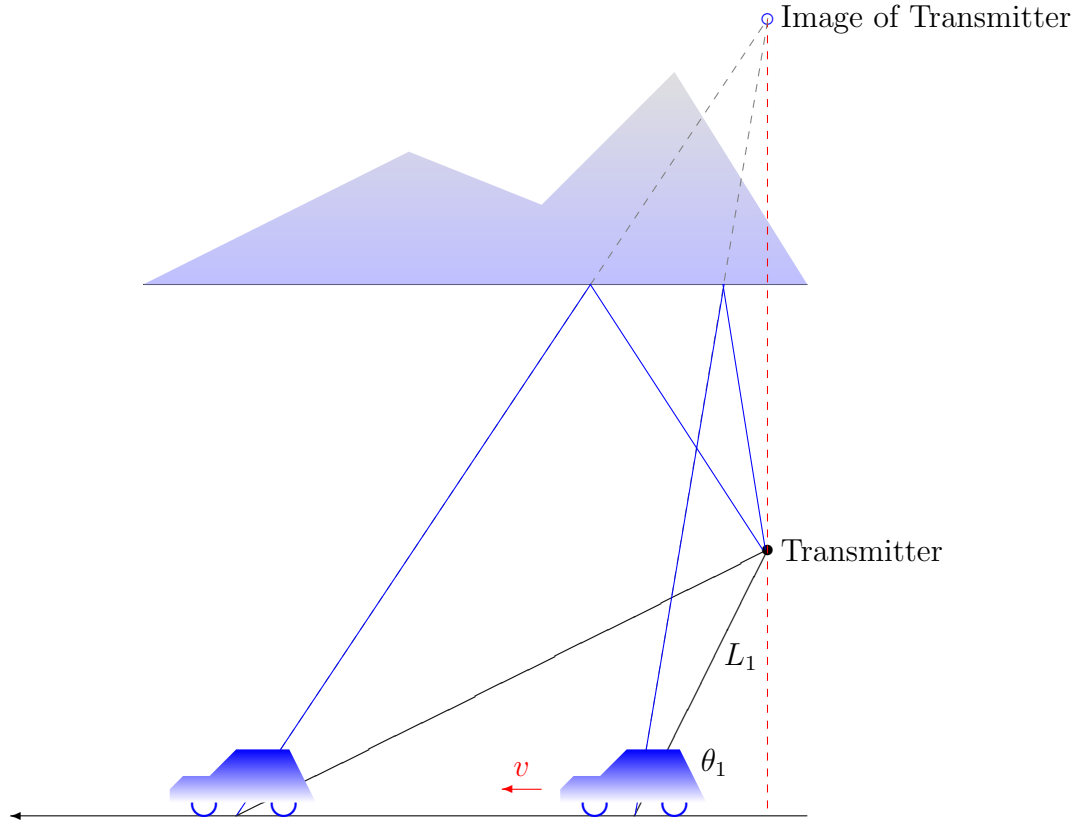


# 2016 Spring: The Final Exam of Digital Communications

The total number of points is 131.

1.



As shown in the figure above, a car is receiving a signal from a remote transmitter at distance  $L_1$ . The car is moving at speed  $v$  parallel to a series of high mountains. The transmitter is located between the mountain line and car directional line. Thus a two-path fading channel is formed. Assume that the length of the reflection path, which is reflected with respect to the high mountain line, is  $L_2$ . Let the attenuation of the direct path be  $\alpha_1(t)$  and that of the reflection path  $\alpha_2(t)$ . Denote the the path delay of the direct path by  $\tau_1(t)$  and that of the reflection path  $\tau_2(t)$ . Also denote the angle between the car directional line and the incoming signal from the reflection path by  $\theta_2$ .

- (a) (6 pt.) By setting that  $\tau_1(0) = L_1/c$  and  $f_m \triangleq v f_c/c$ , where  $c$  is the light speed and  $f_c$  is the carrier frequency of the signal wave, prove that

$$\tau_1(t) \approx \tau_1'(0) \cdot t + \tau_1(0) = \frac{f_m}{f_c} \cos(\theta_1) t + \frac{f_m}{f_c} \frac{L_1}{v}. \quad (1)$$

Note: Similarly, with  $\tau_2(0) = L_2/c$ , we have

$$\tau_2(t) \approx \tau_2'(0) \cdot t + \tau_2(0) = \frac{f_m}{f_c} \cos(\theta_2) t + \frac{f_m}{f_c} \frac{L_2}{v}. \quad (2)$$

- (b) (6 pt.) The passband channel impulse response of the two-path fading channel is given by

$$c(\tau; t) = \alpha_1(t) \cdot \delta(\tau - \tau_1(t)) + \alpha_2(t) \cdot \delta(\tau - \tau_2(t)).$$

Its lowpass equivalent channel impulse response is

$$\begin{aligned} c_\ell(\tau; t) &= c(\tau; t)e^{-\imath 2\pi f_c \tau} \\ &= \alpha_1(t) \cdot \delta(\tau - \tau_1(t))e^{-\imath 2\pi f_c \tau} + \alpha_2(t) \cdot \delta(\tau - \tau_2(t))e^{-\imath 2\pi f_c \tau} \\ &= \alpha_1(t) \cdot \delta(\tau - \tau_1(t))e^{-\imath 2\pi f_c \tau_1(t)} + \alpha_2(t) \cdot \delta(\tau - \tau_2(t))e^{-\imath 2\pi f_c \tau_2(t)} \end{aligned}$$

For the path delays, since the time we consider meets  $t \ll \min\{L_1, L_2\}/c$ , we can well approximate  $\tau_1(t) \approx T_1 \triangleq \frac{f_m L_1}{f_c v} = \frac{L_1}{c}$  and  $\tau_2(t) \approx T_2 \triangleq \frac{f_m L_2}{f_c v} = \frac{L_2}{c}$ . However, for the phases, since  $f_c$  is often very large, we cannot ignore the first order terms in (1) and (2) and hence

$$f_c \tau_1(t) \approx f_m \cos(\theta_1) t + f_m \frac{L_1}{v} = f_m \cos(\theta_1) t + f_c T_1$$

and

$$f_c \tau_2(t) \approx f_m \cos(\theta_2) t + f_m \frac{L_2}{v} = f_m \cos(\theta_2) t + f_c T_2.$$

This reduces  $c_\ell(\tau; t)$  to

$$c_\ell(\tau; t) \approx \alpha_1(t) e^{-\imath 2\pi f_m \cos(\theta_1) t} e^{-\imath 2\pi f_c T_1} \delta(\tau - T_1) + \alpha_2(t) e^{-\imath 2\pi f_m \cos(\theta_2) t} e^{-\imath 2\pi f_c T_2} \delta(\tau - T_2).$$

Determine  $R_{c_\ell}(\bar{\tau}, \tau; t + \Delta t, t) = \mathbb{E}\{c_\ell(\bar{\tau}; t + \Delta t)c_\ell^*(\tau; t)\}$  under the assumption that

$$\{\alpha_1(t)e^{-\imath 2\pi f_m \cos(\theta_1) t}, t \in \mathcal{R}\} \quad \text{and} \quad \{\alpha_2(t)e^{-\imath 2\pi f_m \cos(\theta_2) t}, t \in \mathcal{R}\}$$

are independent of each other and are both zero-mean wide-sense stationary (WSS) processes with autocorrelation functions

$$\begin{aligned} \zeta_k(\Delta t) &= \mathbb{E}[\alpha_k(t + \Delta t)e^{-\imath 2\pi f_m \cos(\theta_k)(t + \Delta t)} \alpha_k(t)e^{\imath 2\pi f_m \cos(\theta_k)t}] \\ &= \mathbb{E}[\alpha_k(t + \Delta t)\alpha_k(t)e^{-\imath 2\pi f_m \cos(\theta_k)(\Delta t)}] \quad k = 1, 2. \end{aligned}$$

Note:  $f_c$ ,  $T_1$  and  $T_2$  are constants.

