ell}(\Delta t) = J_0(2\pi f_m \cdot \Delta t) \]

and

\[ S_{C_\ell}(\lambda) = \begin{cases} 
\frac{1}{\pi f_m} \frac{1}{\sqrt{1-(\lambda/f_m)^2}}, & |\lambda| \leq f_m \\
0, & \text{otherwise}
\end{cases} \]

where

\[ f_m = \frac{v f_c}{c} \] is the maximum Doppler shift
\[ v \] is the vehicle speed (m/s)
\[ c \] is the light speed (3 \times 10^8 m/s)
\[ f_c \] is the carrier frequency
\[ J_0(\cdot) \] is the zero-order Bessel function of the first kind.
Jakes’ model: Example 13.1-3

\[ J_0(2\pi f_m \cdot \Delta t) \]

\[ f_m \cdot S_{C_t}(\Delta f = 0; \lambda) \]
Difference in path length

\[ \Delta L = \sqrt{(L \sin(\theta))^2 + (L \cos(\theta) + v \cdot \Delta t)^2} - L \]

\[ = \sqrt{L^2 + v^2 (\Delta t)^2 + 2L \cdot v \cdot \Delta t \cdot \cos(\theta)} - L \]

Phase change \( \Delta \phi = 2\pi \frac{\Delta L}{(c/f_c)} \)
Estimated Doppler shift

\[
\lambda_m = \lim_{\Delta t \to 0} \frac{1}{2\pi} \frac{\Delta \phi}{\Delta t} = \frac{1}{c/f_c} \lim_{\Delta t \to 0} \frac{\sqrt{L^2 + v^2 (\Delta t)^2 + 2L \cdot v \cdot \Delta t \cdot \cos(\theta)} - L}{\Delta t} = \frac{vf_c}{c} \cos(\theta) = f_m \cos(\theta)
\]

**Example.** \(v = 108 \text{ km/hour}, f_c = 5 \text{ GHz and } c = 1.08 \times 10^9 \text{ km/hour.}\)

\[\implies \lambda_m = 500 \cos(\theta) \text{ Hz.}\]

This is ok because \(\frac{500 \text{ Hz}}{5 \text{ GHz}} = 0.1 \text{ ppm.}\)
Here, a rough (and not so rigorous) derivation is provided for Jakes’ model.

Just to give you a rough idea of how this model is obtained.

Suppose $\tau(t)$ is the delay of some path.

\[
\tau'(t) = \lim_{\Delta t \to 0} \frac{\tau(t+\Delta t) - \tau(t)}{\Delta t} \\
= \lim_{\Delta t \to 0} \frac{L + \Delta L - L}{\Delta t} \\
= \lim_{\Delta t \to 0} \frac{\Delta L}{c \Delta t} \\
= \frac{v}{c} \cos(\theta) \\
\Rightarrow \tau(t) \approx \frac{v}{c} \cos(\theta) t + \tau_0
\]

(Assume for simplicity $\tau_0 = 0$.)
Assume that  

\[ c(\tau; t) \approx a \cdot \delta(\tau - \tau(t)) \]  

(a constant attenuation single path system);  

\[ c(\tau; t) \approx a \cdot \delta(\tau - \tau(t)) \]  

WSS.

\[
c_\ell(\tau; t) = c(\tau; t) e^{-i 2\pi f_c \tau}
\]

\[
\approx a \cdot \delta(\tau - \tau(t)) e^{-i 2\pi f_c \cdot \tau(t)}
\]

\[
\approx a \cdot \delta(\tau - (v/c) \cos(\theta) t) e^{-i 2\pi f_c \left(\frac{v}{c} \cos(\theta) t\right)}
\]

\[
= a \cdot \delta(\tau - (f_m/f_c) \cos(\theta) t) e^{-i 2\pi f_m \cos(\theta) t}
\]

\[
R_{c_\ell}(\tau; \Delta t)
\]

\[
= \int_{-\infty}^{\infty} \mathbb{E}\left[ c_\ell(\bar{\tau}; t + \Delta t) c_\ell^*(\tau; t) \right] d\bar{\tau}
\]

\[
= \int_{-\infty}^{\infty} \mathbb{E}\left[ a \cdot \delta(\bar{\tau} - (f_m/f_c) \cos(\theta)(t + \Delta t)) e^{-i 2\pi f_m \cos(\theta)(t + \Delta t)}
\right.
\]

\[
\cdot a \cdot \delta(\tau - (f_m/f_c) \cos(\theta) t) e^{i 2\pi f_m \cos(\theta) t} \bigg] d\bar{\tau}
\]

\[
= a^2 \cdot \mathbb{E}\left[ e^{-i 2\pi f_m \cos(\theta) \cdot \Delta t} \right] \delta(\tau - (f_m/f_c) \cos(\theta) t)
\]
Since \( R_{c_\ell}(\tau; \Delta t) \) is nothing to do with \( t \), we take \( t = 0 \) for simplicity; thus

\[
R_{c_\ell}(\tau; \Delta t) = a^2 \cdot \mathbb{E} \left[ e^{-i 2\pi f_m \cos(\theta) \cdot \Delta t} \right] \delta(\tau).
\]

\[
R_{c_\ell}(\Delta f = 0; \Delta t) \quad \left( = \int_{-\infty}^{\infty} R_{c_\ell}(\tau; \Delta t) e^{i 2\pi (\Delta f) \tau} d\tau \right)
\]

\[
\begin{align*}
&= \int_{-\infty}^{\infty} R_{c_\ell}(\tau; \Delta t) d\tau \\
&= \int_{-\infty}^{\infty} a^2 \cdot \mathbb{E} \left[ e^{-i 2\pi f_m \cos(\theta) \cdot \Delta t} \right] \delta(\tau) d\tau \\
&= a^2 \cdot \mathbb{E} \left[ e^{-i 2\pi f_m \cos(\theta) \cdot \Delta t} \right] \\
&= J_0(2\pi f_m \cdot \Delta t),
\end{align*}
\]

where the last step is valid if \( \theta \) uniformly distributed over \([–\pi, \pi)\), and \( a = 1 \).
\( \theta \) can be treated as uniformly distributed over \([-\pi, \pi)\) and independent of attenuation \( \alpha \) and delay path \( \tau \).
A consistent channel model is required to allow comparison among different WLAN systems.

The IEEE 802.11 Working Group adopted the following channel model as the baseline for predicting multipath for modulations used in IEEE 802.11a and IEEE 802.11b, which is ideal for software simulations.

- The phase is uniformly distributed.
- The magnitude is Rayleigh distributed with average power decaying exponentially.
\[
c_{\ell}(\tau; t) = \sum_{i=0}^{i_{\text{max}}-1} \alpha_i e^{-i \phi_i} \delta(\tau - iT_s)
\]

where

\[
\begin{align*}
T_s & \text{ sampling period} \\
\alpha_i e^{i \phi_i} & \equiv \mathcal{N}(0, \sigma_i^2/2) + i \mathcal{N}(0, \sigma_i^2/2) \\
\sigma_i^2 & = \sigma_0^2 e^{-iT_s/\tau_{\text{rms}}} \\
\sigma_0^2 & = 1 - e^{-T_s/\tau_{\text{rms}}}
\end{align*}
\]
\[ R_{c \ell}(\tau) = \int_{-\infty}^{\infty} \mathbb{E} \left[ c_\ell(\bar{\tau}; t) c_\ell^*(\tau; t) \right] d\bar{\tau} \]

\[ = \sum_{i=0}^{i_{\text{max}}-1} \int_{-\infty}^{\infty} \mathbb{E} \left[ \alpha_i^2 \delta(\tau - iT_s) \delta(\bar{\tau} - \tau) \right] d\bar{\tau} \]

\[ = \sum_{i=0}^{i_{\text{max}}-1} \mathbb{E} \left[ \alpha_i^2 \right] \delta(\tau - iT_s) \]

\[ = \sum_{i=0}^{i_{\text{max}}-1} \sigma_0^2 e^{-iT_s/\tau_{\text{rms}}} \delta(\tau - iT_s) \]

