2016 Spring: The First Midterm of Digital Communications

The total points of this exam is 112.

1. The passband signal $x(t) = x_i(t)\cos(2\pi f_0 t) - x_q(t)\sin(2\pi f_0 t)$ introduced in our lectures can be regarded as a linear combination in the form of

$$x(t) = x_i(t) \cdot \psi_i(t) + x_q(t) \cdot \psi_q(t),$$

where $\psi_i(t) = \cos(2\pi f_0 t)$ and $\psi_q(t) = -\sin(2\pi f_0 t)$. By this view, we can relate $x(t)$ and $x_\ell(t) = x_i(t) + \imath\, x_q(t)$ without using the Hilbert transform (or analytic signal) by following the procedure below.

   (a) (6 pt.) Use the relation of

$$\mathcal{F}\{\psi_i(t)\} = \frac{1}{2}[\delta(f+f_0) + \delta(f-f_0)] \text{ and } \mathcal{F}\{\psi_q(t)\} = \frac{1}{2\imath}[\delta(f+f_0) - \delta(f-f_0)],$$

   where $\mathcal{F}\{\cdot\}$ is the Fourier transform, to prove that

$$X(f) = \mathcal{F}\{x(t)\} = \frac{1}{2}\left[X_i(f-f_0) + \imath\, X_q(f-f_0)\right] + \frac{1}{2}\left[X_i(f+f_0) - \imath\, X_q(f+f_0)\right].$$

   (b) (6 pt.) By noting that $x_i(t)$ and $x_q(t)$ are both real-valued signals that should satisfy $X_i(-f) = X_i^*(f)$ and $X_q(-f) = X_q^*(f)$, respectively, prove from (a) that

$$X(f) = \frac{1}{2}\left[X_\ell(f-f_0) + X_\ell^*(-f-f_0)\right].$$

   (c) (6 pt.) Use (b) to prove that $x(t) = \mathbf{Re}\left\{x_\ell(t)e^{\imath 2\pi f_0 t}\right\}$.

Solutions.

(a)

$$\begin{aligned}
X(f) &= \mathcal{F}\{x(t)\} \\
&= \mathcal{F}\{x_i(t)\cdot\psi_i(t) + x_q(t)\cdot\psi_q(t)\} \\
&= \mathcal{F}\{x_i(t)\cdot\psi_i(t)\} + \mathcal{F}\{x_q(t)\cdot\psi_q(t)\} \\
&= X_i(f) \star \mathcal{F}\{\psi_i(t)\} + X_q(f) \star \mathcal{F}\{\psi_q(t)\} \\
&= X_i(f) \star \frac{1}{2}[\delta(f+f_0)+\delta(f-f_0)] + X_q(f) \star \frac{1}{2\imath}[\delta(f+f_0)-\delta(f-f_0)] \\
&= \frac{1}{2}[X_i(f)\star\delta(f+f_0) + X_i(f)\star\delta(f-f_0)] + \frac{1}{2\imath}[X_q(f)\star\delta(f+f_0) - X_q(f)\star\delta(f-f_0)] \\
&= \frac{1}{2}[X_i(f+f_0) + X_i(f-f_0)] - \frac{1}{2}\imath\,[X_q(f+f_0) - X_q(f-f_0)] \\
&= \frac{1}{2}\left[X_i(f-f_0) + \imath\, X_q(f-f_0)\right] + \frac{1}{2}\left[X_i(f+f_0) - \imath\, X_q(f+f_0)\right]
\end{aligned}$$

where "$\star$" is the convolution operator.

(b)

$$\begin{aligned} X(f) &= \frac{1}{2}\left[X_i(f-f_0) + \imath X_q(f-f_0)\right] + \frac{1}{2}\left[X_i(f+f_0) - \imath X_q(f+f_0)\right] \\ &= \frac{1}{2}\left[X_i(f-f_0) + \imath X_q(f-f_0)\right] + \frac{1}{2}\left[X_i^*(-f-f_0) - \imath X_q^*(-f-f_0)\right] \\ &= \frac{1}{2}\left[X_i(f-f_0) + \imath X_q(f-f_0)\right] + \frac{1}{2}\left[X_i(-f-f_0) + \imath X_q(-f-f_0)\right]^* \\ &= \frac{1}{2}X_\ell(f-f_0) + \frac{1}{2}X_\ell^*(-f-f_0) \end{aligned}$$