- (c) (3 pt.) Justify that this  $c_\ell(\tau; t)$  is an uncorrelated scattering channel.  
(d) (3 pt.) What is the delay spread of the channel?  
(e) (4 pt.) Determine the spaced-frequency, spaced-time correlation function of the channel.  
(f) (4 pt.) Determine the Doppler power spectrum of the channel, provided that for  $k = 1, 2$ ,

$$\Gamma_k(\lambda) = \int_{-\infty}^{\infty} \zeta_k(\Delta t) e^{-\imath 2\pi \lambda (\Delta t)} d(\Delta t).$$

- (g) (4 pt.) Determine the scattering function of the channel.  
(h) (6 pt.) Determine the frequency of fade minimum, i.e.,  $f_0(t) = \arg \min_{f \in \mathcal{R}} |C_\ell(f; t)|$ . What is the minimum difference between two fade-minimum frequencies?

Hint: For nonnegative real numbers  $\alpha_1$  and  $\alpha_2$ ,

$$|\alpha_1 e^{\imath \beta_1} + \alpha_2 e^{\imath \beta_2}|^2 = |\alpha_2|^2 \left| \frac{\alpha_1}{\alpha_2} e^{\imath(\beta_1 - \beta_2)} + 1 \right|^2 = |\alpha_2|^2 \left( 1 + \frac{\alpha_1^2}{\alpha_2^2} + 2 \frac{\alpha_1}{\alpha_2} \cos(\beta_1 - \beta_2) \right),$$

where the minimum occurs when  $\cos(\beta_1 - \beta_2) = -1$ .

Hint:

$$\begin{aligned} C_\ell(f; t) &= \int_{-\infty}^{\infty} c_\ell(\tau; t) e^{-i2\pi f\tau} d\tau \\ &= \alpha_1(t) e^{-i2\pi f_m \cos(\theta_1)t} e^{-i2\pi(f+f_c)T_1} + \alpha_2(t) e^{-i2\pi f_m \cos(\theta_2)t} e^{-i2\pi(f+f_c)T_2}. \end{aligned}$$

- (i) (3 pt.) With  $L_2 - L_1 = 3$  km, we learn that  $T_2 - T_1 = 10^{-5}$  seconds because  $c = 3 \times 10^8$  meter/second. Can we avoid the fade minimum (or the occurrence of deep fade) by carefully selecting the carrier frequency of the transmission signal if the required transmission bandwidth is 20 MHz. Justify your answer.

### Solutions.

(a)

$$\begin{aligned} \Delta L_1 &= \sqrt{(L_1 \sin(\theta_1))^2 + (L_1 \cos(\theta_1) + v \cdot \Delta t)^2} - L_1 \\ &= \sqrt{L_1^2 + v^2(\Delta t)^2 + 2L_1 \cdot v \cdot \Delta t \cdot \cos(\theta_1)} - L_1, \end{aligned}$$

which implies

$$\lim_{\Delta t \rightarrow 0} \frac{\Delta L_1}{\Delta t} = v \cos(\theta_1).$$

Thus,

$$\begin{aligned} \tau_1'(0) &= \lim_{\Delta t \rightarrow 0} \frac{\tau_1(t + \Delta t) - \tau_1(t)}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \frac{\frac{L_1 + \Delta L_1}{c} - \frac{L_1}{c}}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \frac{1}{c} \frac{\Delta L_1}{\Delta t} \\ &= \frac{v}{c} \cos(\theta_1) \end{aligned}$$

As a result,

$$\tau_1(t) \approx \frac{v}{c} \cos(\theta_1) t + \tau_1(0) = \frac{f_m}{f_c} \cos(\theta_1) t + \frac{f_m L_1}{f_c v}.$$

Note: Since we can treat the reflection path as a direct path from the image of the transmitter with respect to the mountain line, we can similarly obtain

$$\tau_2(t) \approx \frac{v}{c} \cos(\theta_2) t + \tau_2(0) = \frac{f_m}{f_c} \cos(\theta_2) t + \frac{f_m L_2}{f_c v}.$$

(b)

$$\begin{aligned}
R_{c_\ell}(\bar{\tau}, \tau; t + \Delta t, t) &= \mathbb{E}[c_\ell(\bar{\tau}; t + \Delta t)c_\ell^*(\tau; t)] \\
&= \mathbb{E}\left[\left(\alpha_1(t + \Delta t)e^{-\imath 2\pi f_m \cos(\theta_1)(t + \Delta t)}e^{-\imath 2\pi f_c T_1}\delta(\bar{\tau} - T_1)\right.\right. \\
&\quad \left.\left. + \alpha_2(t + \Delta t)e^{-\imath 2\pi f_m \cos(\theta_2)(t + \Delta t)}e^{-\imath 2\pi f_c T_2}\delta(\bar{\tau} - T_2)\right)\right. \\
&\quad \left.\times \left(\alpha_1(t)e^{\imath 2\pi f_m \cos(\theta_1)t}e^{\imath 2\pi f_c T_1}\delta(\tau - T_1) + \alpha_2(t)e^{\imath 2\pi f_m \cos(\theta_2)t}e^{\imath 2\pi f_c T_2}\delta(\tau - T_2)\right)\right] \\
&= \mathbb{E}[\alpha_1(t + \Delta t)\alpha_1(t)e^{-\imath 2\pi f_m \cos(\theta_1)(\Delta t)}] \cdot \delta(\bar{\tau} - T_1)\delta(\tau - T_1) \\
&\quad + \mathbb{E}[\alpha_2(t + \Delta t)e^{-\imath 2\pi f_m \cos(\theta_2)(t + \Delta t)}] \mathbb{E}[\alpha_1(t)e^{\imath 2\pi f_m \cos(\theta_1)t}] e^{\imath 2\pi f_c(T_1 - T_2)} \cdot \delta(\bar{\tau} - T_2)\delta(\tau - T_1) \\
&\quad + \mathbb{E}[\alpha_1(t + \Delta t)e^{-\imath 2\pi f_m \cos(\theta_1)(t + \Delta t)}] \mathbb{E}[\alpha_2(t)e^{\imath 2\pi f_m \cos(\theta_2)t}] e^{\imath 2\pi f_c(T_2 - T_1)} \cdot \delta(\bar{\tau} - T_1)\delta(\tau - T_2) \\
&\quad + \mathbb{E}[\alpha_2(t + \Delta t)\alpha_2(t)e^{-\imath 2\pi f_m \cos(\theta_2)(\Delta t)}] \cdot \delta(\bar{\tau} - T_2)\delta(\tau - T_2) \\
&= \mathbb{E}[\alpha_1(t + \Delta t)\alpha_1(t)e^{-\imath 2\pi f_m \cos(\theta_1)(\Delta t)}] \cdot \delta(\bar{\tau} - T_1)\delta(\tau - T_1) \\
&\quad + \mathbb{E}[\alpha_2(t + \Delta t)\alpha_2(t)e^{-\imath 2\pi f_m \cos(\theta_2)(\Delta t)}] \cdot \delta(\bar{\tau} - T_2)\delta(\tau - T_2) \\
&= \left( \mathbb{E}[\alpha_1(t + \Delta t)\alpha_1(t)e^{-\imath 2\pi f_m \cos(\theta_1)(\Delta t)}] \cdot \delta(\tau - T_1) \right. \\
&\quad \left. + \mathbb{E}[\alpha_2(t + \Delta t)\alpha_2(t)e^{-\imath 2\pi f_m \cos(\theta_2)(\Delta t)}] \cdot \delta(\tau - T_2) \right) \delta(\bar{\tau} - \tau) \\
&= (\zeta_1(\Delta t)\delta(\tau - T_1) + \zeta_2(\Delta t)\delta(\tau - T_2)) \delta(\bar{\tau} - \tau).
\end{aligned}$$