By this example, I want to introduce the \textit{rms} delay. By definition, the “effective” \textit{rms} delay is

\[ T_{\text{rms}}^2 = \frac{\int_{-\infty}^{\infty} \tau^2 R_{c \ell}(\tau) d\tau}{\int_{-\infty}^{\infty} R_{c \ell}(\tau) d\tau} - \left( \frac{\int_{-\infty}^{\infty} \tau R_{c \ell}(\tau) d\tau}{\int_{-\infty}^{\infty} R_{c \ell}(\tau) d\tau} \right)^2 \]

\[ = \frac{\sum_{i=0}^{i_{\text{max}}-1} (iT_s)^2 \sigma_0^2 e^{-iT_s/\tau_{\text{rms}}}}{\sum_{i=0}^{i_{\text{max}}-1} \sigma_0^2 e^{-iT_s/\tau_{\text{rms}}}} - \left( \frac{\sum_{i=0}^{i_{\text{max}}-1} iT_s \sigma_0^2 e^{-iT_s/\tau_{\text{rms}}}}{\sum_{i=0}^{i_{\text{max}}-1} \sigma_0^2 e^{-iT_s/\tau_{\text{rms}}}} \right)^2 \]
We wish to choose \( i_{\text{max}} \) such that \( T_{\text{rms}} \approx \tau_{\text{rms}} \).

Let \( \tilde{\tau}_{\text{rms}} = \frac{\tau_{\text{rms}}}{T_s} \), \( \tilde{T}_{\text{rms}} = \frac{T_{\text{rms}}}{T_s} \) and \( \tilde{i}_{\text{max}} = \frac{i_{\text{max}}}{\tilde{\tau}_{\text{rms}}} \).

We obtain

\[
\tilde{T}_{\text{rms}} = \sqrt{\frac{e^{-1/\tilde{\tau}_{\text{rms}}}}{(1 - e^{-1/\tilde{\tau}_{\text{rms}}})^2} - \frac{\tilde{i}_{\text{max}}^2 e^{-\tilde{i}_{\text{max}}}}{(1 - e^{-\tilde{i}_{\text{max}}})^2} \frac{1}{\tilde{\tau}_{\text{rms}}^2}}
\]

\[
= \sqrt{\left(\frac{\tilde{T}_{\text{rms}}^2}{12} + \frac{1}{240} \frac{1}{\tilde{T}_{\text{rms}}^2} + \ldots\right) - \frac{\tilde{i}_{\text{max}}^2 e^{-\tilde{i}_{\text{max}}}}{(1 - e^{-\tilde{i}_{\text{max}}})^2} \frac{1}{\tilde{T}_{\text{rms}}^2}}
\]

Taking \( \tilde{T}_{\text{rms}} \approx \sqrt{\tilde{T}_{\text{rms}}^2 - \frac{1}{12}} \) and \( \frac{\tilde{i}_{\text{max}}^2 e^{-\tilde{i}_{\text{max}}}}{(1 - e^{-\tilde{i}_{\text{max}}})^2} \approx \frac{1}{240} \) yield

\[
\tilde{i}_{\text{max}} = 10.1072\ldots \quad \text{(or equivalently, } i_{\text{max}} = 10\frac{T_{\text{rms}}}{T_s})\].
Typical multipath delay spread for indoor environment (Table 8-1 in IEEE 802.11 Handbook)

<table>
<thead>
<tr>
<th>Environment</th>
<th>Delay Spread</th>
</tr>
</thead>
<tbody>
<tr>
<td>Home</td>
<td>&lt; 50 nsec</td>
</tr>
<tr>
<td>Office</td>
<td>~ 100 nsec</td>
</tr>
<tr>
<td>Manufacturing floor</td>
<td>200-300 nsec</td>
</tr>
</tbody>
</table>

\[
i_{\text{max}} = 10 \frac{\tau_{\text{rms}}}{T_s} = 10 \frac{50 \times 10^{-9}}{1/(20 \times 10^6)} = 10
\]

\[
i_{\text{max}} = 10 \frac{\tau_{\text{rms}}}{T_s} = 10 \frac{100 \times 10^{-9}}{1/(20 \times 10^6)} = 20
\]

\[
i_{\text{max}} = 10 \frac{\tau_{\text{rms}}}{T_s} = 10 \frac{200 \times 10^{-9}}{1/(20 \times 10^6)} = 40
\]
13.1-2 Statistical models for fading channels
In addition to zero-mean Gaussian (Rayleigh), non-zero-mean Gaussian (Rice) and Nakagami-\(m\) distributions, there are other models for \(c_\ell(\tau; t)\) have been proposed in literature.

**Example.**

- Channels with a direct path and a single multipath component, such as airplane-to-ground communications

\[
c_\ell(\tau; t) = \alpha \delta(\tau) + \beta(t) \delta(\tau - \tau_0(t))
\]

where \(\alpha\) controls the power in the direct path and is named *specular component*, and \(\beta(t)\) is modeled as zero-mean Gaussian.
Example.

- Microwave LOS radio channels used for long-distance voice and video transmission by telephone companies in the 6 GHz band (Rummler 1979)

\[
c_{\ell}(\tau) = \alpha \left[ \delta(\tau) - \beta e^{i2\pi f_0\tau} \delta(\tau - \tau_0) \right]
\]

where
\[
\begin{align*}
\alpha & \text{ overall attenuation parameter} \\
\beta & \text{ shape parameter due to multipath components} \\
\tau_0 & \text{ time delay} \\
f_0 & \text{ frequency of the fade minimum, i.e., } f_0 = \arg \min_{f \in \mathbb{R}} |C_{\ell}(f)|
\end{align*}
\]
Rummler found that

1. $\alpha \perp \beta$ (Independent)
2. $f(\beta) \approx (1 - \beta)^{2.3}$ (pdf)
3. $-\log(\alpha)$ Gaussian distributed (i.e., $\alpha$ lognormal distributed)
4. $\tau_0 \approx 6.3$ ns

**Deep fading phenomenon:** At $f = f_0$, the so-called **deep fading** occurs.
13.2 The effect of signal characteristics on the choice of a channel model
Usually, we prefer **slowly fading** and **frequency non-selectivity**.