(c)

$$\begin{aligned} x(t) &= \mathcal{F}^{-1}\{X(f)\} \\ &= \mathcal{F}^{-1}\left\{\frac{1}{2}\left[X_\ell(f-f_0) + X_\ell^*(-f-f_0)\right]\right\} \\ &= \frac{1}{2}\left[\mathcal{F}^{-1}\left\{X_\ell(f-f_0)\right\} + \mathcal{F}^{-1}\left\{X_\ell^*(-f-f_0)\right\}\right] \\ &= \frac{1}{2}\left[\int_{-\infty}^{\infty} X_\ell(f-f_0)e^{\imath 2\pi f t}df + \int_{-\infty}^{\infty} X_\ell^*(-f-f_0)e^{\imath 2\pi f t}df\right] \\ &= \frac{1}{2}\left[\int_{-\infty}^{\infty} X_\ell(f')e^{\imath 2\pi (f'+f_0) t}df + \int_{-\infty}^{\infty} X_\ell^*(f'')e^{\imath 2\pi (-f''-f_0) t}df\right] \\ &= \frac{1}{2}\left[\int_{-\infty}^{\infty} X_\ell(f')e^{\imath 2\pi (f'+f_0) t}df + \left(\int_{-\infty}^{\infty} X_\ell(f'')e^{\imath 2\pi (f''+f_0) t}df\right)^*\right] \\ &= \frac{1}{2}\left[x_\ell(t)e^{\imath 2\pi f_0 t} + \left(x_\ell(t)e^{\imath 2\pi f_0 t}\right)^*\right] \\ &= \mathbf{Re}\left\{x_\ell(t)e^{\imath 2\pi f_0 t}\right\} \end{aligned}$$

2. Let $\boldsymbol{U}$ be a random variable uniformly distributed over $[-\pi/2, \pi/2)$. Define a continuous random process as:

$$\boldsymbol{X}(t) = \cos\left(2\pi f_c t + \boldsymbol{U}\right).$$

   (a) (8 pt.) Determine the mean function and autocorrelation function of $\boldsymbol{X}(t)$.

   (b) (4 pt.) Is $\boldsymbol{X}(t)$ a wide-sense stationary process? Justify your answer.

   (c) (4 pt.) Is $\boldsymbol{X}(t)$ cyclostationary? Justify your answer.

   (d) (4 pt.) Is $\boldsymbol{X}(t)$ a band-limited bandpass stochastic signal? Justify your answer.
   Hint: $\mathcal{F}\left\{\cos(2\pi f_c t)\right\} = \frac{1}{2}[\delta(f+f_c) + \delta(f-f_c)]$.

   (e) (6 pt.) Determine the lowpass equivalent signal $\boldsymbol{X}_\ell(t)$ of the passband signal $\boldsymbol{X}(t)$ with respect to carrier frequent f_c.
   Hint: $x(t) = \mathbf{Re}\left\{x_\ell(t)e^{\imath 2\pi f_c t}\right\}$.

   (f) (6 pt.) Are $\boldsymbol{X}_i(t+\tau)$ and $\boldsymbol{X}_q(t)$ uncorrelated for any τ? Justify your answer.

Solutions.

(a)

$$\begin{aligned}
\mathbb{E}[\boldsymbol{X}(t)] &= \mathbb{E}[\cos(2\pi f_c t + \boldsymbol{U})] = \int_{-\pi/2}^{\pi/2} \cos(2\pi f_c t + \theta)\, d\theta = \sin(2\pi f_c t + \theta)|_{-\pi/2}^{\pi/2} \\
&= \sin(2\pi f_c t + \pi/2) - \sin(2\pi f_c t - \pi/2) \\
&= \cos(2\pi f_c t) + \cos(2\pi f_c t) \\
&= 2\cos(2\pi f_c t)
\end{aligned}$$

$$\begin{aligned}
R_{\boldsymbol{X}}(t+\tau, t) = \mathbb{E}[\boldsymbol{X}(t+\tau)\boldsymbol{X}^*(t)] &= \mathbb{E}[\cos(2\pi f_c(t+\tau) + \boldsymbol{U})\cos(2\pi f_c t + \boldsymbol{U})] \\
&= \frac{1}{2}\mathbb{E}\left[\cos(2\pi f_c(2t+\tau) + 2\boldsymbol{U})\right] + \frac{1}{2}\mathbb{E}\left[\cos(2\pi f_c \tau)\right] \\
&= \frac{1}{2}\cos(2\pi f_c \tau) = R_{\boldsymbol{X}}(\tau).
\end{aligned}$$

(b) No because the mean function is not a constant.

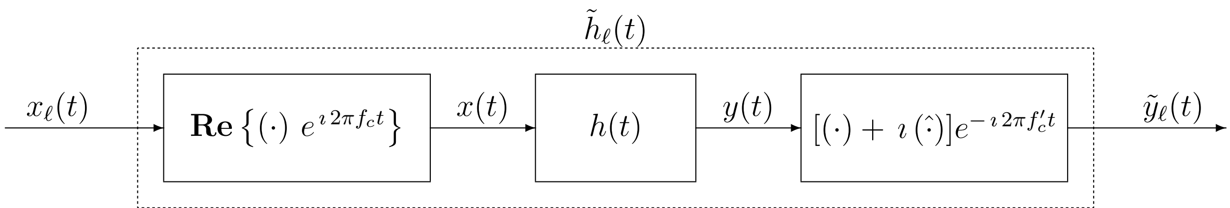
(c) Yes because both the mean function and the autocorrelation function are periodic with period $1/f_c$. In fact, the autocorrelation function is only a function of the time difference τ.