(c) It is an uncorrelated scattering channel because the correlation of the two paths are zero. In other words,

$$\mathbb{E}[c_\ell(\bar{\tau} = T_1; t + \Delta t)c_\ell^*(\tau = T_2; t)] = 0.$$

(d)  $R_{c_\ell}(\tau) = R_{c_\ell}(\tau; \Delta t = 0) = \zeta_1(0)\delta(\tau - T_1) + \zeta_2(0)\delta(\tau - T_2)$ . Thus the delay spread is  $T_2$ .

(e)

$$\begin{aligned}
R_{C_\ell}(\Delta f; \Delta t) &= \int_{-\infty}^{\infty} R_{c_\ell}(\tau; \Delta t) e^{-\imath 2\pi(\Delta f)\tau} d\tau \\
&= \int_{-\infty}^{\infty} (\zeta_1(\Delta t)\delta(\tau - T_1) + \zeta_2(\Delta t)\delta(\tau - T_2)) e^{-\imath 2\pi(\Delta f)\tau} d\tau \\
&= \zeta_1(\Delta t)e^{-\imath 2\pi(\Delta f)T_1} + \zeta_2(\Delta t)e^{-\imath 2\pi(\Delta f)T_2}.
\end{aligned}$$

(f)

$$\begin{aligned}
S_{C_\ell}(\lambda) &= \int_{-\infty}^{\infty} R_{C_\ell}(\Delta f = 0; \Delta t) e^{-\imath 2\pi\lambda(\Delta t)} d(\Delta t) \\
&= \int_{-\infty}^{\infty} (\zeta_1(\Delta t) + \zeta_2(\Delta t)) e^{-\imath 2\pi\lambda(\Delta t)} d(\Delta t) \\
&= \Gamma_1(\lambda) + \Gamma_2(\lambda).
\end{aligned}$$

(g)

$$\begin{aligned}
S(\tau; \lambda) &= \int_{-\infty}^{\infty} R_{c_\ell}(\tau; \Delta t) e^{-\imath 2\pi\lambda(\Delta t)} d(\Delta t) \\
&= \int_{-\infty}^{\infty} (\zeta_1(\Delta t)\delta(\tau - T_1) + \zeta_2(\Delta t)\delta(\tau - T_2)) e^{-\imath 2\pi\lambda(\Delta t)} d(\Delta t) \\
&= \Gamma_1(\lambda)\delta(\tau - T_1) + \Gamma_2(\lambda)\delta(\tau - T_2).
\end{aligned}$$

(h)

$$\begin{aligned}
C_\ell(f; t) &= \int_{-\infty}^{\infty} c_\ell(\tau; t) e^{-\imath 2\pi f \tau} d\tau \\
&= \int_{-\infty}^{\infty} (\alpha_1(t) e^{-\imath 2\pi f_m \cos(\theta_1) t} e^{-\imath 2\pi f_c T_1} \delta(\tau - T_1) \\
&\quad + \alpha_2(t) e^{-\imath 2\pi f_m \cos(\theta_2) t} e^{-\imath 2\pi f_c T_2} \delta(\tau - T_2)) e^{-\imath 2\pi f \tau} d\tau \\
&= \alpha_1(t) e^{-\imath 2\pi f_m \cos(\theta_1) t} e^{-\imath 2\pi(f+f_c)T_1} + \alpha_2(t) e^{-\imath 2\pi f_m \cos(\theta_2) t} e^{-\imath 2\pi(f+f_c)T_2}
\end{aligned}$$

Thus,

$$\begin{aligned}
|C_\ell(f; t)|^2 &= \alpha_2^2(t) \left| \frac{\alpha_1(t)}{\alpha_2(t)} e^{\imath 2\pi f_m [\cos(\theta_2) - \cos(\theta_1)] t} e^{\imath 2\pi(f+f_c)|T_2 - T_1|} + 1 \right|^2 \\
&= \alpha_2^2(t) \left| A(t) e^{\imath [2\pi(f+f_c)|T_2 - T_1| + \phi(t)]} + 1 \right|^2 \\
&= \alpha_2^2(t) [1 + A^2(t) + 2A(t) \cos(2\pi(f+f_c)|T_2 - T_1| + \phi(t))]
\end{aligned}$$

where  $A(t) = \frac{\alpha_1(t)}{\alpha_2(t)}$  and  $\phi(t) = 2\pi f_m [\cos(\theta_2) - \cos(\theta_1)] t$ . Hence, the fade minimum occurs when

$$\cos(2\pi(f+f_c)|T_2 - T_1| + \phi(t)) = -1,$$

equivalently,

$$2\pi(f+f_c)|T_2 - T_1| + 2\pi f_m [\cos(\theta_2) - \cos(\theta_1)] t = (2k+1)\pi.$$

Accordingly,

$$f_0(t) = \frac{1}{(T_2 - T_1)} k + \frac{1}{2(T_2 - T_1)} - \frac{f_m [\cos(\theta_2) - \cos(\theta_1)]}{(T_2 - T_1)} t - f_c \quad \text{for integer } k.$$

The minimum difference between two fade-minimum frequencies is  $\frac{1}{T_2 - T_1}$ .

