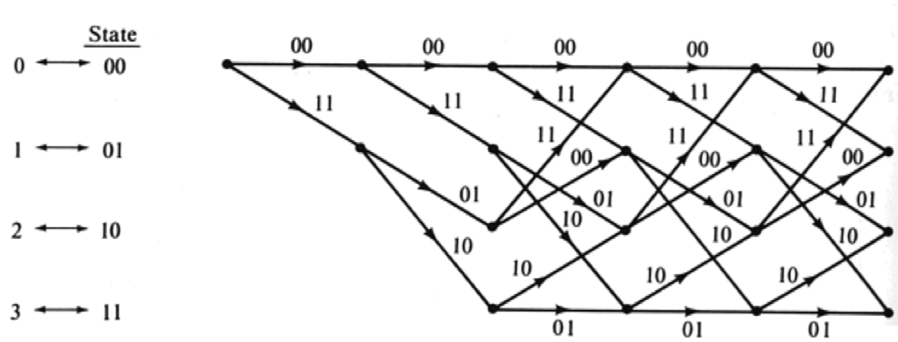
2016 Spring: The First Midterm of Digital Communications

The total points of this exam is 108.

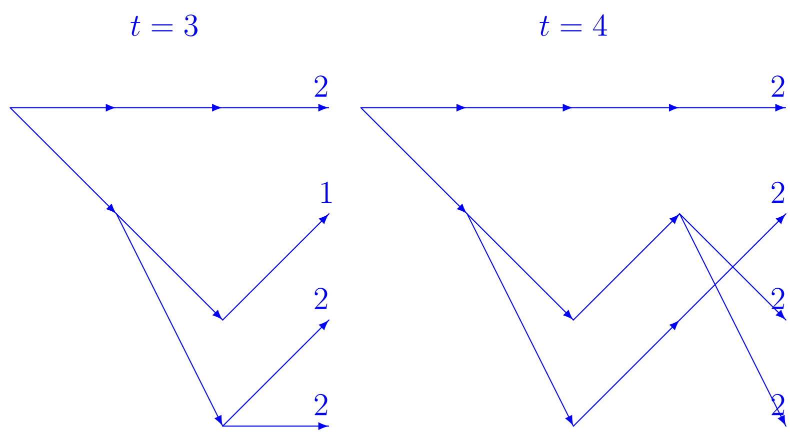
1. (12 pt)



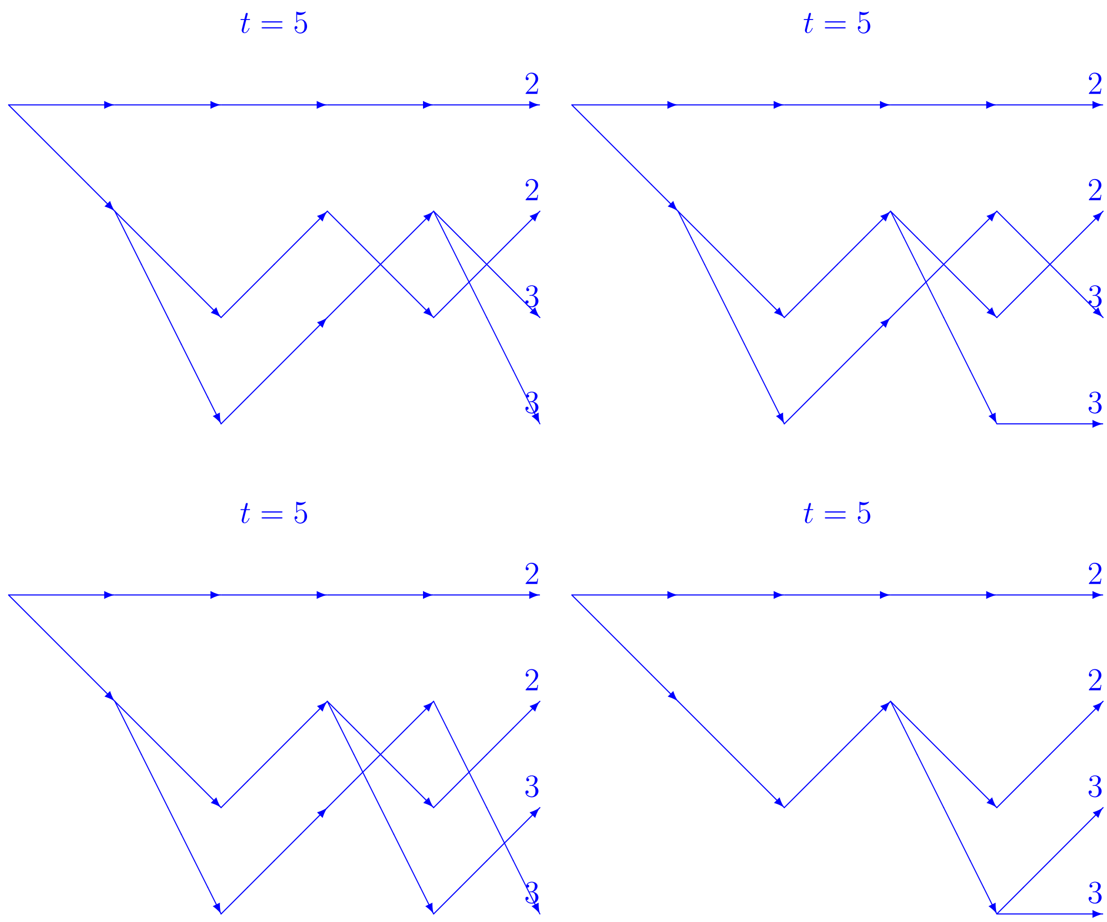
Based on the trellis above, draw the survivor paths at time instance 3, 4 and 5 if the received vector is 11 00 00 00 00 . . .. Please mark the accumulated Hamming metric of each survivor path.