(d) From (a), we learn that the time-averaged autocorrelation function of $\boldsymbol{X}(t)$ is equal to its autocorrelation function and hence $S_{\boldsymbol{X}}(f) = \mathcal{F}\{R_{\boldsymbol{X}}(\tau)\} = \frac{1}{4}[\delta(f+f_c) + \delta(f-f_c)]$. Since $S_{\boldsymbol{X}}(f) = 0$ for all $|f \pm f_c| > W = 0$, it is by definition a band-limited bandpass stochastic signal.

(e) $\boldsymbol{X}(t) = \mathbf{Re}\left[e^{\imath(2\pi f_c t + \boldsymbol{U})}\right] = \mathbf{Re}\left[e^{\imath \boldsymbol{U}} e^{\imath 2\pi f_c t}\right]$ implies that $\boldsymbol{X}_\ell(t) = \exp(\imath \boldsymbol{U})$.

(f) $\boldsymbol{X}_i(t) = \cos(\boldsymbol{U})$ and $\boldsymbol{X}_q(t) = \sin(\boldsymbol{U})$. Hence the two conditions of (i) $\mathbb{E}[\boldsymbol{X}_i(t+\tau)\boldsymbol{X}_q(t)] = \mathbb{E}[\cos(\boldsymbol{U})\sin(\boldsymbol{U})] = \frac{1}{2}\mathbb{E}[\sin(2\boldsymbol{U})] = 0$ and (ii) $\mathbb{E}[\boldsymbol{X}_q(t)] = \mathbb{E}[\sin(\boldsymbol{U})] = 0$ jointly imply $\mathbb{E}[\boldsymbol{X}_i(t+\tau)\boldsymbol{X}_q(t)] = \mathbb{E}[\boldsymbol{X}_i(t+\tau)]\mathbb{E}[\boldsymbol{X}_q(t)]$. So $\boldsymbol{X}_i(t+\tau)$ and $\boldsymbol{X}_q(t)$ are uncorrelated for any τ.

3. As shown in the figure below, suppose that in a communication system, the receiver unawarely assumes a different carrier frequency f_c' from the carrier frequency f_c used by the transmitter, where $x(t) = \mathbf{Re}\left\{x_\ell(t)\, e^{\imath 2\pi f_c t}\right\}$, $y(t) = x(t) \star h(t)$, $\tilde{y}_\ell(t) = [y(t) + \imath\, \hat{y}(t)]e^{-\imath 2\pi f_c' t}$, and $(\hat{\cdot})$ denotes the Hilbert transform output.



(a) (6 pt.) Prove that

$$\tilde{Y}_\ell(f) = u_{-1}(f + f_c')\, H(f + f_c') \cdot X_\ell(f + (f_c' - f_c)).$$

Hint: The transfer function of the Hilbert transformer is equal to $\hat{H}(f) = -\imath\, \mathrm{sgn}(f)$, and $x_\ell(t) = [x(t) + \imath\, \hat{x}(t)]e^{-\imath 2\pi f_c t}$. It is suggested to use spectrum view when doing this problem.

(b) (6 pt.) Can we relate $\tilde{y}_\ell(t)$ with $x_\ell(t)$ via a linear operation such as convolution? If your answer is yes, prove it. If your answer is negative, justify your answer by giving a counterexample.

Hint: If your answer is yes, then you should show $\tilde{Y}_\ell(f) = \tilde{H}_\ell(f)\, X_\ell(f)$ for some $\tilde{H}_\ell(f)$ and for every $f \in \mathcal{R}$.

(c) (6 pt.) Determine the power spectrum density (PSD) $S_{\tilde{y}_\ell}(f)$ of the output $\tilde{y}_\ell(t)$ as a function of the transfer function $H(f)$ and the PSD of the wide-sense stationary $x_\ell(t)$.

Hint: If $\tilde{Y}_\ell(f) = \tilde{H}_\ell(f)\tilde{X}_\ell(f)$, where $\tilde{H}_\ell(f) = u_{-1}(f+f_c')\,H(f+f_c')$ and $\tilde{X}_\ell(f) = X_\ell(f + (f_c' - f_c))$, then $S_{\tilde{y}_\ell}(f) = |\tilde{H}_\ell(f)|^2 S_{\tilde{x}_\ell}(f)$.

Solutions.