- (i) The minimum difference between two fade-minimum frequencies is  $\frac{1}{T_2 - T_1} = 0.1$  MHz. Thus for a signal bandwidth of 20 MHz, it is not possible to avoid suffering fade minimum.

2. Suppose we have a two-path channel model with time-invariant impulse response:

$$c_\ell(\tau) = \alpha_1 e^{-\imath 2\pi f_c T_1} \delta(\tau - T_1) + \alpha_2 e^{-\imath 2\pi f_c T_2} \delta(\tau - T_2),$$

where  $\alpha_1$  and  $\alpha_2$  are random variables having certain joint distribution, and  $T_1$  and  $T_2$  are constant path delays with  $T_2 > T_1$ . The received signal is thus given by

$$r_\ell(t) = \int_{-\infty}^{\infty} c_\ell(\tau) s_{m,\ell}(t - \tau) d\tau + z_\ell(t),$$

where  $z_\ell(t)$  is zero-mean white Gaussian noise with two-sided PSD of height  $2N_0$ . Let

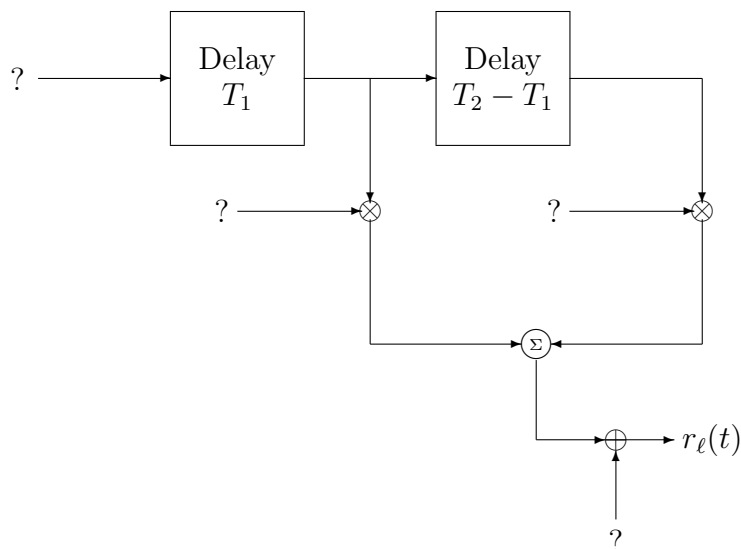
$$s_{m,\ell}(t) = \begin{cases} -\sqrt{2\mathcal{E}} g(t), & m = 1; \\ +\sqrt{2\mathcal{E}} g(t), & m = 2 \end{cases}$$

be an antipodal modulated signal, where

$$g(t) = \begin{cases} \frac{1}{\sqrt{T}}, & 0 \leq t < T; \\ 0, & \text{otherwise} \end{cases}$$

and  $0 < T < T_2 - T_1$ . Assume equal prior probability and that the receiver knows the constant path delays  $T_1$  and  $T_2$ . Also assume that the receiver can perfectly estimate  $\alpha_1$  and  $\alpha_2$ .

- (a) (4 pt.) Draw the tapped delay line model of this channel based on the below structure.



Hint: You shall replace each of the seven question marks by a proper term.

- (b) (4 pt.) Show that

$$r_\ell(t) = \pm\sqrt{2\mathcal{E}} \cdot \tilde{g}(t) + z_\ell(t),$$

where  $\tilde{g}(t) = \alpha_1 e^{-i2\pi f_c T_1} g(t - T_1) + \alpha_2 e^{-i2\pi f_c T_2} g(t - T_2)$ .

- (c) (4 pt.) When doing signal demodulation (See for example Slide 4-5), what should be the basis  $\phi(t)$  chosen for vectorization? Note that when doing vectorization, the range of integration is set to be from  $-\infty$  to  $\infty$ .

Hint:  $\|\tilde{g}(t)\|^2 = \int_{-\infty}^{\infty} \tilde{g}(t)\tilde{g}^*(t)dt$ .

- (d) (6 pt.) Determine the ML decision rule for detection (after the vectorization).

Hint: The decision rule should be a function of  $r = \int_{-\infty}^{\infty} r_\ell(t)\phi^*(t)dt$ , where  $\phi(t)$  is the basis chosen in (c). Note that  $r$  is in general a complex number. You may need to consider whether the real or complex part of  $r$  is necessary in the ML decision rule.

- (e) (8 pt.) Can we realize the ML decision rule in (d) by the Rake receiver? If positive, draw the realization diagram. If negative, justify why.

Hint: Check whether you can formulate the ML decision rule as the form on Slide 13-99.

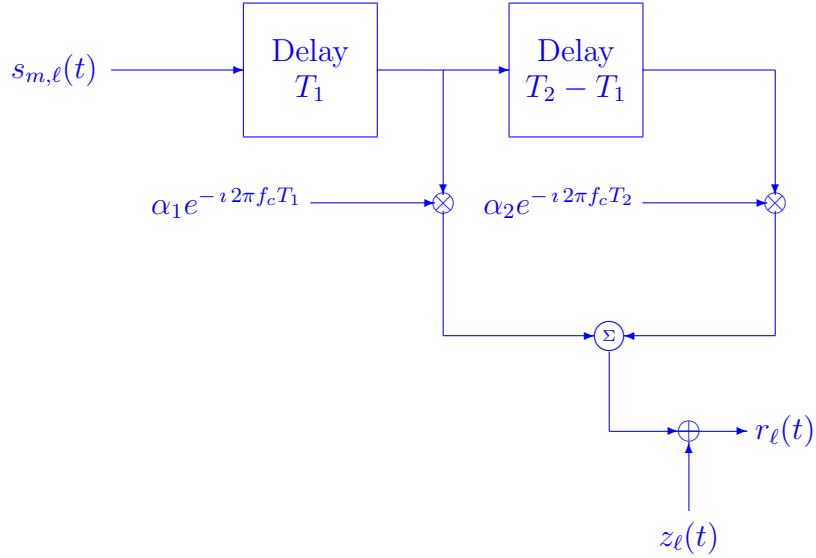
- (f) (8 pt.) Determine the error probability of the ML decision rule in (d) if  $\alpha_1$  and  $\alpha_2$  are independent random variables having common distribution as

$$\Pr[\alpha_1 = 0] = \Pr[\alpha_1 = 1] = \frac{1}{2}.$$

Hint: For binary transmission, recall that  $\Pr(\text{error}) = Q\left(\sqrt{\frac{d_{12}^2}{4\sigma^2}}\right)$ .