So we wish to choose symbol time $T$ and transmission bandwidth $B$ such that

$$T < (\Delta t)_c \quad \text{and} \quad B < (\Delta f)_c$$

Hence, using $BT = 1$, we wish

$$\frac{T}{(\Delta t)_c} \frac{B}{(\Delta f)_c} = B_d T_m < 1.$$  

The term $B_d T_m$ is an essential channel parameter and is called **spread factor**.
Underspread versus overspread

Underspread $\equiv B_d T_m < 1$
Overspread $\equiv B_d T_m > 1$

MULTIPATH SPREAD, DOPPLER SPREAD, AND SPREAD FACTOR FOR SEVERAL TIME-VARIANT MULTIPATH CHANNELS

<table>
<thead>
<tr>
<th>Type of channel</th>
<th>Multipath duration (sec)</th>
<th>Doppler spread (Hz)</th>
<th>Spread factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shortwave ionospheric propagation (HF)</td>
<td>$10^{-3} - 10^{-2}$</td>
<td>$10^{-1} - 1$</td>
<td>$10^{-4} - 10^{-2}$</td>
</tr>
<tr>
<td>Ionospheric propagation under disturbed auroral conditions (HF)</td>
<td>$10^{-3} - 10^{-2}$</td>
<td>$10 - 100$</td>
<td>$10^{-2} - 1$</td>
</tr>
<tr>
<td>Ionospheric forward scatter (VHF)</td>
<td>$10^{-4}$</td>
<td>$10$</td>
<td>$10^{-3}$</td>
</tr>
<tr>
<td>Tropospheric scatter (SHF)</td>
<td>$10^{-6}$</td>
<td>$10$</td>
<td>$10^{-5}$</td>
</tr>
<tr>
<td>Orbital scatter ($X$ band)</td>
<td>$10^{-4}$</td>
<td>$10^3$</td>
<td>$10^{-1}$</td>
</tr>
<tr>
<td>Moon at max. libration</td>
<td>$10^{-2}$</td>
<td>$10$</td>
<td>$10^{-1}$</td>
</tr>
</tbody>
</table>
13.3 Frequency-nonslective, slowly fading channel
For a frequency-nonslective, slowly fading channel, i.e.,

\[ T_m < \frac{1}{B} = T < (\Delta t)_c, \]

the signal spectrum \( s_\ell(f) \) is almost unchanged by \( C_\ell(f; t) \); hence,

\[ C_\ell(f; t) \approx C_\ell(0; t) \text{ within the signal bandwidth} \]

and it is almost time-invariant; hence,

\[ C_\ell(f; t) \approx C_\ell(0) \text{ within the signal bandwidth} \]

This gives

\[
\begin{align*}
r_\ell(t) &= c_\ell(\tau; t) \ast s_\ell(t) + z(t) \\
        &= \int_{-\infty}^{\infty} C_\ell(f; t)s_\ell(f)e^{-j2\pi ft} df + z(t) \\
        &\approx \int_{-\infty}^{\infty} C_\ell(0)s_\ell(f)e^{-j2\pi ft} df + z(t) \\
        &= C_\ell(0)s_\ell(t) + z(t)
\end{align*}
\]
Assume that the phase of $C_\ell(0) = \alpha e^{i\phi}$ can be perfectly estimated and compensated. The channel model becomes:

$$r_\ell(t) = \alpha s_\ell(t) + z(t).$$

After demodulation, we obtain

$$r_\ell = \alpha s_\ell + n_\ell.$$

**Question:** What will the error probability be under random $\alpha$?
Case 1: Equal-prior BPSK

\[ r = \pm \alpha \sqrt{E} + n \quad \text{(passband vectorization with } E[n^2] = \frac{N_0}{2}) \]
\[ r_{\ell,\text{real}} = \pm \alpha \sqrt{2E} + n_{\ell,\text{real}} \quad \text{(baseband vectorization with } E[n_{\ell,\text{real}}^2] = N_0) \]

The optimal decision is \( r \leq 0 \), regardless of \( \alpha \) (due to equal-prior).

Thus,

\[
\Pr\{\text{error}|\alpha\} = Q\left(\sqrt{2\gamma_b}\right)
\]

where \( \gamma_b = \gamma_b(\alpha) = \alpha^2 E/N_0 \).

Given that \( \alpha \) is Rayleigh distributed, \( \gamma_b(\alpha) \) is \( \chi^2 \)-distributed with two degrees of freedom; hence,

\[
P_{e,BPSK} = \int_0^\infty \Pr\{\text{error}|\alpha\} f(\alpha) \, d\alpha
\]
\[
= \int_0^\infty Q\left(\sqrt{2\gamma_b}\right) f(\gamma_b) \, d\gamma_b
\]
\[ P_{e,BPSK} = \int_0^\infty Q\left(\sqrt{2\bar{\gamma}_b}\right) \frac{1}{\bar{\gamma}_b} e^{-\gamma_b/\bar{\gamma}_b} d\gamma_b, \quad \text{where } \bar{\gamma}_b = \mathbb{E}[\gamma_b] \]

\[ = \frac{1}{2} - \frac{1}{2} \sqrt{\frac{\bar{\gamma}_b}{1 + \bar{\gamma}_b}} \]

\[ = \frac{1}{2(1 + \bar{\gamma}_b + \sqrt{\bar{\gamma}_b^2 + \bar{\gamma}_b})} \approx \frac{1}{4\bar{\gamma}_b} \quad \text{when } \bar{\gamma}_b \text{ large} \]
Case 2: Equal-prior BFSK

Similarly, for BFSK,

\[ r = \begin{cases} \left[ \begin{array}{c} \alpha \sqrt{\mathcal{E}} \\ 0 \end{array} \right] & \text{or} & \left[ \begin{array}{c} 0 \\ \alpha \sqrt{\mathcal{E}} \end{array} \right] \end{cases} + n \]

The optimal decision is \( r_1 \preceq r_2 \), regardless of \( \alpha \).

\[
P_{e,BFSK} = \int_0^\infty \Pr\{\text{error}|\alpha\} f(\alpha) d\alpha
\]

\[
= \int_0^\infty Q(\sqrt{\gamma_b}) f(\gamma_b) d\gamma_b
\]

\[
= \frac{1}{2} - \frac{1}{2} \sqrt{\frac{\gamma_b}{2 + \gamma_b}}
\]

\[
= \frac{1}{2 + \gamma_b + \sqrt{\gamma_b^2 + 2\gamma_b}} \approx \frac{1}{2\gamma_b} \quad \text{when } \gamma_b \text{ large}
\]
\[ P_{e,BDPSK} = \int_0^\infty \Pr\{\text{error}|\alpha\} f(\alpha) d\alpha \]
\[ = \int_0^\infty \frac{1}{2} e^{-\gamma_b} f(\gamma_b) d\gamma_b \]
\[ = \frac{1}{2(1 + \bar{\gamma}_b)} \approx \frac{1}{2\bar{\gamma}_b} \quad \text{when } \bar{\gamma}_b \text{ large} \]

and

\[ P_{e,\text{noncoherent BFSK}} = \int_0^\infty \Pr\{\text{error}|\alpha\} f(\alpha) d\alpha \]
\[ = \int_0^\infty \frac{1}{2} e^{-\gamma_b/2} f(\gamma_b) d\gamma_b \]
\[ = \frac{1}{2 + \bar{\gamma}_b} \approx \frac{1}{\bar{\gamma}_b} \quad \text{when } \bar{\gamma}_b \text{ large} \]
<table>
<thead>
<tr>
<th></th>
<th>$P_e$ under AWGN</th>
<th>$P_e$ under Rayleigh fading</th>
<th>Approx $P_e$ under Rayleigh fading</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>BPSK</strong></td>
<td>$Q(\sqrt{2\gamma_b})$</td>
<td>$\frac{1}{2} \left( 1 - \sqrt{\frac{\gamma_b}{1+\gamma_b}} \right)$</td>
<td>$\frac{1}{4\bar{\gamma}_b}$</td>
</tr>
<tr>
<td><strong>BFSK</strong></td>
<td>$Q(\sqrt{\gamma_b})$</td>
<td>$\frac{1}{2} \left( 1 - \sqrt{\frac{\gamma_b}{2+\gamma_b}} \right)$</td>
<td>$\frac{1}{2\bar{\gamma}_b}$</td>
</tr>
<tr>
<td><strong>BDPSK</strong></td>
<td>$\frac{1}{2} e^{-\gamma_b}$</td>
<td>$\frac{1}{2(1+\gamma_b)}$</td>
<td>$\frac{1}{2\bar{\gamma}_b}$</td>
</tr>
<tr>
<td><strong>Noncoherent BFSK</strong></td>
<td>$\frac{1}{2} e^{-\gamma_b/2}$</td>
<td>$\frac{1}{2+\gamma_b}$</td>
<td>$\frac{1}{\bar{\gamma}_b}$</td>
</tr>
</tbody>
</table>
- BPSK is 3dB better than BDPSK/BFSK; 6dB better than noncoherent BFSK.