(a) From spectrum view, we have

$$\begin{aligned}
\tilde{Y}_\ell(f) &= \mathcal{F}\left\{[y(t) + \imath\, \hat{y}(t)]e^{-\imath 2\pi f_c' t}\right\} \\
&= \mathcal{F}\left\{y(t)e^{-\imath 2\pi f_c' t}\right\} + \imath\, \mathcal{F}\left\{\hat{y}(t)e^{-\imath 2\pi f_c' t}\right\} \\
&= Y(f+f_c') + \imath\, \hat{Y}(f+f_c') \\
&= Y(f+f_c') + \imath\, \hat{H}(f+f_c')Y(f+f_c') \\
&= [1 + \imath\, \hat{H}(f+f_c')]Y(f+f_c') \\
&= [1 + \mathrm{sgn}(f+f_c')]Y(f+f_c') \\
&= 2\, u_{-1}(f+f_c')\, Y(f+f_c').
\end{aligned}$$

Similarly, we obtain $X_\ell(f) = 2\, u_{-1}(f+f_c)\, X(f+f_c)$. Since

$$\begin{aligned}
X_\ell(f - f_c + f_c') &= 2\, u_{-1}((f - f_c + f_c') + f_c)\, X((f - f_c + f_c') + f_c) \\
&= 2\, u_{-1}(f+f_c')\, X(f+f_c'),
\end{aligned}$$

we derive

$$\begin{aligned}
\tilde{Y}_\ell(f) &= 2\, u_{-1}(f+f_c')\, Y(f+f_c') \\
&= u_{-1}(f+f_c')\, H(f+f_c') \cdot 2u_{-1}(f+f_c')\, X(f+f_c') \\
&= u_{-1}(f+f_c')\, H(f+f_c') \cdot X_\ell(f + (f_c' - f_c)).
\end{aligned}$$

(b) Since $X_\ell(f) = 2\, u_{-1}(f+f_c)\, X(f+f_c) \neq 2\, u_{-1}(f+f_c')\, X(f+f_c')$, we have

$$\begin{aligned}
\tilde{Y}_\ell(f) &= 2\, u_{-1}(f+f_c')\, Y(f+f_c') \\
&= u_{-1}(f+f_c')\, H(f+f_c') \cdot 2u_{-1}(f+f_c')\, X(f+f_c') \\
&\neq u_{-1}(f+f_c')\, H(f+f_c') \cdot X_\ell(f).
\end{aligned}$$

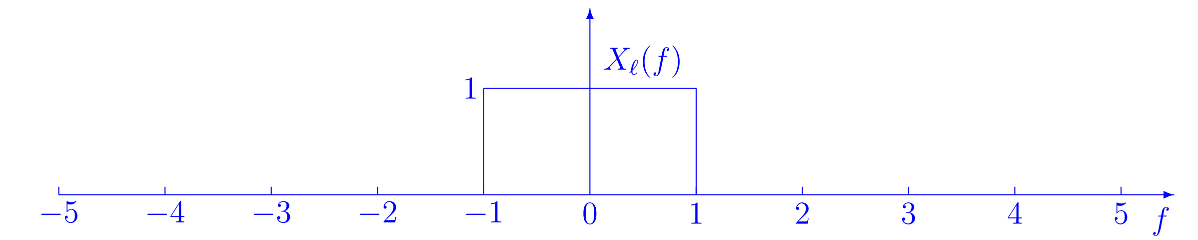
Basically, we cannot relate $\tilde{Y}_\ell(f)$ with $X_\ell(f)$ via a linear operation such as convolution.

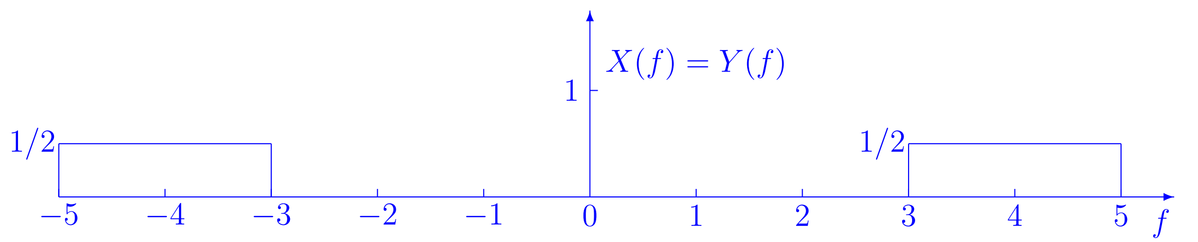
A simple counterexample is that $X_\ell(f) = u_{-t}(f-1) - u_{-1}(f+1)$, $h(t) = \delta(t)$ and $f_c' = 2f_c = 8$ as shown in the figures below. In this counterexample,

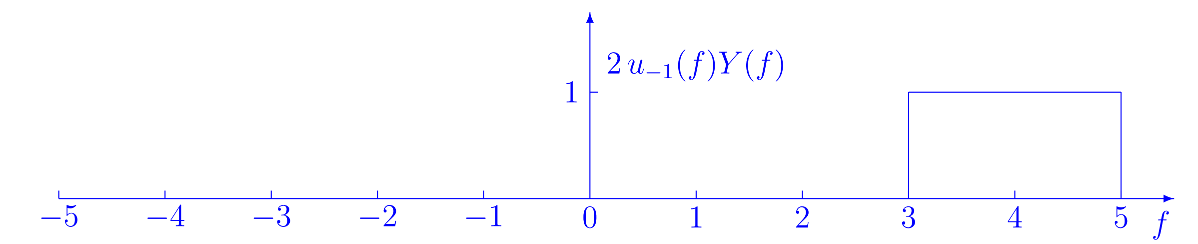
$$X_\ell(f) = 0 \text{ and } \tilde{Y}_\ell(f) = 1 \quad \text{for } -5 < f < -3.$$