## Solutions.

- (a)



(b)

$$\begin{aligned}
r_\ell(t) &= \int_{-\infty}^{\infty} c_\ell(\tau) s_\ell(t - \tau) d\tau + z_\ell(t) \\
&= \pm\sqrt{2\mathcal{E}} \int_{-\infty}^{\infty} [\alpha_1 e^{-i2\pi f_c T_1} \delta(\tau - T_1) + \alpha_2 e^{-i2\pi f_c T_2} \delta(\tau - T_2)] g(t - \tau) d\tau + z_\ell(t) \\
&= \pm\sqrt{2\mathcal{E}} [\alpha_1 e^{-i2\pi f_c T_1} g(t - T_1) + \alpha_2 e^{-i2\pi f_c T_2} g(t - T_2)] + z_\ell(t)
\end{aligned}$$

(c)  $\phi(t) = \frac{\tilde{g}(t)}{\|\tilde{g}(t)\|}$ . Now derive

$$\begin{aligned}
\|\tilde{g}(t)\|^2 &= \int_{-\infty}^{\infty} \tilde{g}(t) \tilde{g}^*(t) dt \\
&= \int_{-\infty}^{\infty} (\alpha_1 e^{-i2\pi f_c T_1} g(t - T_1) + \alpha_2 e^{-i2\pi f_c T_2} g(t - T_2)) \\
&\quad (\alpha_1 e^{i2\pi f_c T_1} g(t - T_1) + \alpha_2 e^{i2\pi f_c T_2} g(t - T_2)) dt \\
&= \alpha_1^2 \int_{-\infty}^{\infty} g^2(t - T_1) dt + \alpha_2^2 \int_{-\infty}^{\infty} g^2(t - T_2) dt \\
&= \alpha_1^2 + \alpha_2^2.
\end{aligned}$$

Thus the basis is

$$\phi(t) = \frac{\tilde{g}(t)}{\sqrt{\alpha_1^2 + \alpha_2^2}} = \frac{\alpha_1}{\sqrt{\alpha_1^2 + \alpha_2^2}} e^{-i2\pi f_c T_1} g(t - T_1) + \frac{\alpha_2}{\sqrt{\alpha_1^2 + \alpha_2^2}} e^{-i2\pi f_c T_2} g(t - T_2).$$

(d)

$$\begin{aligned}
r &= \int_{-\infty}^{\infty} r_\ell(t) \phi^*(t) dt \\
&= \int_{-\infty}^{\infty} r_\ell(t) \frac{\tilde{g}^*(t)}{\|\tilde{g}(t)\|} dt \\
&= \pm\sqrt{2\mathcal{E}} \cdot \int_{-\infty}^{\infty} \tilde{g}(t) \frac{\tilde{g}^*(t)}{\|\tilde{g}(t)\|} dt + \int_{-\infty}^{\infty} z_\ell(t) \frac{\tilde{g}^*(t)}{\|\tilde{g}(t)\|} dt \\
&= \pm\sqrt{2\mathcal{E}} \cdot \|\tilde{g}(t)\| + z \\
&= \pm\sqrt{2\mathcal{E}(\alpha_1^2 + \alpha_2^2)} + z,
\end{aligned}$$

where  $z$  is complex Gaussian random variable with mean zero and variance  $E[|z|^2] = 2N_0$ . Since  $\pm\sqrt{2\mathcal{E}} \cdot \|\tilde{g}(t)\|$  is real, we can ignore the complex part of  $r$  without sacrificing the optimality. This gives

$$\text{Re}\{r\} = \pm\sqrt{2\mathcal{E}} \cdot \|\tilde{g}(t)\| + \text{Re}\{z\}.$$

Because  $\text{Re}\{z\}$  is Gaussian distributed with mean 0 and variance  $N_0$ , the ML decision rule should be

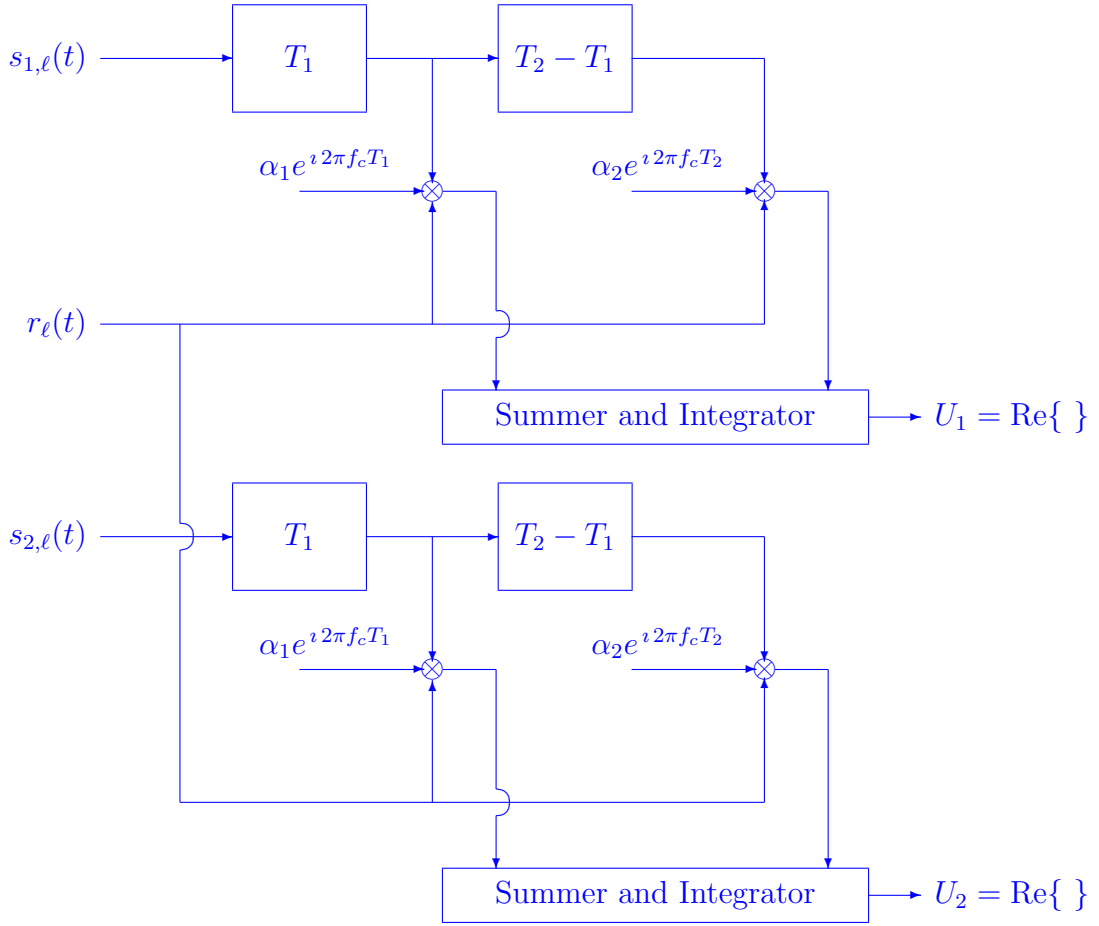
$$g_{\text{ML}}(\text{Re}\{r\}) = \begin{cases} 1, & \text{Re}\{r\} < 0; \\ 2, & \text{Re}\{r\} > 0. \end{cases}$$