- $P_e$ decreases **inversely** proportional with SNR under fading.

- $P_e$ decreases **exponentially** with SNR when no fading.
• To achieve $P_e = 10^{-4}$, the system must provide an SNR higher than 35dB, which is not practically possible!

• So alternative solution should be used to compensate the fading such as the diversity technique.
If $\alpha \equiv$ Nakagami-$m$ fading,

Turin et al. (1972) and Suzuki (1977) have shown that the Nakagami-$m$ distribution is the best-fit for urban radio multipath channels.

$$f(\gamma_b) = \frac{m^m}{\Gamma(m) \bar{\gamma}_b^m} \gamma_b^{m-1} e^{-m\gamma_b/\bar{\gamma}_b}, \text{ where } \bar{\gamma}_b = E[\alpha^2] \mathcal{E}_b / N_0.$$  

- $m < 1$: Worse than Rayleigh fading in performance
- $m = 1$: Rayleigh fading
- $m > 1$: Better than Rayleigh fading in performance
Prob density function of Nakagami-\(m\)
BPSK performance under Nakagami-\(m\) fading

\[
P_{e,BPSK} = \int_0^\infty Q\left(\sqrt{2\gamma_b}\right) \frac{m^m \gamma_b^{m-1} e^{-m\gamma_b/\bar{\gamma}_b}}{\Gamma(m)\bar{\gamma}_b^m} f(\gamma_b) \ d\gamma_b
\]

where \(f(\gamma_b)\) is the pdf of the Nakagami-\(m\) distribution.
In some channel, the system performance may degrade even worse, such as Rummler’s model in slide 13-48, where **deep fading** occurs at some frequency.

\[ |C_\ell(f)| \]

The lowest is equal to \( \alpha(1 - \beta) \), which is itself a random variable (which makes the performance worse as we have seen from Rayleigh example study) even if \( f_0 \) is fixed!
13.4 Diversity techniques for fading multipath channels
Solutions to compensate deep fading

- **Frequency diversity**
  - Separation of carriers $\geq (\Delta f)_c = 1/T_m$ to obtain uncorrelation in signal replicas.

- **Time diversity**
  - Separation of time slots $\geq (\Delta t)_c = 1/B_d$ to obtain uncorrelation in signal replicas.

- **Space diversity (Multiple receiver antennas)**
  - Spaced sufficiently far apart to ensure received signals faded independently (usually, $> 10$ wavelengths)

- **RAKE correlator or RAKE matched filter (Price and Green 1958)**
  - It is named **wideband approach**, since it is usually applied to situation where signal bandwidth is **much greater** than the coherent bandwidth $(\Delta f)_c$. 
It is clear for the first three diversities, we will have $L$ identical replicas at the Rx.

The idea is that as long as not all of them are deep-faded, the demodulation is “ok.”

For the last one (i.e., RAKE), where $B \gg (\Delta f)_c$, which results in a frequency selective channel, we have

$$L = \frac{B}{(\Delta f)_c}.$$  

Detail will be given later.
13.4-1 Binary signals
Assumption

1. \( L \) identical and independent channels.
2. Each channel is \textbf{frequency-nonselective} and \textbf{slowly fading} with Rayleigh-distributed envelope.
3. Zero-mean additive white Gaussian background noise.
4. Assume the phase-shift can be perfectly compensated.
5. Assume the attenuation \( \{\alpha_k\}_{k=1}^L \) can be perfectly estimated at Rx.

Hence,

\[ r_k = \alpha_k s + n_k \quad k = 1, 2, \ldots, L \]

How to combine these \( L \) outputs when making decision? \textbf{Maximal ratio combiner} (Brennan 1959)

\[ r = \sum_{k=1}^L \alpha_k r_k = \sum_{k=1}^L \alpha_k^2 s + \sum_{k=1}^L \alpha_k n_k. \]
Idea behind maximal ratio combiner

- Trust more on the strong signals and trust less on the weak signal.

Advantage of maximal ratio combiner

- Theoretically tractable; so we can predict how “good” the system can achieve without performing simulations.
Case 1: Equal-prior BPSK

\[ r = \pm \alpha^2 \sqrt{\mathcal{E}} + n, \text{ where } \alpha = \sqrt{\sum_{k=1}^{L} \alpha_k^2} \text{ and } n = \sum_{k=1}^{L} \alpha_k n_k \]

The optimal decision is \( r \leq 0 \), regardless of \( \alpha \).

Thus, \( \left\{ \begin{array}{l}
\begin{align*}
\quad r &= (s_1 \text{ or } s_2) + n \\
\quad n &= \text{0-mean Gaussian with } \mathbb{E}[nn^\dagger] = \sigma^2 \mathbb{I}
\end{align*}
\end{array} \right. \Rightarrow P_e = Q \left( \sqrt{\frac{d_1^2}{4\sigma^2}} \right) \)

\[
\Pr\{\text{error}|\{\alpha_k\}_{k=1}^{L}\} = Q \left( \sqrt{\frac{(2\alpha^2 \sqrt{\mathcal{E}})^2}{4(\alpha^2 (N_0/2))}} \right) = Q \left( \sqrt{2\gamma_b} \right)
\]

where \( \gamma_b = \gamma_b(\alpha) = \alpha^2 \mathcal{E}/N_0 \).

Given that \( \{\alpha_k\}_{k=1}^{L} \) is i.i.d. Rayleigh distributed, \( \gamma_b(\alpha) \) is \( \chi^2 \)-distributed with \( 2L \) degrees of freedom; hence,

\[
P_{e,BPSK} = \int_0^\infty \cdots \int_0^\infty \Pr\{\text{error}|\{\alpha_k\}_{k=1}^{L}\} f(\alpha_1, \ldots, \alpha_L) \, d\alpha_1 \cdots d\alpha_L
\]

\[
= \int_0^\infty Q \left( \sqrt{2\gamma_b} \right) f(\gamma_b) \, d\gamma_b
\]
\[ P_{e, BPSK} = \int_0^\infty Q\left(\sqrt{2\gamma_b}\right) \frac{1}{(L-1)!\bar{\gamma}_c} \gamma_b^{L-1} e^{-\gamma_b/\bar{\gamma}_c} \, d\gamma_b \]

where \( \bar{\gamma}_c = \mathbb{E}[\alpha_k^2] \mathcal{E}_b / N_0 \)

\[ = \left( \frac{1-\mu}{2} \right)^L \cdot \sum_{k=0}^{L-1} \binom{L-1+k}{k} \left( \frac{1+\mu}{2} \right)^k \]

where \( \mu = \sqrt{\frac{\bar{\gamma}_c}{1+\bar{\gamma}_c}} \)

\[ \left( \approx \left( \frac{2L-1}{L} \right) \left( \frac{1}{4\bar{\gamma}_c} \right)^L \text{ when } \bar{\gamma}_c \text{ large} \right) \]

where we have \( \frac{1-\mu}{2} = \frac{1}{2(1+\bar{\gamma}_c+\sqrt{\bar{\gamma}_c^2+\bar{\gamma}_c})} \approx \frac{1}{4\bar{\gamma}_c} \) and \( \frac{1+\mu}{2} \approx 1. \)
Case 2: Equal-prior BFSK