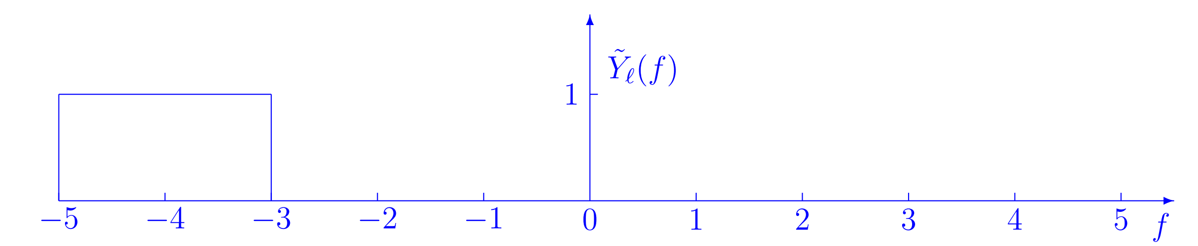
Hence, there exists no $\tilde{H}_\ell(f)$ satisfying

$$1 = \tilde{Y}_\ell(f) = \tilde{H}_\ell(f)X_\ell(f) = 0 \quad \text{for } -5 < f < -3.$$









(c) From (a), we can let $\tilde{H}_\ell(f) = u_{-1}(f + f_c')\,H(f + f_c')$ and $\tilde{X}_\ell(f) = X_\ell(f + (f_c' - f_c))$. Hence,

$$
\begin{aligned}
\tilde{x}_\ell(t) &= \int_{-\infty}^{\infty} \tilde{X}_\ell(f) e^{\imath 2\pi f t} df \\
&= \int_{-\infty}^{\infty} X_\ell(f + (f_c' - f_c)) e^{\imath 2\pi f t} df \quad (\text{Let } s = f + (f_c' - f_c)) \\
&= \int_{-\infty}^{\infty} X_\ell(s) e^{\imath 2\pi (s - (f_c' - f_c)) t} ds \\
&= e^{-\imath 2\pi (f_c' - f_c) t} \int_{-\infty}^{\infty} X_\ell(s) e^{\imath 2\pi s t} ds \\
&= x_\ell(t) e^{-\imath 2\pi (f_c' - f_c) t}.
\end{aligned}
$$

We then derive

$$
\begin{aligned}
R_{\tilde{x}_\ell}(t+\tau, t) &= \mathbb{E}[\tilde{x}_\ell(t+\tau)\tilde{x}_\ell^*(t)] \\
&= \mathbb{E}[x_\ell(t+\tau) e^{-\imath 2\pi (f_c' - f_c)(t+\tau)} x_\ell^*(t) e^{\imath 2\pi (f_c' - f_c) t}] \\
&= e^{-\imath 2\pi (f_c' - f_c)\tau} \mathbb{E}[x_\ell(t+\tau) x_\ell^*(t)] \\
&= e^{-\imath 2\pi (f_c' - f_c)\tau} R_{x_\ell}(\tau) \quad \big(= R_{\tilde{x}_\ell}(\tau)\big)
\end{aligned}
$$

and

$$\begin{aligned} S_{\tilde{x}_\ell}(f) &= \int_{-\infty}^{\infty} R_{\tilde{x}_\ell}(\tau) e^{-\imath 2\pi f\tau} d\tau \\ &= \int_{-\infty}^{\infty} e^{-\imath 2\pi (f_c'-f_c)\tau} R_{x_\ell}(\tau) e^{-\imath 2\pi f\tau} d\tau \\ &= \int_{-\infty}^{\infty} R_{x_\ell}(\tau) e^{-\imath 2\pi (f+(f_c'-f_c))\tau} d\tau \\ &= S_{x_\ell}(f+(f_c'-f_c)). \end{aligned}$$

As a result,

$$S_{\tilde{y}_\ell}(f) = S_{x_\ell}(f+(f_c'-f_c)) \cdot \left| u_{-1}(f+f_c')\, H(f+f_c') \right|^2.$$

4. (a) (6 pt.) Prove that if $\{\phi_k(t)\}_{k=1}^K$ is a complete orthonormal basis for signal $s(t)$, then the energy of $s(t)$, i.e., $\|s(t)\|^2$, is equal to $\sum_{k=1}^K |a_k|^2$, provided that

$$a_k = \langle s(t), \phi_k(t) \rangle = \int_0^T s(t)\phi_k^*(t)\, dt.$$

Hint: $\|s(t)\|^2 = \left\langle \sum_{j=1}^K a_j\phi_j(t), \sum_{k=1}^K a_k\phi_k(t) \right\rangle$.

