2016 Spring: The First Midterm of Digital Communications

The total points of this exam is 112.

1. The passband signal $x(t) = x_i(t)\cos(2\pi f_0 t) - x_q(t)\sin(2\pi f_0 t)$ introduced in our lectures can be regarded as a linear combination in the form of

$$x(t) = x_i(t) \cdot \psi_i(t) + x_q(t) \cdot \psi_q(t),$$

where $\psi_i(t) = \cos(2\pi f_0 t)$ and $\psi_q(t) = -\sin(2\pi f_0 t)$. By this view, we can relate $x(t)$ and $x_\ell(t) = x_i(t) + \imath\, x_q(t)$ without using the Hilbert transform (or analytic signal) by following the procedure below.

   (a) (6 pt.) Use the relation of

$$\mathcal{F}\{\psi_i(t)\} = \frac{1}{2}[\delta(f+f_0) + \delta(f-f_0)] \text{ and } \mathcal{F}\{\psi_q(t)\} = \frac{1}{2\imath}[\delta(f+f_0) - \delta(f-f_0)],$$

   where $\mathcal{F}\{\cdot\}$ is the Fourier transform, to prove that

$$X(f) = \mathcal{F}\{x(t)\} = \frac{1}{2}\left[X_i(f-f_0) + \imath\, X_q(f-f_0)\right] + \frac{1}{2}\left[X_i(f+f_0) - \imath\, X_q(f+f_0)\right].$$

   (b) (6 pt.) By noting that $x_i(t)$ and $x_q(t)$ are both real-valued signals that should satisfy $X_i(-f) = X_i^*(f)$ and $X_q(-f) = X_q^*(f)$, respectively, prove from (a) that

$$X(f) = \frac{1}{2}\left[X_\ell(f-f_0) + X_\ell^*(-f-f_0)\right].$$

   (c) (6 pt.) Use (b) to prove that $x(t) = \mathbf{Re}\left\{x_\ell(t)e^{\imath 2\pi f_0 t}\right\}$.

2. Let $\boldsymbol{U}$ be a random variable uniformly distributed over $[-\pi/2, \pi/2)$. Define a continuous random process as:

$$\boldsymbol{X}(t) = \cos(2\pi f_c t + \boldsymbol{U}).$$

   (a) (8 pt.) Determine the mean function and autocorrelation function of $\boldsymbol{X}(t)$.

   (b) (4 pt.) Is $\boldsymbol{X}(t)$ a wide-sense stationary process? Justify your answer.

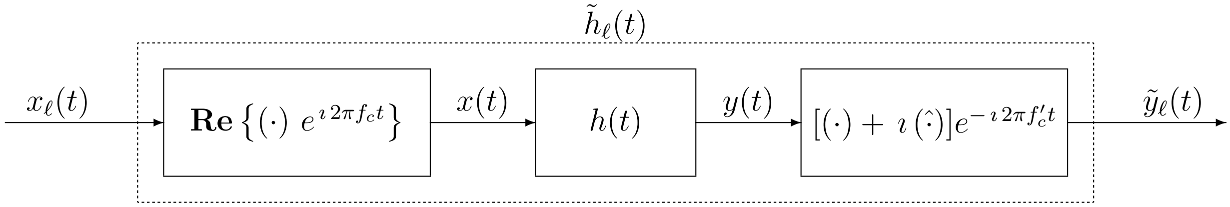
   (c) (4 pt.) Is $\boldsymbol{X}(t)$ cyclostationary? Justify your answer.

   (d) (4 pt.) Is $\boldsymbol{X}(t)$ a band-limited bandpass stochastic signal? Justify your answer.
   Hint: $\mathcal{F}\{\cos(2\pi f_c t)\} = \frac{1}{2}[\delta(f+f_c) + \delta(f-f_c)]$.