Similarly, for BFSK,

\[ r = \left\{ \begin{bmatrix} \alpha^2 \sqrt{\mathcal{E}} \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ \alpha^2 \sqrt{\mathcal{E}} \end{bmatrix} \right\} + n, \text{ where } \alpha = \sqrt{\sum_{k=1}^{L} \alpha_k^2} \]

The optimal decision is \( r_1 \leq r_2 \), regardless of \( \alpha \).

\[
P_{e,BFSK} = \int_0^\infty Q(\sqrt{\gamma_b}) f(\gamma_b) d\gamma_b
\]

\[
= \left( \frac{1 - \mu}{2} \right)^L \sum_{k=0}^{L-1} \binom{L - 1 + k}{k} \left( \frac{1 + \mu}{2} \right)^k
\]

\[
\approx \left( \frac{1}{2\bar{\gamma}_c} \right)^L \binom{2L - 1}{L} \text{ when } \bar{\gamma}_c \text{ large}
\]

but \( \mu = \sqrt{\frac{\bar{\gamma}_c}{2 + \bar{\gamma}_c}} \)
Case 3: BDPSK

From Slide 4-184 with \( t \) being the time index,

\[
\hat{m} = \arg \max_{1 \leq m \leq 2} \text{Re} \left\{ \left( r_{\ell}^{(t-1)} \right)^* r_{\ell}^{(t)} e^{-j \theta_m} \right\}
\]

\[
= \arg \max \left( \underbrace{\text{Re} \left\{ \left( r_{\ell}^{(t-1)} \right)^* r_{\ell}^{(t)} \right\}}_{m=1}, \underbrace{-\text{Re} \left\{ \left( r_{\ell}^{(t-1)} \right)^* r_{\ell}^{(t)} \right\}}_{m=2} \right)
\]

Now with \( L \) previous receptions and \( L \) current receptions,

\[
\hat{m} = \arg \max ( U_{\ell} , -U_{\ell} )
\]

where

\[
U_{\ell} = \sum_{k=1}^{L} \text{Re} \left\{ \left( r_{k,\ell}^{(t-1)} \right)^* r_{k,\ell}^{(t)} \right\}
\]

\[
= \sum_{k=1}^{L} \text{Re} \left\{ \left( \alpha_k s_{\ell}^{(t-1)} + n_{k,\ell}^{(t-1)} \right)^* \left( \alpha_k s_{\ell}^{(t)} + n_{k,\ell}^{(t)} \right) \right\}
\]

which closely resembles maximal ratio combining.
With some lengthy derivation, we obtain

\[
P_{e, \text{BDPSK}} = \left( \frac{1 - \mu}{2} \right)^L \cdot \sum_{k=0}^{L-1} \binom{L - 1 + k}{k} \left( \frac{1 + \mu}{2} \right)^k
\]

but \( \mu = \frac{\bar{\gamma}_c}{1 + \bar{\gamma}_c} \)

\[
\approx \left( \frac{1}{2 \bar{\gamma}_c} \right)^L \left( \frac{2L - 1}{L} \right) \text{ when } \bar{\gamma}_c \text{ large.}
\]
Case 4: Noncoherent FSK

Recall from Slide 4-164:

The noncoherent ML computes

\[ \hat{m} = \arg \max_{1 \leq m \leq M} \left| r_{\ell}^\dagger s_{m,\ell} \right| \]

From Slide 4-173,

\[
\begin{align*}
  s_{1,\ell} &= \begin{pmatrix}
  \sqrt{2E_s} & 0 & \ldots & 0
  \end{pmatrix} \\
  s_{2,\ell} &= \begin{pmatrix}
  0 & \sqrt{2E_s} & \ldots & 0
  \end{pmatrix} \\
  \vdots &= \begin{pmatrix}
  \vdots & \vdots & \ddots & \vdots
  \end{pmatrix} \\
  s_{M,\ell} &= \begin{pmatrix}
  0 & 0 & \ldots & \sqrt{2E_s}
  \end{pmatrix}
\end{align*}
\]

Hence,

\[ \hat{m} = \arg \max_{1 \leq m \leq M} \left| r_{m,\ell} \right| = \arg \max_{1 \leq m \leq M} \left| r_{m,\ell} \right|^2. \]
Now we have $k$ diversities/channels:

$$
\mathbf{r}_{k,\ell} = \begin{bmatrix} 
  r_{k,1,\ell} \\
  \vdots \\
  r_{k,M,\ell}
\end{bmatrix} = \alpha_k \mathbf{s}_{m,\ell} + \mathbf{n}_{k,\ell} \quad k = 1, 2, \ldots, L
$$

Instead of maximal ratio combining, we do square-law combining:

$$
\hat{m} = \arg \max_{1 \leq m \leq M} \sum_{k=1}^{L} |r_{k,m,\ell}|^2.
$$

$$
P_{e,\text{noncoherent BFSK}} = \left(\frac{1 - \mu}{2}\right)^L \cdot \sum_{k=0}^{L-1} \binom{L-1+k}{k} \left(\frac{1+\mu}{2}\right)^k
$$

but $\mu = \frac{\tilde{\gamma}_c}{2 + \tilde{\gamma}_c}$

$$
\approx \left(\frac{1}{\tilde{\gamma}_c}\right)^L \left(\frac{2L-1}{L}\right) \text{ when } \tilde{\gamma}_c \text{ large.}
$$
Summary (what the theoretical results indicate?)

- With $L$th order diversity, the POE decreases inversely with $L$th power of the SNR.
Comparing the prob density functions of $\gamma_b$ for 1-diversity (no diversity) Nakagami fading and $L$-diversity Rayleigh fading, we found that

$$f(\gamma_b) = \frac{1}{\Gamma(m)(\bar{\gamma}_b/m)^m} \gamma_b^{m-1} e^{-\gamma_b/(\bar{\gamma}_b/m)}$$ 1-divert Nakagami

$$f(\gamma_b) = \frac{1}{\Gamma(L)(\bar{\gamma}_b/L)^L} \gamma_b^{L-1} e^{-\gamma_b/(\bar{\gamma}_b/L)}$$ $L$-divert Rayleigh.

We can then conclude:

$L$-diversity in Rayleigh fading = 1-diversity in Nakagami-$L$

or further

$mL$-diversity in Rayleigh fading = $L$-diversity in Nakagami-$m$
13.4-2 Multiphase signals
For $M$-ary phase signal over $L$ Rayleigh fading channels, the symbol error rate $P_e$ can be derived as (Appendix C)

\[
P_e = \frac{(-1)^{L-1}(1 - \mu^2)^L}{\pi(L - 1)!} \left( \frac{\partial^{L-1}}{\partial b^{L-1}} \left\{ \frac{1}{b - \mu^2} \left[ \frac{\pi}{M} (M - 1) - \frac{\mu \sin(\pi/M)}{\sqrt{b - \mu^2 \cos^2(\pi/M)}} \cot^{-1} \frac{-\mu \cos(\pi/M)}{\sqrt{b - \mu^2 \cos^2(\pi/M)}} \right] \right\} \right)_{b=1}
\]

\[
\approx \begin{cases} 
\frac{M-1}{\log_2(M) \sin^2(\pi/M)} & M\text{-ary PSK & } L=1 \\
\frac{M-1}{\log_2(M) \sin^2(\pi/M)} & \frac{1}{M\gamma_b} \text{ M-ary DPSK & } L=1
\end{cases}
\]

where

\[
\mu = \begin{cases} 
\sqrt{\frac{\tilde{\gamma}_c}{1 + \tilde{\gamma}_c}} & M\text{-ary PSK} \\
\frac{\tilde{\gamma}_c}{1 + \tilde{\gamma}_c} & M\text{-ary DPSK}
\end{cases}
\]

and in this case, the system SNR $\tilde{\gamma}_t = \tilde{\gamma}_b \log_2(M) = L\tilde{\gamma}_c$. 
PSK is about 3dB better than DPSK for all $M$. 