(b) (6 pt.) When vectorizing the ASK modulation signals given by

$$s_m(t) = \mathbf{Re}\left\{ A_m g(t) e^{\imath 2\pi f_c t} \right\} = A_m g(t) \cos(2\pi f_c t)\,,\ t \in [0, T),$$

where $g(t)$ is a real-valued pulse shaping function, we use the basis

$$\phi_1(t) = \frac{g(t)}{\|g(t)\|} \sqrt{2} \cos(2\pi f_c t)$$

and obtain a one-dimensional vector $\boldsymbol{s}_m = \left[\frac{A_m}{\sqrt{2}} \cdot \|g(t)\| \right]$. Explain why $\|s_m(t)\|^2$ is not necessarily equal to $\frac{A_m}{2}\|g(t)\|^2$ as what has been obtained in (a)? Give an example that yields $\|s_m(t)\|^2 < \frac{A_m}{2}\|g(t)\|^2$ and also construct an example that results in $\|s_m(t)\|^2 > \frac{A_m}{2}\|g(t)\|^2$.

Hint: For convenience, you may wish to let $g(t) = u_{-1}(t) - u_{-1}(t-1)$ with $T = 1$ in your two examples.

Solutions.

(a)

$$\begin{aligned} \|s(t)\|^2 &= \left\langle \sum_{j=1}^K a_j\phi_j(t), \sum_{k=1}^K a_k\phi_k(t) \right\rangle \\ &= \sum_{j=1}^K \sum_{k=1}^K a_j a_k^* \langle \phi_j(t), \phi_k(t) \rangle \\ &= \sum_{j=1}^K a_j a_j^* \\ &= \sum_{j=1}^K |a_j|^2 \end{aligned}$$

(b) $\|s_m(t)\|^2$ is not necessarily equal to $\frac{A_m}{2}\|g(t)\|^2$ because $\phi_1(t)$ is not necessarily of unit power as required by an orthonormal basis. From the derivation in (a), it is clear that

$$\|s(t)\|^2 = |a_1|^2\|\phi_1(t)\|^2 = \frac{A_m}{2}\|g(t)\|^2\|\phi_1(t)\|^2.$$

Thus,

$$\|s(t)\|^2 < \frac{A_m}{2}\|g(t)\|^2 \text{ if, and only if, } \|\phi_1(t)\|^2 < 1.$$

By

$$\begin{aligned}
\|\phi_1(t)\|^2 &= \frac{2}{\|g(t)\|^2}\int_0^T g^2(t)\cos^2(2\pi f_c t)\,dt \\
&= \frac{2}{\|g(t)\|^2}\int_0^T g^2(t)\left(\frac{1+\cos(4\pi f_c t)}{2}\right)dt \\
&= \frac{1}{\|g(t)\|^2}\int_0^T g^2(t)\,dt + \frac{1}{\|g(t)\|^2}\int_0^T g^2(t)\cos(4\pi f_c t)\,dt \\
&= 1 + \frac{1}{\|g(t)\|^2}\int_0^T g^2(t)\cos(4\pi f_c t)\,dt \\
&= 1 + \int_0^1 \cos(4\pi f_c t)\,dt = 1 + \frac{\sin(2\pi f_c)}{2\pi f_c},
\end{aligned}$$

where the last step follows from $T = 1$ and $g(t) = u_{-1}(t) - u_{-1}(t-1)$, an example for $\|\phi_1\|^2 = 1 - \frac{2}{3\pi} < 1$ is to take $f_c = \frac{3}{4}$, and an example for $\|\phi_1\|^2 = 1 + \frac{2}{\pi} > 1$ is to set $f_c = \frac{1}{4}$. As f_c is sufficiently large, $\frac{|\sin(2\pi f_c)|}{2\pi f_c} \leq \frac{1}{2\pi f_c} \to 0$; accordingly, $\|\phi_1(t)\|^2 \to 1$.

5. A baseband 16-QAM signal can be represented as

$$s_\ell(t) = \sum_{n=-\infty}^{\infty}\left[A_{\mathrm{i},n}\,g(t-nT_s) + \imath A_{\mathrm{q},n}\,g(t-nT_s)\right],$$

where $\{A_{\mathrm{i},n}\}_{n=-\infty}^{\infty}$ and $\{A_{\mathrm{q},n}\}_{n=-\infty}^{\infty}$ are both i.i.d. with uniform marginal distribution and are independent of each other. Each of $\{A_{\mathrm{i},n}\}_{n=-\infty}^{\infty}$ and $\{A_{\mathrm{q},n}\}_{n=-\infty}^{\infty}$ takes values from $\{\pm 1, \pm 3\}$.