   (e) (6 pt.) Determine the lowpass equivalent signal $\boldsymbol{X}_\ell(t)$ of the passband signal $\boldsymbol{X}(t)$ with respect to carrier frequent f_c.
   Hint: $x(t) = \mathbf{Re}\left\{x_\ell(t)e^{\imath 2\pi f_c t}\right\}$.

   (f) (6 pt.) Are $\boldsymbol{X}_i(t+\tau)$ and $\boldsymbol{X}_q(t)$ uncorrelated for any τ? Justify your answer.

3. As shown in the figure below, suppose that in a communication system, the receiver unawarely assumes a different carrier frequency f_c' from the carrier frequency f_c used by the transmitter, where $x(t) = \mathbf{Re}\left\{x_\ell(t)\, e^{\imath 2\pi f_c t}\right\}$, $y(t) = x(t) \star h(t)$, $\tilde{y}_\ell(t) = [y(t) + \imath\, \hat{y}(t)]e^{-\imath 2\pi f_c' t}$, and $(\hat{\cdot})$ denotes the Hilbert transform output.



(a) (6 pt.) Prove that

$$\tilde{Y}_\ell(f) = u_{-1}(f + f_c')\, H(f + f_c') \cdot X_\ell(f + (f_c' - f_c)).$$

Hint: The transfer function of the Hilbert transformer is equal to $\hat{H}(f) = -\imath\,\mathrm{sgn}(f)$, and $x_\ell(t) = [x(t) + \imath\,\hat{x}(t)]e^{-\imath 2\pi f_c t}$. It is suggested to use spectrum view when doing this problem.

(b) (6 pt.) Can we relate $\tilde{y}_\ell(t)$ with $x_\ell(t)$ via a linear operation such as convolution? If your answer is yes, prove it. If your answer is negative, justify your answer by giving a counterexample.

Hint: If your answer is yes, then you should show $\tilde{Y}_\ell(f) = \tilde{H}_\ell(f)\, X_\ell(f)$ for some $\tilde{H}_\ell(f)$ and for every $f \in \mathcal{R}$.

(c) (6 pt.) Determine the power spectrum density (PSD) $S_{\tilde{y}_\ell}(f)$ of the output $\tilde{y}_\ell(t)$ as a function of the transfer function $H(f)$ and the PSD of the wide-sense stationary $x_\ell(t)$.

Hint: If $\tilde{Y}_\ell(f) = \tilde{H}_\ell(f)\tilde{X}_\ell(f)$, where $\tilde{H}_\ell(f) = u_{-1}(f + f_c')\, H(f + f_c')$ and $\tilde{X}_\ell(f) = X_\ell(f + (f_c' - f_c))$, then $S_{\tilde{y}_\ell}(f) = |\tilde{H}_\ell(f)|^2 S_{\tilde{x}_\ell}(f)$.

4. (a) (6 pt.) Prove that if $\{\phi_k(t)\}_{k=1}^K$ is a complete orthonormal basis for signal $s(t)$, then the energy of $s(t)$, i.e., $\|s(t)\|^2$, is equal to $\sum_{k=1}^K |a_k|^2$, provided that

$$a_k = \langle s(t), \phi_k(t) \rangle = \int_0^T s(t)\phi_k^*(t)\, dt.$$

Hint: $\|s(t)\|^2 = \left\langle \sum_{j=1}^K a_j\phi_j(t), \sum_{k=1}^K a_k\phi_k(t) \right\rangle$.

(b) (6 pt.) When vectorizing the ASK modulation signals given by

$$s_m(t) = \mathbf{Re}\left\{A_m g(t) e^{\imath 2\pi f_c t}\right\} = A_m g(t)\cos(2\pi f_c t),\ t \in [0, T),$$

where $g(t)$ is a real-valued pulse shaping function, we use the basis

$$\phi_1(t) = \frac{g(t)}{\|g(t)\|}\sqrt{2}\cos(2\pi f_c t)$$

and obtain a one-dimensional vector $\boldsymbol{s}_m = \left[\frac{A_m}{\sqrt{2}} \cdot \|g(t)\|\right]$. Explain why $\|s_m(t)\|^2$ is not necessarily equal to $\frac{A_m}{2}\|g(t)\|^2$ as what has been obtained in (a)? Give an example that yields $\|s_m(t)\|^2 < \frac{A_m}{2}\|g(t)\|^2$ and also construct an example that results in $\|s_m(t)\|^2 > \frac{A_m}{2}\|g(t)\|^2$.