**FIGURE 13.4-3**

Probability of symbol error for PSK and DPSK for Rayleigh fading.
DPSK performance with diversity

- Bit error $P_b$ is calculated based on Gray coding.
- Larger $M$, worse $P_b$ except for equal $P_b$ at $M = 2, 4$. 

**FIGURE 13.4-4**
Probability of a bit error for DPSK with diversity for Rayleigh fading.
13.4-3 $M$-ary orthogonal signals
Here, the derivation assumes that both passband and lowpass equivalent signals are orthogonal; hence, the frequency separation is \(1/T\) rather than \(1/(2T)\).

Based on lowpass (baseband) orthogonality, \(L\)-diversity square-law combining gives

\[
P_e = \frac{1}{(L-1)!} \sum_{m=1}^{M-1} \frac{(-1)^{m+1} (M-1)}{m! (1 + m + m\bar{\gamma}_c)^L} \sum_{k=0}^{m(L-1)} \beta_{k,m} (L - 1 + k)! \left(\frac{1 + \bar{\gamma}_c}{1 + m + m\bar{\gamma}_c}\right)^k
\]

where \(\beta_{k,m}\) satisfies

\[
\left(\sum_{k=0}^{L-1} \frac{U^k}{k!}\right)^m = \sum_{k=0}^{m(L-1)} \beta_{k,m} U^k.
\]
$M = 2$ case:

- Let $\bar{\gamma}_t = L\bar{\gamma}_c$ be the total system power. For fixed $\bar{\gamma}_t$, there is an $L$ that minimizes $P_e$.
- This hints that $\bar{\gamma}_c = 3$ will give the best performance.
$M = 4$ case:

- Let $\tilde{\gamma}_t = L\tilde{\gamma}_c$ be the total system power. For fixed $\tilde{\gamma}_t$, there is an $L$ that minimizes $P_e$.
- This hints that $\tilde{\gamma}_c = 3$ will give the best performance.
Discussions:

- Larger $M$, better performance but larger bandwidth.
- Larger $L$, better performance.

- An increase in $L$ is more efficient in performance gain than an increase in $M$. 
13.5 Digital signaling over a frequency-selective, slowly fading channel
13.5.1 A tapped-delay-line channel model
Assumption (Time-invariant channel)

\[ c_\ell(\tau; t) = c_\ell(\tau) \]

Assumption (Bandlimited signal)

\[ s_\ell(t) \text{ is band-limited, i.e., } |s_\ell(f)| = 0 \text{ for } |f| > W/2 \]

In such case, we shall add a lowpass filter at the Rx.

\[ L(f) = \begin{cases} 
1, & |f| \leq W/2 \\
0, & \text{otherwise}
\end{cases} \]
$$r_{\ell}(t) = \int_{-\infty}^{\infty} s_{\ell}(f) C_\ell(f) e^{i2\pi ft} df + z_W(t)$$

Equivalent channel with $C_\ell(f)$ random and bandlimited, and $z_W(t)$ bandlimited white noise
For a bandlimited \( C_\ell(f) \), sampling theorem gives:

\[
\begin{align*}
C_\ell(f) &= \int_{-\infty}^{\infty} c_\ell(t) e^{-\jmath 2\pi ft} dt \\
&= \left\{
\begin{array}{ll}
\frac{1}{W} \sum_{n=-\infty}^{\infty} c_\ell \left( \frac{n}{W} \right) e^{-\jmath 2\pi fn/W}, & |f| \leq W/2 \\
0, & \text{otherwise}
\end{array}
\right.
\end{align*}
\]

\[
c_\ell(t) = \sum_{n=-\infty}^{\infty} c_\ell \left( \frac{n}{W} \right) \text{sinc} \left( W \left( t - \frac{n}{W} \right) \right)
\]
\[ r_\ell(t) = \int_{-\infty}^{\infty} s_\ell(f) C_\ell(f) e^{i 2\pi f t} df + z_W(t) \]
\[ = \frac{1}{W} \sum_{n=-\infty}^{\infty} c_\ell \left( \frac{n}{W} \right) \int_{-W/2}^{W/2} s_\ell(f) e^{i 2\pi f (t-n/W)} df + z_W(t) \]
\[ = \frac{1}{W} \sum_{n=-\infty}^{\infty} c_\ell \left( \frac{n}{W} \right) s_\ell \left( t - \frac{n}{W} \right) + z_W(t) \]
\[ = \sum_{n=-\infty}^{\infty} c_n \cdot s_\ell \left( t - \frac{n}{W} \right) + z_W(t), \text{ where } c_n = \frac{1}{W} c_\ell \left( \frac{n}{W} \right) \]

For a time-varying channel, we replace \( c_\ell(\tau) \) and \( C_\ell(f) \) by \( c_\ell(\tau; t) \) and \( C_\ell(f; t) \) and obtain

\[ r_\ell(t) = \sum_{n=-\infty}^{\infty} c_n(t) \cdot s_\ell \left( t - \frac{n}{W} \right) + z_W(t) \]

where \( c_n(t) = \frac{1}{W} c_\ell \left( \frac{n}{W}; t \right) \).
Statistically, \( c_\ell(\tau) = 0 \) for \( \tau > T_m \) and \( \tau < 0 \) with probability one.

So, \( c_\ell(\tau) \) is assumed band-limited and is also statistically time-limited!

Hence, \( c_n(t) = 0 \) for \( n < 0 \) and \( n > T_m W \) (since \( \tau = n/W > T_m \)).

\[
r_\ell(t) = \sum_{n=0}^{\lfloor T_m W \rfloor} c_n(t) \cdot s_\ell \left( t - \frac{n}{W} \right) + z_W(t)
\]

For convenience, the text re-indexes the system as

\[
r_\ell(t) = \sum_{k=1}^{L} c_k(t) \cdot s_\ell \left( t - \frac{k}{W} \right) + z_W(t).
\]
13.5-2 The RAKE demodulator
Assumption (Gaussian and US (uncorrelated scattering))

\[ \{ c_k(t) \}_{k=1}^L \text{ complex i.i.d. Gaussian and can be perfectly estimated by } Rx. \]

So the RX can regard the “transmitted signal” as one of

\[
\begin{align*}
    v_{1,\ell}(t) &= \sum_{k=1}^L c_k(t) \cdot s_{1,\ell}(t - \frac{k}{W}) \\
    \vdots \\
    v_{M,\ell}(t) &= \sum_{k=1}^L c_k(t) \cdot s_{M,\ell}(t - \frac{k}{W})
\end{align*}
\]

So Slide 4-165 said:

Coherent MAP detection

\[
\hat{m} = \arg \max_{1 \leq m \leq M} \text{Re} \left[ r_{\ell}^\dagger v_m,\ell \right] = \arg \max_{1 \leq m \leq M} \text{Re} \left[ \int_0^T r_{\ell}(t) v_{m,\ell}^*(t) dt \right]
\]

\[
= \arg \max_{1 \leq m \leq M} \text{Re} \left[ \sum_{k=1}^L \int_0^T r_{\ell}(t) c_k^*(t) s_{m,\ell}^*(t - \frac{k}{W}) dt \right] \quad \text{under } U_{m,\ell}
\]
Discussions on assumptions: We assume:

- $s_\ell(t)$ is band-limited to $W$.
- $c_\ell(\tau)$ is causal and (statistically) time-limited to $T_m$ and, at the same time, band-limited to $W$.
- $W \gg (\Delta f)_c = \frac{1}{T_m}$ (i.e., $L \approx WT_m \gg 1$) as stated in page 879 in textbook.
- The definition of $U_m$ requires $T \gg T_m$ (See page 871 in textbook) such that the longest delayed version
  \[ s_\ell(t - L/W) = s_\ell(t - WT_m/W) = s_\ell(t - T_m) \]
  is still well-confined within the integration range $[0, T)$. As a result, the signal bandwidth is much larger than $1/T$; RAKE is used in the demodulation of “spread-spectrum” signals!

\[
WT \gg WT_m \gg 1 \implies W \gg \frac{1}{T}.
\]
The receiver collects the signal energy from all received paths, which is somewhat analogous to the garden rake that is used to gather leaves, hays, etc. Consequently, the name “RAKE receiver” has been coined for this receiver structure by Price and Green (1958). (I use $s_m, \ell$, but the text uses $s_{\ell,m}$.)

**FIGURE 13.5–2**
Optimum demodulator for wideband binary signals (delayed reference configuration).
Alternative realization of RAKE receiver

The previous structure requires $M$ delay lines.

We can reduce the number of the delay lines to one by the following derivation.

Let $u = t - \frac{k}{W}$.

\[
U_{m,\ell} = \text{Re} \left[ \sum_{k=1}^{L} \int_{0}^{T} r_\ell(t) c_k^*(t) s_{m,\ell}^* \left( t - \frac{k}{W} \right) dt \right]
\]

\[
= \text{Re} \left[ \sum_{k=1}^{L} \int_{-k/W}^{T-k/W} r_\ell \left( u + \frac{k}{W} \right) c_k^* \left( u + \frac{k}{W} \right) s_{m,\ell}^* (u) du \right]
\]

\[
\approx \text{Re} \left[ \sum_{k=1}^{L} \int_{0}^{T} r_\ell \left( t + \frac{k}{W} \right) c_k^* \left( t + \frac{k}{W} \right) s_{m,\ell}^* (t) dt \right]
\]

where the last approximation follows from

\[ \left| \frac{k}{W} \right| \leq \left| \frac{L}{W} \right| \approx \left| \frac{T_m W}{W} \right| = T_m \ll T \] (See Slide 13-100).
\[ r_\ell (t + \frac{L}{W}) \quad r_\ell (t + \frac{L-1}{W}) \quad \ldots \quad r_\ell (t + \frac{1}{W}) \]

**FIGURE 13.5–3**
Optimum demodulator for wideband binary signals (delayed received signal configuration).

\[ c_k^* (t + \frac{k}{W}) = \frac{1}{W} c_\ell^* (\frac{k}{W}; t + \frac{k}{W}) \] is abbreviated as \( c_k (t) \) in the above figure.
Suppose $c_k(t) = c_k$ and the signal corresponding to $m = 1$ is transmitted. Then, letting $\tilde{U}_{m,\ell} = \frac{1}{\sqrt{2E_s}} U_{m,\ell}$ and

$$\tilde{s}_{m,\ell}^* \left( t - \frac{k}{W} \right) = \frac{1}{\sqrt{2E_s}} s_{m,\ell}^* \left( t - \frac{k}{W} \right) \text{ (normalization)},$$

we have

$$\tilde{U}_{m,\ell} = \text{Re} \left[ \sum_{k=1}^{L} \int_{0}^{T} r_\ell(t) c_k^* \tilde{s}_{m,\ell}^* \left( t - \frac{k}{W} \right) dt \right]$$

$$= \text{Re} \left[ \sum_{k=1}^{L} \int_{0}^{T} \left( \sum_{n=1}^{L} c_n s_{1,\ell} \left( t - \frac{n}{W} \right) + z_W(t) \right) c_k^* \tilde{s}_{m,\ell}^* \left( t - \frac{k}{W} \right) dt \right]$$

$$= \text{Re} \left[ \sum_{k=1}^{L} \sum_{n=1}^{L} c_n c_k^* \int_{0}^{T} s_{1,\ell} \left( t - \frac{n}{W} \right) \tilde{s}_{m,\ell}^* \left( t - \frac{k}{W} \right) dt \right]$$

$$+ \text{Re} \left[ \sum_{k=1}^{L} c_k^* \int_{0}^{T} z_W(t) \tilde{s}_{m,\ell}^* \left( t - \frac{k}{W} \right) dt \right]$$
Assumption (Add-and-delay property)

The transmitted signal is orthogonal to the shifted counterparts of all signals, including itself.

\[
\{ z_k = \int_0^T z(t) \tilde{s}_m,\ell (t - \frac{k}{W}) dt \}_{k=1}^L \text{ complex Gaussian with } \quad E[|z_k|^2] = 2N_0 \text{ because } \{ \tilde{s}_m,\ell (t - \frac{k}{W}) \}_{k=1}^L \text{ orthonormal.}
\]

Hence, with \( \alpha_k = |c_k| \),

\[
\tilde{U}_{m,\ell} = \text{Re} \left[ \sum_{k=1}^L |c_k|^2 \int_0^T s_{1,\ell} (t - \frac{k}{W}) \tilde{s}_m,\ell (t - \frac{k}{W}) dt \right] + \text{Re} \left[ \sum_{k=1}^L c_k^* z_k \right]
\]

\[
= \sum_{k=1}^L \alpha_k^2 \text{Re} \left[ \left( s_{1,\ell} (t - \frac{k}{W}) , \tilde{s}_m,\ell (t - \frac{k}{W}) \right) \right] + \sum_{k=1}^L \alpha_k n_{k,\ell},
\]

where \( \{ n_{k,\ell} = \text{Re} [e^{\frac{i\pi}{2} c_k z_k}] \}_{k=1}^L \) i.i.d. Gaussian with \( E[n_{k,\ell}^2] = N_0 \).

Under \( T \gg T_m \), \( \int_0^T s_{1,\ell} (t - \frac{k}{W}) \tilde{s}_m,\ell (t - \frac{k}{W}) dt \) is almost independent of \( k \); so,

\[
\left< s_{1,\ell} (t - \frac{k}{W}) , \tilde{s}_m,\ell (t - \frac{k}{W}) \right> \approx \left< s_{1,\ell} (t) , \tilde{s}_m,\ell (t) \right>.
\]
Therefore, the performance of RAKE is the same as the $L$-diversity maximal ratio combiner if $\{\alpha_k\}_{k=1}^{L}$ i.i.d. However, $\{\alpha_k = |c_k|\}_{k=1}^{L}$ may not be identically distributed.