(e) The ML decision rule can be re-written as

$$\begin{aligned} \hat{m} &= \arg \max_{1 \leq m \leq 2} \left\{ \underbrace{-\text{Re}[r]}_{m=1}, \underbrace{\text{Re}[r]}_{m=2} \right\} \\ &= \arg \max_{1 \leq m \leq 2} \left\{ \underbrace{\int_{-\infty}^{\infty} r_{\ell}(t) [-\phi^*(t)] dt}_{m=1}, \underbrace{\int_{-\infty}^{\infty} r_{\ell}(t) \phi^*(t) dt}_{m=2} \right\} \\ &= \arg \max_{1 \leq m \leq 2} \text{Re} \left\{ \int_{-\infty}^{\infty} r_{\ell}(t) [\alpha_1 e^{i2\pi f_c T_1} s_{1,\ell}^*(t - T_1) + \alpha_2 e^{i2\pi f_c T_2} s_{1,\ell}^*(t - T_2)] dt, \right. \\ &\quad \left. \int_{-\infty}^{\infty} r_{\ell}(t) [\alpha_1 e^{i2\pi f_c T_1} s_{2,\ell}^*(t - T_1) + \alpha_2 e^{i2\pi f_c T_2} s_{2,\ell}^*(t - T_2)] dt \right\}. \end{aligned}$$

This matches exactly the coherent MAP detection on Slide 13-99. We can then draw the Rake receiver as follows.





(f) Under BPSK transmission,

$$\text{Re}\{r\} = \pm \sqrt{2\mathcal{E}(\alpha_1^2 + \alpha_2^2)} + \text{Re}\{z\}$$

the error rate given  $\alpha_1$  and  $\alpha_2$  is equal to

$$\begin{aligned} \text{Pr}(\text{error}|\alpha_1, \alpha_2) &= Q\left(\sqrt{\frac{d_{12}^2}{4\sigma^2}}\right) \\ &= Q\left(\sqrt{\frac{(2\sqrt{2\mathcal{E}(\alpha_1^2 + \alpha_2^2)})^2}{4N_0}}\right) \\ &= Q\left(\sqrt{2\alpha^2 \frac{\mathcal{E}}{N_0}}\right) \end{aligned}$$

where  $\alpha^2 = \alpha_1^2 + \alpha_2^2$ . Thus,

$$\begin{aligned} P_e &= \frac{1}{4}Q(0) + \frac{1}{2}Q\left(\sqrt{2\frac{\mathcal{E}}{N_0}}\right) + \frac{1}{4}Q\left(\sqrt{4\frac{\mathcal{E}}{N_0}}\right) \\ &= \frac{1}{8} + \frac{1}{2}Q\left(\sqrt{2\frac{\mathcal{E}}{N_0}}\right) + \frac{1}{4}Q\left(\sqrt{4\frac{\mathcal{E}}{N_0}}\right). \end{aligned}$$

Note: The error probability is very bad because  $\mathcal{E}/N_0 \rightarrow \infty$  implies  $P_e \rightarrow \frac{1}{8}$ . Hence, we have a “floor” for the error probability due to fading.

3. (a) (6 pt.) Show that the product of two independent complex random processes  $p_{\text{PN}}(t)$  and  $c(t)$  is cyclostationary if  $p_{\text{PN}}(t)$  is zero-mean wide-sense stationary and  $c(t)$  is cyclostationary.  
 (b) (6 pt.) Show that the time-averaged power spectrum density of the product process is given by

$$\bar{S}_{p \times c}(f) = \int_{-\infty}^{\infty} S_p(u) \bar{S}_c(f - u) du.$$

- (c) (4 pt.) Show that if  $p_{\text{PN}}(t)$  is a white process, so is the product process  $p_{\text{PN}}(t)c(t)$ , provided that  $\int_{-\infty}^{\infty} \bar{S}_c(f) df$  is finite.

Hint: Use (b).

### Solutions.

- (a) The mean function of the product process  $p_{\text{PN}}(t)c(t)$  is

$$\mathbb{E}[p_{\text{PN}}(t)c(t)] = \mathbb{E}[p_{\text{PN}}(t)]\mathbb{E}[c(t)] = 0,$$

which is a constant and hence can be regarded as periodic.

The autocorrelation function of the product process  $p_{\text{PN}}(t)c(t)$  is

$$\begin{aligned} R_{p \times c}(t + \tau, t) &= \mathbb{E}[p_{\text{PN}}(t + \tau)c(t + \tau)p_{\text{PN}}^*(t)c^*(t)] \\ &= \mathbb{E}[p_{\text{PN}}(t + \tau)p_{\text{PN}}^*(t)]\mathbb{E}[c(t + \tau)c^*(t)] \\ &= R_p(\tau)R_c(t + \tau, t). \end{aligned}$$

Thus  $R_{p \times c}(t + \tau, t)$  is periodic in  $t$  if  $R_c(t + \tau, t)$  is periodic in  $t$ .

- (b) The time-averaged autocorrelation function of the product process is

$$\begin{aligned} \bar{R}_{p \times c}(\tau) &= \frac{1}{T} \int_0^T R_{p \times c}(t + \tau, t) dt = \frac{1}{T} \int_0^T R_p(\tau)R_c(t + \tau, t) dt \\ &= R_p(\tau)\bar{R}_c(\tau). \end{aligned}$$

By this, we can directly infer that the time-averaged power spectrum density of the product process is equal to the convolution of the two individual time-averaged power spectrums.

Note: For your information, the detail derivation is as follows.

$$\begin{aligned} \bar{S}_{p \times c}(f) &= \int_{-\infty}^{\infty} R_p(\tau)\bar{R}_c(\tau)e^{-i2\pi f\tau} d\tau \\ &= \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} S_p(u)e^{i2\pi u\tau} du \right) \left( \int_{-\infty}^{\infty} \bar{S}_c(v)e^{i2\pi v\tau} dv \right) e^{-i2\pi f\tau} d\tau \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_p(u)\bar{S}_c(v) \left( \int_{-\infty}^{\infty} e^{-i2\pi(f-u-v)\tau} d\tau \right) dv du \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_p(u)\bar{S}_c(v)\delta(f - u - v) dv du \\ &= \int_{-\infty}^{\infty} S_p(u)\bar{S}_c(f - u) du \end{aligned}$$

(c) When  $S_p(f) = P_0$  is a constant,

$$\begin{aligned}\bar{S}_{p \times c}(f) &= \int_{-\infty}^{\infty} S_p(u) \bar{S}_c(f-u) du \\ &= P_0 \int_{-\infty}^{\infty} \bar{S}_c(f-u) du \\ &= P_0 \int_{-\infty}^{\infty} \bar{S}_c(u') du' \quad (u' = f-u) \\ &= \text{constant.}\end{aligned}$$

Thus the product process is also white.

Note: Multiplication with a white process leads automatically to a white product process.

4. (a) (4 pt.) Suppose a direct sequence spread spectrum (DSSS) system employs the  $(n_1, k) = (7, 4)$  Hamming code as the outer code and  $(n_2, 1)$  repetition code as the inner code. Select a processing gain  $L_c$ , which lies between 100 and 110 and which can result in an integer  $n_2$  that fits the DSSS system.
- (b) (4 pt.) The 16 codewords of the  $(7, 4)$  Hamming code are listed below.