(a) (8 pt.) Determine the time-averaged power spectrum density of $s_\ell(t)$.
Hint: We can rewrite

$$s_\ell(t) = \sum_{n=-\infty}^{\infty} \boldsymbol{a}_n\, g(t-nT_s)$$

for a properly defined complex-valued $\boldsymbol{a}_n$.

(b) (8 pt.) Re-do (a) for the offset 16QAM signal below:

$$s_\ell(t) = \sum_{n=-\infty}^{\infty}\left[A_{\mathrm{i},n}\,g(t-2nT) + \imath A_{\mathrm{q},n}\,g(t-(2n+1)T)\right],$$

where $T = T_s/2$.
Hint: Again, $s_\ell(t) = \sum_{m=-\infty}^{\infty} \boldsymbol{a}_m\, g(t-mT)$ for a properly defined $\boldsymbol{a}_m$.

(c) (4 pt.) Which one has a better bandwidth efficiency, the "non-offset" 16QAM in (a) or the offset 16QAM in (b)? Justify your answer.

(d) (8 pt.) With $g(t) = \sin\left(\pi \frac{t}{2T}\right)[u_{-1}(t) - u_{-1}(t-2T)]$ in (b), prove that

$$\begin{aligned} s_\ell(t) &= \sum_{n=-\infty}^{\infty} [A_{\mathrm{i},n}\, g(t-2nT) + \imath A_{\mathrm{q},n}\, g(t-(2n+1)T)] \\ &= \frac{\imath}{2}\left([A_{\mathrm{q},-1} - A_{\mathrm{i},0}]\, e^{\imath 2\pi \frac{t}{4T}} + [A_{\mathrm{q},-1} + A_{\mathrm{i},0}]\, e^{-\imath 2\pi \frac{t}{4T}}\right), \quad t \in [0, T). \end{aligned}$$

In addition to the given $g(t)$, we force $A_{\mathrm{q},n} = \pm A_{\mathrm{i},n}$ with $A_{\mathrm{i},n}, A_{\mathrm{q},n} \in \{\pm 3, \pm 1\}$ (i.e., $(A_{\mathrm{i},n}, A_{\mathrm{q},n}) \in \{(-3,-3),(-3,3),(-1,-1),(1-,1),(1,-1),(1,1),(3,-3),(3,3)\}$). Does this change the ASK/PSK (QAM) signal to an ASK/FSK signal? Justify your answer.

(e) (4 pt.) What is the modulation index of the ASK/FSK signal in (d)?

Solutions.

(a) Re-writing

$$s_\ell(t) = \sum_{n=-\infty}^{\infty} (A_{\mathrm{i},n} + \imath A_{\mathrm{q},n})\, g(t - nT_s)$$

gives exactly the same form as the one on Slide 3-111 with $\boldsymbol{a}_n = A_{\mathrm{i},n} + \imath A_{\mathrm{q},n}$. We thus have

$$\begin{aligned} R_{s_\ell}(t_1, t_2) &= \mathbb{E}[s_\ell(t_1) s_\ell^*(t_2)] \\ &= \mathbb{E}\left\{\left(\sum_{n=-\infty}^{\infty} \boldsymbol{a}_n g(t-nT_s)\right)\left(\sum_{m=-\infty}^{\infty} \boldsymbol{a}_m^* g^*(t-mT_s)\right)\right\} \\ &= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \mathbb{E}\{\boldsymbol{a}_n \boldsymbol{a}_m^*\}\, g(t-nT_s) g^*(t-mT_s) \end{aligned}$$

Continuing the same procedure as Slide 3-115 $\sim$ 3-117, we obtain:

$$\overline{S}_{s_\ell}(f) = \frac{1}{T_s} S_{\boldsymbol{a}}(f) |G(f)|^2,$$

where

$$\begin{aligned} S_{\boldsymbol{a}}(f) &= \sum_{k=-\infty}^{\infty} R_{\boldsymbol{a}}(k) e^{-\imath 2\pi k f T_s} = R_{\boldsymbol{a}}(0) = \mathbb{E}[|\boldsymbol{a}_n|^2] = \mathbb{E}\left[|A_{\mathrm{i},n} + \imath A_{\mathrm{q},n}|^2\right] \\ &= \mathbb{E}[A_{\mathrm{i},n}^2] + \mathbb{E}[A_{\mathrm{q},n}^2] = 10. \end{aligned}$$

As a result, $\overline{S}_{s_\ell}(f) = \frac{10}{T_s}|G(f)|^2$.