In such case, we can still obtain the pdf of $\gamma_b = \sum_{k=1}^{L} \gamma_k$ from

\[
\begin{align*}
&\quad \text{characteristic function of } \gamma_k \equiv \Psi_k(i\nu) = \frac{1}{1 - i\nu \bar{\gamma}_k} \\
&\quad \text{characteristic function of } \gamma_b = \sum_{k=1}^{L} \gamma_k \equiv \prod_{k=1}^{L} \Psi_k(i\nu) = \prod_{k=1}^{L} \frac{1}{1 - i\nu \bar{\gamma}_k}
\end{align*}
\]

The pdf of $\gamma$ is then given by the Fourier transform of characteristic function:

\[
f(\gamma) = \sum_{k=1}^{L} \frac{\pi_k}{\bar{\gamma}_k} e^{-\gamma/\bar{\gamma}_k}
\]

where $\pi_k = \prod_{i=1, i \neq k}^{L} \frac{\bar{\gamma}_k}{\bar{\gamma}_k - \bar{\gamma}_i}$, provided $\bar{\gamma}_k \neq \bar{\gamma}_i$ for $k \neq i$. 
\[
BPSK: \begin{cases} 
U_{1,\ell} \approx \sum_{k=1}^{L} \alpha_k^2 \Re \left[ \langle s_{1,\ell}(t), \tilde{s}_{1,\ell}(t) \rangle \right] + \sum_{k=1}^{L} \alpha_k n_{k,\ell} \\
U_{2,\ell} \approx \sum_{k=1}^{L} \alpha_k^2 \Re \left[ \langle s_{1,\ell}(t), \tilde{s}_{2,\ell}(t) \rangle \right] + \sum_{k=1}^{L} \alpha_k n_{k,\ell}
\end{cases}
\]

\[
BFSK: \begin{cases} 
U_{1,\ell} \approx \sum_{k=1}^{L} \alpha_k^2 \Re \left[ \langle s_{1,\ell}(t), \tilde{s}_{1,\ell}(t) \rangle \right] + \sum_{k=1}^{L} \alpha_k n_{k,\ell} \\
U_{2,\ell} \approx \sum_{k=1}^{L} \alpha_k^2 \Re \left[ \langle s_{1,\ell}(t), \tilde{s}_{2,\ell}(t) \rangle \right] + \sum_{k=1}^{L} \alpha_k n_{k,\ell}
\end{cases}
\]

\[
BPSK: \begin{cases} 
U_{1,\ell} \approx \sum_{k=1}^{L} \alpha_k^2 \sqrt{2E_s} + \sum_{k=1}^{L} \alpha_k n_{k,\ell}
\end{cases}
\]

\[
BFSK: \begin{cases} 
U_{1,\ell} \approx \sum_{k=1}^{L} \alpha_k^2 \sqrt{2E_s} + \sum_{k=1}^{L} \alpha_k n_{k,\ell}
\end{cases}
\]

Thus,

\[P_e = \begin{cases} 
\frac{1}{2} \sum_{k=1}^{L} \pi_k \left( 1 - \sqrt{\frac{\gamma_k}{1+\gamma_k}} \right) \approx \left( \frac{2L-1}{L} \right) \prod_{k=1}^{L} \frac{1}{4\gamma_k}, & \text{BPSK, RAKE} \\
\frac{1}{2} \sum_{k=1}^{L} \pi_k \left( 1 - \sqrt{\frac{\gamma_k}{2+\gamma_k}} \right) \approx \left( \frac{2L-1}{L} \right) \prod_{k=1}^{L} \frac{1}{2\gamma_k}, & \text{BFSK, RAKE}
\end{cases}\]
Estimation of $c_k$

For orthogonal signaling, we can estimate $c_n$ via

$$
\int_0^T r_\ell \left( t + \frac{n}{W} \right) (s_{1,\ell}^*(t) + \cdots + s_{M,\ell}^*(t)) \, dt
$$

$$
= \sum_{k=1}^L c_k \int_0^T s_{m,\ell} \left( t + \frac{n}{W} - \frac{k}{W} \right) (s_{1,\ell}^*(t) + \cdots + s_{M,\ell}^*(t)) \, dt
$$

$$
+ \int_0^T z \left( t + \frac{n}{W} \right) (s_{1,\ell}^*(t) + \cdots + s_{M,\ell}^*(t)) \, dt
$$

$$
= \sum_{k=1}^L c_k \int_0^T s_{m,\ell} \left( t + \frac{n}{W} - \frac{k}{W} \right) s_{m,\ell}^*(t) \, dt
$$

$$
+ \int_0^T z \left( t + \frac{n}{W} \right) (s_{1,\ell}^*(t) + \cdots + s_{M,\ell}^*(t)) \, dt \quad \text{(Orthogonality)}
$$

$$
= c_n \int_0^T |s_{m,\ell}(t)|^2 \, dt + \text{noise term} \quad \text{(Add-and-delay)}
$$
$M = 2$ case

$B_d = \text{Doppler spread}$
Decision-feedback estimator

The previous estimator only works for orthogonal signaling. For, e.g., PAM signal with

\[ s_\ell(t) = I \cdot g(t) \text{ where } I \in \{\pm 1, \pm 3, \ldots, \pm (M - 1)\}, \]

we can estimate \( c_n \) via

\[
\int_0^T r_\ell \left( t + \frac{n}{W} \right) g^*(t) dt \\
= \int_0^T \left( \sum_{k=1}^L c_k \cdot I \cdot g \left( t + \frac{n}{W} - \frac{k}{W} \right) + z \left( t + \frac{n}{W} \right) \right) g^*(t) dt \\
= \sum_{k=1}^L c_k \cdot I \cdot \int_0^T g \left( t + \frac{n}{W} - \frac{k}{W} \right) g^*(t) dt + \text{noise term} \\
= c_n \cdot I \cdot \int_0^T |g(t)|^2 dt + \text{noise term} \quad \text{(Add-and-delay)}
\]
Usually it requires \( \frac{(\Delta t)_c}{T} > 100 \) in order to have an accurate estimate of \( \{c_n\}_{n=1}^L \).

Note that for DPSK and FSK with square-law combiner, it is unnecessary to estimate \( \{c_n\}_{n=1}^L \).

So, they have no further performance loss (due to an inaccurate estimate of \( \{c_n\}_{n=1}^L \)).
What you learn from Chapter 13

- Statistical model of (US-WSS) (linear) multipath fading channels:
  - \( c_\ell(\tau; t) = c(\tau; t)e^{-i2\pi f_c}\tau \) and \( c(\tau; t) = |c_\ell(\tau; t)| \)
  - Multipath intensity profile or delay power spectrum
    \[
    R_{c_\ell}(\tau) = R_{c_\ell}(\tau; \Delta t = 0).
    \]
- Multipath delay spread \( T_m \) vs coherent bandwidth \((\Delta f)_c\)
- Frequency-selective vs frequency-nonselective
- Spaced-frequency, spaced-time correlation function
  \[
  R_{C_\ell}(\Delta f; \Delta t) = \mathbb{E}\{C_\ell(f + \Delta f; t + \Delta t)C_\ell^*(f; t)\}
  \]
• Doppler power spectrum

\[ S_{C_\ell}(\lambda) = \int_{-\infty}^{\infty} R_{C_\ell}(\Delta f = 0; \Delta t) e^{-\frac{\lambda}{2} \pi (\Delta t)} d(\Delta t) \]

• Doppler spread \( B_d \) vs coherent time \( (\Delta t)_c \)
• Slow fading versus fast fading
• Scattering function

\[ S(\tau; \lambda) = \mathcal{F}_{\Delta t} \{ R_{C_\ell}(\tau; \Delta t) \} \]

• (Good to know) Jakes’ model
• Rayleigh, Rice and Nakagami-\( m \), Rummler’s 3-path model
• Deep fading phenomenon
• \( B_d T_m \) spread factor: Underspread vs overspread
Analysis of error rate under frequency-nonselective, slowly Rayleigh- and Nakagami-$m$-distributed fading channels (≡diversity under Rayleigh) with $M = 2$

(Good to know) Analysis of the error rate ... with $M > 2$.

Rake receiver under frequency-selective, slowly fading channels

- Assumption: Bandlimited signal with ideal lowpass filter and perfect channel estimator at the receiver
- This assumption results in a (finite-length) tapped-delay-line channel model under a finite delay spread.
- Error analysis under add-and-delay assumption on the transmitted signals