Message	Codeword	Message	Codeword
0 0 0 0	0 0 0 0 0 0 0	1 0 0 0	1 1 0 1 0 0 0
0 0 0 1	1 0 1 0 0 0 1	1 0 0 1	0 1 1 1 0 0 1
0 0 1 0	1 1 1 0 0 1 0	1 0 1 0	0 0 1 1 0 1 0
0 0 1 1	0 1 0 0 0 1 1	1 0 1 1	1 0 0 1 0 1 1
0 1 0 0	0 1 1 0 1 0 0	1 1 0 0	1 0 1 1 1 0 0
0 1 0 1	1 1 0 0 1 0 1	1 1 0 1	0 0 0 1 1 0 1
0 1 1 0	1 0 0 0 1 1 0	1 1 1 0	0 1 0 1 1 1 0
0 1 1 1	0 0 1 0 1 1 1	1 1 1 1	1 1 1 1 1 1 1

Derive the coding gain of the DSSS system.

- (c) (6 pt.) Given that  $\gamma_b = 9.6$  dB satisfies  $Q(\sqrt{2\gamma_b}) = 10^{-5}$ , find the jamming margin to achieve error rate  $1.5 \times 10^{-4}$  for this DSSS system via the union bound formula:

$$\Pr(\text{error}) \leq (M-1)Q\left(\sqrt{4\frac{L_c}{J_{av}/P_{av}} \min_{2 \leq m \leq M} R_c w_m}\right),$$

where  $M$  is the number of (outer) codewords.

Hint: Follow Slide 12-39 and check

$$\Pr(\text{error}) \leq (M-1)Q\left(\sqrt{4\frac{L_c}{J_{av}/P_{av}} \min_{2 \leq m \leq M} R_c w_m}\right) \leq 1.5 \times 10^{-4}.$$

Hint:  $10 \log_{10}(2) = 3.01$  dB and  $10 \log_{10}(12/7) = 2.34$  dB.

$L_c$	101	102	103	104	105	106	107	108	109	110
$10 \log_{10}(L_c)$ dB	20.04	20.09	20.13	20.17	20.21	20.25	20.29	20.33	20.37	20.41

**Solutions.**

(a)

$$4T_b = nT_c = n_1n_2T_c = 7n_2T_c \Rightarrow n_2 = \frac{4T_b}{7T_c} = \frac{4}{7}L_c.$$

Therefore,  $L_c$  must be a multiple of 7. The only integer that is a multiple of 7 and that lies between 100 and 110 is 105. In such case,  $n_2 = 60$ .

(b)

$$\text{Coding gain} = \min_{2 \leq m \leq M} R_c^{(out)} w_m^{(out)} = R_c^{(out)} \min_{2 \leq m \leq M} w_m^{(out)} = \frac{4}{7} \cdot 3 = \frac{12}{7}.$$

(c)

$$15 Q \left( \sqrt{4 \frac{L_c}{J_{av}/P_{av}} \min_{2 \leq m \leq M} R_c w_m} \right) \leq 1.5 \times 10^{-4}$$

is equivalent to

$$Q \left( \sqrt{4 \frac{L_c}{J_{av}/P_{av}} \min_{2 \leq m \leq M} R_c w_m} \right) \leq 10^{-5} = Q(\sqrt{2\gamma_b})$$

Hence,

$$2 \frac{L_c}{J_{av}/P_{av}} \min_{2 \leq m \leq M} R_c w_m \geq 9.6 \text{ dB}$$

or equivalently,

$$10 \log_{10}(2) \text{ dB} + L_c \text{ dB} + \min_{2 \leq m \leq M} R_c w_m \text{ dB} - (J_{av}/P_{av}) \text{ dB} \geq 9.6 \text{ dB}.$$

As a result,

$$3.01 \text{ dB} + 20.21 \text{ dB} + 2.34 \text{ dB} - (J_{av}/P_{av}) \text{ dB} \geq 9.6 \text{ dB} \Rightarrow (J_{av}/P_{av}) \text{ dB} \leq 15.96 \text{ dB}.$$

5. (a) (6 pt.) Denote a set of  $Q$  waveforms as

$$\left\{ \kappa e^{i2\pi \frac{k}{U} t} : t \in [0, T), \quad k = 0, 1, \dots, Q-1 \right\}$$

where  $T$  is the symbol duration,  $\kappa = 1/\sqrt{T}$  and  $U$  is a waveform parameter satisfying  $U > T$ . Are these waveforms orthogonal signals? Justify your answer.

Hint: The inner product of two signals are defined as  $\langle f(t), g(t) \rangle = \int_0^T f(t)g^*(t)dt$ .

(b) (6 pt.) Form the transmission signal via a linear combination of the  $Q$  signal waveforms in (a) as follows:

$$s_\ell(t) = \begin{cases} \kappa \sum_{k=0}^{Q-1} X_k e^{i2\pi \frac{k}{U} t}, & t \in [0, T) \\ 0, & \text{otherwise} \end{cases}$$

and transmit it via a noiseless link. Let the received signal be

$$r_\ell(t) = \tilde{s}_\ell(t) \star c_\ell(t) = \int_{-\infty}^{\infty} c_\ell(\tau) \tilde{s}_\ell(t - \tau) d\tau,$$

where  $\tilde{s}_\ell(t)$  is the periodic counterpart of  $s_\ell(t)$  with period  $T$ . Show that

$$r_\ell(t) = \kappa \sum_{k=0}^{Q-1} X_k e^{i2\pi \frac{k}{U} t} C_\ell \left( \frac{k}{U} \right),$$

where  $C_\ell(f) = \int_{-\infty}^{\infty} c_\ell(\tau) e^{-i2\pi f \tau} d\tau$ .

(c) (8 pt.) Sampling  $r_\ell(t)$  with sampling period  $T/N$ , where  $N > Q$ , we obtain

$$r_m = r_\ell\left(\frac{m}{N}T\right) = \kappa \sum_{k=0}^{Q-1} C_\ell\left(\frac{k}{U}\right) X_k e^{i2\pi\frac{kmT}{UN}} \quad m = 0, 1, \dots, N-1$$

We then perform  $N$ -point DFT onto  $\{r_m\}_{m=0}^{N-1}$ , i.e.,

$$R_n = \sum_{m=0}^{N-1} r_m e^{-i2\pi\frac{mn}{N}}.$$

Can we recover each of  $\{X_k\}_{k=0}^{Q-1}$  from  $\{R_n\}_{n=0}^{N-1}$  for every  $T$  and  $U$  satisfying  $U > T$  if  $\{C_\ell(k/U)\}_{k=0}^{Q-1}$  are known? Justify your answer.