Hint: For convenience, you may wish to let $g(t) = u_{-1}(t) - u_{-1}(t-1)$ with $T = 1$ in your two examples.

5. A baseband 16-QAM signal can be represented as

$$s_\ell(t) = \sum_{n=-\infty}^{\infty} \left[A_{\mathrm{i},n}\, g(t-nT_s) + \imath A_{\mathrm{q},n}\, g(t-nT_s)\right],$$

where $\{A_{\mathrm{i},n}\}_{n=-\infty}^{\infty}$ and $\{A_{\mathrm{q},n}\}_{n=-\infty}^{\infty}$ are both i.i.d. with uniform marginal distribution and are independent of each other. Each of $\{A_{\mathrm{i},n}\}_{n=-\infty}^{\infty}$ and $\{A_{\mathrm{q},n}\}_{n=-\infty}^{\infty}$ takes values from $\{\pm 1, \pm 3\}$.

(a) (8 pt.) Determine the time-averaged power spectrum density of $s_\ell(t)$.
Hint: We can rewrite

$$s_\ell(t) = \sum_{n=-\infty}^{\infty} \boldsymbol{a}_n\, g(t-nT_s)$$

for a properly defined complex-valued $\boldsymbol{a}_n$.

(b) (8 pt.) Re-do (a) for the offset 16QAM signal below:

$$s_\ell(t) = \sum_{n=-\infty}^{\infty} \left[A_{\mathrm{i},n}\, g(t-2nT) + \imath A_{\mathrm{q},n}\, g(t-(2n+1)T)\right],$$

where $T = T_s/2$.
Hint: Again, $s_\ell(t) = \sum_{m=-\infty}^{\infty} \boldsymbol{a}_m\, g(t-mT)$ for a properly defined $\boldsymbol{a}_m$.

(c) (4 pt.) Which one has a better bandwidth efficiency, the "non-offset" 16QAM in (a) or the offset 16QAM in (b)? Justify your answer.

(d) (8 pt.) With $g(t) = \sin\left(\pi \frac{t}{2T}\right)[u_{-1}(t) - u_{-1}(t-2T)]$ in (b), prove that

$$\begin{aligned} s_\ell(t) &= \sum_{n=-\infty}^{\infty} \left[A_{\mathrm{i},n}\, g(t-2nT) + \imath A_{\mathrm{q},n}\, g(t-(2n+1)T)\right] \\ &= \frac{\imath}{2}\left(\left[A_{\mathrm{q},-1} - A_{\mathrm{i},0}\right] e^{\imath 2\pi \frac{t}{4T}} + \left[A_{\mathrm{q},-1} + A_{\mathrm{i},0}\right] e^{-\imath 2\pi \frac{t}{4T}}\right), \quad t \in [0, T). \end{aligned}$$

In addition to the given $g(t)$, we force $A_{\mathrm{q},n} = \pm A_{\mathrm{i},n}$ with $A_{\mathrm{i},n}, A_{\mathrm{q},n} \in \{\pm 3, \pm 1\}$ (i.e., $(A_{\mathrm{i},n}, A_{\mathrm{q},n}) \in \{(-3,-3), (-3,3), (-1,-1), (1-,1), (1,-1), (1,1), (3,-3), (3,3)\}$). Does this change the ASK/PSK (QAM) signal to an ASK/FSK signal? Justify your answer.

(e) (4 pt.) What is the modulation index of the ASK/FSK signal in (d)?