(b) Re-writing the offset 16QAM as

$$s_\ell(t) = \sum_{m=-\infty}^{\infty} \boldsymbol{a}_m\, g(t - mT)$$

with

$$\boldsymbol{a}_m = \begin{cases} A_{\mathrm{i},n}, & m = 2n; \\ \imath A_{\mathrm{q},n}, & m = 2n+1. \end{cases} = \begin{cases} A_{\mathrm{i},m/2}, & m = 2n; \\ \imath A_{\mathrm{q},(m-1)/2}, & m = 2n+1. \end{cases}$$

Continuing the same procedure as Slide 3-115 ∼ 3-117, we obtain:

$$\overline{S}_{s_\ell}(f) = \frac{1}{T} S_{\boldsymbol{a}}(f)|G(f)|^2,$$

where

$$\begin{aligned} S_{\boldsymbol{a}}(f) &= \sum_{k=-\infty}^{\infty} R_{\boldsymbol{a}}(k) e^{-\imath 2\pi k f T_s} = R_{\boldsymbol{a}}(0) = \mathbb{E}[|\boldsymbol{a}_m|^2] \\ &= \begin{cases} \mathbb{E}[A_{\mathrm{i},n}^2], & m = 2n; \\ \mathbb{E}[A_{\mathrm{q},n}^2], & m = 2n+1 \end{cases} \\ &= 5. \end{aligned}$$

As a result, $\overline{S}_{s_\ell}(f) = \frac{5}{T}|G(f)|^2$.

(c) Since the (non-offset) 16QAM and the offset 16QAM have the same PSD under uniform i.i.d. $\{A_{\mathrm{i},n}\}_{n=-\infty}^{\infty}$ and $\{A_{\mathrm{q},n}\}_{n=-\infty}^{\infty}$ and $T_s = 2T$, they have the same (ideal) bandwidth efficiency.

Note that in practice, since the (non-offset) 16QAM may have an abrupt transition from $(-3,-3)$ to $(3,3)$, it requires a highly linear amplifier to prevent from spectral growth (from its idea spectrum), while the offset 16QAM has less demand on linearity of amplifiers.

(d)

$$\begin{aligned} s_\ell(t) &= \sum_{n=-\infty}^{\infty} \left[A_{\mathrm{i},n}\, g(t-2nT) + \imath A_{\mathrm{q},n}\, g(t-(2n+1)T)\right] \\ &= A_{\mathrm{i},0}\, g(t) + \imath A_{\mathrm{q},-1}\, g(t+T), \quad t \in [0,T) \\ &= A_{\mathrm{i},0} \sin\left(\pi \frac{t}{2T}\right) + \imath A_{\mathrm{q},-1} \sin\left(\pi \frac{(t+T)}{2T}\right), \quad t \in [0,T) \\ &= A_{\mathrm{i},0} \sin\left(\pi \frac{t}{2T}\right) + \imath A_{\mathrm{q},-1} \cos\left(\pi \frac{t}{2T}\right), \quad t \in [0,T) \\ &= A_{\mathrm{i},0} \frac{1}{2\imath}\left(e^{\imath\pi\frac{t}{2T}} - e^{-\imath\pi\frac{t}{2T}}\right) + \imath A_{\mathrm{q},-1} \frac{1}{2}\left(e^{\imath\pi\frac{t}{2T}} + e^{-\imath\pi\frac{t}{2T}}\right), \quad t \in [0,T) \\ &= \frac{\imath}{2}\left(\left[A_{\mathrm{q},-1} - A_{\mathrm{i},0}\right] e^{\imath 2\pi\frac{t}{4T}} + \left[A_{\mathrm{q},-1} + A_{\mathrm{i},0}\right] e^{-\imath 2\pi\frac{t}{4T}}\right), \quad t \in [0,T). \end{aligned}$$

Hence, from the table below,

$A_{\mathrm{i},0}$	$A_{\mathrm{q},-1}$	$A_{\mathrm{q},-1} - A_{\mathrm{i},0}$	$A_{\mathrm{q},-1} + A_{\mathrm{i},0}$	$-(2\imath)s_\ell(t)$
-3	-3	0	-6	$-6e^{-\imath 2\pi\frac{t}{4T}}$
-3	$+3$	6	0	$6e^{\imath 2\pi\frac{t}{4T}}$
-1	-1	0	-2	$-2e^{-\imath 2\pi\frac{t}{4T}}$
-1	$+1$	2	0	$2e^{\imath 2\pi\frac{t}{4T}}$
$+1$	-1	-2	0	$-2e^{\imath 2\pi\frac{t}{4T}}$
$+1$	$+1$	0	2	$2e^{-\imath 2\pi\frac{t}{4T}}$
$+3$	-3	-6	0	$-6e^{\imath 2\pi\frac{t}{4T}}$
$+3$	$+3$	0	6	$6e^{-\imath 2\pi\frac{t}{4T}}$

we observe that it becomes a combination of ASK and FSK, where two frequencies $f_1 = -\frac{1}{4T}$ and $f_2 = \frac{1}{4T}$ are used for transmission, each of which has four amplitudes $\{-6,-2,2,6\}$. In combination, 3 bits are carried by one continuous-phase ASK/FSK symbol.

(e) $f_d = \frac{1}{4T} \Rightarrow h = 2 f_d T = \frac{1}{2}$.