(d) (8 pt.) Re-do (c) by sampling  $r_\ell(t)$  with sampling period  $U/N$ , where  $N > Q$ . Note that in such case,

$$r_m = r_\ell\left(\frac{m}{N}U\right) = \kappa \sum_{k=0}^{Q-1} C_\ell\left(\frac{k}{U}\right) X_k e^{i2\pi\frac{km}{N}} \quad m = 0, 1, \dots, N-1.$$

### Solutions.

(a)

$$\begin{aligned} \left\langle \kappa e^{i2\pi\frac{k}{U}t}, \kappa e^{i2\pi\frac{j}{U}t} \right\rangle &= \int_0^T \kappa^2 e^{i2\pi\frac{k}{U}t} e^{-i2\pi\frac{j}{U}t} dt \\ &= \kappa^2 \int_0^T e^{i2\pi\frac{(k-j)}{U}t} dt \\ &= \kappa^2 \left( \frac{1}{i2\pi\frac{(k-j)}{U}} e^{i2\pi\frac{(k-j)}{U}t} \Big|_0^T \right) \\ &= \kappa^2 \left( \frac{1}{i2\pi\frac{(k-j)}{U}} e^{i2\pi\frac{(k-j)}{U}T} - \frac{1}{i2\pi\frac{(k-j)}{U}} \right) \\ &= \frac{U}{i2\pi(k-j)T} \left( e^{i2\pi\frac{T}{U}(k-j)} - 1 \right). \end{aligned} \tag{3}$$

From (3), we know that  $\langle \kappa e^{i2\pi\frac{k}{U}t}, \kappa e^{i2\pi\frac{j}{U}t} \rangle = 1$  when  $k = j$ . If these waveforms are orthogonal, then we must have  $\langle \kappa e^{i2\pi\frac{k}{U}t}, \kappa e^{i2\pi\frac{j}{U}t} \rangle = 0$  for every  $k \neq j$ , which implies  $\frac{T}{U}(k-j)$  must be an integer for every  $k \neq j$ . Apparently, when  $k-j=1$ , this is violated! Hence, these waveforms are not necessarily orthogonal to each other.

(b)

$$\begin{aligned}
r_\ell(t) &= \tilde{s}_\ell(t) \star c_\ell(t) \\
&= \kappa \left( \sum_{k=0}^{Q-1} X_k e^{i 2\pi \frac{k}{U} t} \right) \star c_\ell(t) \\
&= \kappa \sum_{k=0}^{Q-1} X_k \int_{-\infty}^{\infty} c_\ell(\tau) e^{i 2\pi \frac{k}{U} (t-\tau)} d\tau \\
&= \kappa \sum_{k=0}^{Q-1} X_k e^{i 2\pi \frac{k}{U} t} \int_{-\infty}^{\infty} c_\ell(\tau) e^{-i 2\pi \frac{k}{U} \tau} d\tau \\
&= \kappa \sum_{k=0}^{Q-1} X_k e^{i 2\pi \frac{k}{U} t} C_\ell \left( \frac{k}{U} \right).
\end{aligned}$$

(c)

$$\begin{aligned}
R_n &= \sum_{m=0}^{N-1} r_m e^{-i 2\pi \frac{mn}{N}} \\
&= \sum_{m=0}^{N-1} \left( \kappa \sum_{k=0}^{Q-1} C_\ell \left( \frac{k}{U} \right) X_k e^{i 2\pi \frac{kmT}{UN}} \right) e^{-i 2\pi \frac{mn}{N}} \\
&= \kappa \sum_{k=0}^{Q-1} C_\ell \left( \frac{k}{U} \right) X_k \sum_{m=0}^{N-1} e^{-i 2\pi \frac{m}{N} (n-kT/U)} \\
&= \kappa \sum_{k=0}^{Q-1} C_\ell \left( \frac{k}{U} \right) X_k \frac{\left( e^{-i 2\pi \frac{1}{N} (n-kT/U)} \right)^N - 1}{e^{-i 2\pi \frac{1}{N} (n-kT/U)} - 1} \\
&= \kappa \sum_{k=0}^{Q-1} C_\ell \left( \frac{k}{U} \right) X_k \frac{e^{-i 2\pi (n-kT/U)} - 1}{e^{-i 2\pi \frac{1}{N} (n-kT/U)} - 1}.
\end{aligned}$$

Apparently, we cannot recover  $X_n$  directly from a single  $R_n$  for  $0 \leq n < Q$ . However, since we only have  $Q$  unknowns but have  $N$  linear equations, we should be able to solve these  $Q$  unknowns via linear algebraic approach as long as  $N > Q$ .

In fact, we can use directly the  $N$  linear equations below

$$r_m = r_\ell \left( \frac{m}{N} T \right) = \kappa \sum_{k=0}^{Q-1} C_\ell \left( \frac{k}{U} \right) X_k e^{i 2\pi \frac{kmT}{UN}} \quad m = 0, 1, \dots, N-1$$

to solve the  $Q$  unknowns. The problem is that when adding background noise and making  $\{C_\ell \left( \frac{k}{U} \right)\}_{k=0}^{Q-1}$  random, performing ML decision simultaneously on these  $Q$  unknowns based on  $N$  linear equations will be very, very computationally intensive!

Hence, if your answer is YES and you mention about that the solution can be obtained via linear algebraic approach, you get the full mark of this subproblem. However, if your answer is NO and you mention that it is practically infeasible due to its complexity, you also get the full mark of this subproblem. Note that you shall justify your answer based on the correct reasoning.

(d) In this case,

$$\begin{aligned}
R_n &= \sum_{m=0}^{N-1} r_m e^{-\iota 2\pi \frac{mn}{N}} \\
&= \sum_{m=0}^{N-1} \left( \kappa \sum_{k=0}^{Q-1} C_\ell \left( \frac{k}{U} \right) X_k e^{\iota 2\pi \frac{km}{N}} \right) e^{-\iota 2\pi \frac{mn}{N}} \\
&= \kappa \sum_{k=0}^{Q-1} C_\ell \left( \frac{k}{U} \right) X_k \sum_{m=0}^{N-1} e^{-\iota 2\pi \frac{m}{N}(n-k)} \\
&= \kappa \sum_{k=0}^{Q-1} C_\ell \left( \frac{k}{U} \right) X_k \frac{\left( e^{-\iota 2\pi \frac{1}{N}(n-k)} \right)^N - 1}{e^{-\iota 2\pi \frac{1}{N}(n-k)} - 1} \\
&= \kappa \sum_{k=0}^{Q-1} C_\ell \left( \frac{k}{U} \right) X_k \frac{e^{-\iota 2\pi(n-k)} - 1}{e^{-\iota 2\pi \frac{1}{N}(n-k)} - 1} \\
&= \kappa \sum_{k=0}^{Q-1} C_\ell \left( \frac{k}{U} \right) X_k \cdot N \delta_{n-k} \\
&= \begin{cases} \kappa N C_\ell \left( \frac{n}{U} \right) X_n, & 0 \leq n < Q \\ 0, & Q \leq n < N \end{cases}
\end{aligned}$$

So we can obtain  $X_n$  from  $R_n$  for  $0 \leq n < Q$  by using “non-orthogonal waveforms.”