Hint: See for example the bottom picture of Slide 4-207.

Solutions.



There are four possible choices of survivor paths at $t = 5$. Just pick one of them for your answer.



2. (a) (8 pt) The channel model for non-coherent transmission is given by

$$\boldsymbol{r}_\ell = e^{\imath\phi}\boldsymbol{s}_{m,\ell} + \boldsymbol{n}_\ell,$$

where ϕ is a random variable with probability density function $f_{\boldsymbol{\phi}}(\phi)$, $\boldsymbol{s}_{m,\ell}$ is the mth channel symbol for transmission, and $\boldsymbol{n}_\ell$ is a zero-mean complex Gaussian random vector with i.i.d. components of variance $2N_0$. Prove that the MAP decision rule for a semiblind receiver is equal to

$$\hat{m} = \arg\max_{1\le m\le M} \Pr\{\boldsymbol{s}_{m,\ell}|\boldsymbol{r}_\ell\} = \arg\max_{1\le m\le M} P_m e^{-\frac{\mathcal{E}_m}{N_0}} \int_0^{2\pi} e^{\frac{\left|\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell}\right|}{N_0}\cos(\alpha_m+\phi)} f_{\boldsymbol{\phi}}(\phi)\, d\phi$$

where P_m is the prior probability for symbol $\boldsymbol{s}_{m,\ell}$, $\mathcal{E}_m = \frac{1}{2}\|\boldsymbol{s}_{m,\ell}\|^2$, and $\alpha_m = \angle(\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell})$.

(b) (8 pt) For equal-energy and equiprobable signaling, can the decision rule in (a) be simplified to

$$\hat{m} = \arg\max_{1\le m\le M} \left|\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell}\right|$$

provided that $\Pr[\phi = \frac{\pi}{2}] = \Pr[\phi = -\frac{\pi}{2}] = \frac{1}{2}$. If your answer is positive, prove it; otherwise, determine the simplified decision rule corresponding to equal-energy and equiprobable signaling.

Hint: By "simplified decision rule," we mean one that requires no knowledge of N_0.

(c) (8 pt) For differential PSK modulation under non-coherent detection, the two consecutive symbols can be expressed as

$$\boldsymbol{s}_\ell^{(k-1)} = \sqrt{2\mathcal{E}_s} e^{\imath \phi_0} \quad \text{and} \quad \boldsymbol{s}_{m,\ell}^{(k)} = \sqrt{2\mathcal{E}_s} e^{\imath(\theta_m + \phi_0)}, \quad m = 1, 2, \dots, M$$

and the channel is modeled as

$$\vec{\boldsymbol{r}}_\ell = \begin{bmatrix} \boldsymbol{r}_\ell^{(k-1)} \\ \boldsymbol{r}_\ell^{(k)} \end{bmatrix} = e^{\imath \phi} \begin{bmatrix} \boldsymbol{s}_\ell^{(k-1)} \\ \boldsymbol{s}_{m,\ell}^{(k)} \end{bmatrix} + \begin{bmatrix} \boldsymbol{n}_\ell^{(k-1)} \\ \boldsymbol{n}_\ell^{(k)} \end{bmatrix} = e^{\imath \phi} \vec{\boldsymbol{s}}_{m,\ell} + \vec{\boldsymbol{n}}_\ell$$

where $\vec{\boldsymbol{n}}_\ell$ is a zero-mean Gaussian vector with i.i.d. components of variance $2N_0$. Then we immediately know from (a) that

$$\hat{m} = \arg \max_{1 \le m \le M} \Pr\{\vec{\boldsymbol{s}}_{m,\ell} | \vec{\boldsymbol{r}}_\ell\} = \arg \max_{1 \le m \le M} P_m \int_0^{2\pi} e^{\frac{\left|\vec{\boldsymbol{r}}_\ell^\dagger \vec{\boldsymbol{s}}_{m,\ell}\right|}{N_0} \cos(\alpha_m + \phi)} f_{\boldsymbol{\phi}}(\phi)\, d\phi,$$

where P_m is the prior probability for symbol $\boldsymbol{s}_{m,\ell}^{(k)}$, and $\alpha_m = \angle(\vec{\boldsymbol{r}}_\ell^\dagger \vec{\boldsymbol{s}}_{m,\ell})$. Now under equal prior probability, is the decision rule above equivalent to

$$\arg \max_{1 \le m \le M} \mathbf{Re}\left\{ \left(\boldsymbol{r}_\ell^{(k-1)}\right)^* \boldsymbol{r}_\ell^{(k)} e^{-\imath \theta_m} \right\}.$$

Justify your answer.

Hint: Check the decision rule in (b).

Solutions.

(a)

$$\begin{aligned}
\hat{m} &= \arg \max_{1 \le m \le M} P_m \int_0^{2\pi} f_{\boldsymbol{n}_\ell}\left(\boldsymbol{r}_\ell - \boldsymbol{s}_{m,\ell} e^{\imath \phi}\right) f_{\boldsymbol{\phi}}(\phi)\, d\phi \\
&= \arg \max_{1 \le m \le M} P_m \frac{1}{(2\pi N_0)^N} \int_0^{2\pi} e^{-\frac{\left\|\boldsymbol{r}_\ell - \boldsymbol{s}_{m,\ell} e^{\imath \phi}\right\|^2}{2N_0}} f_{\boldsymbol{\phi}}(\phi)\, d\phi \\
&= \arg \max_{1 \le m \le M} P_m e^{-\frac{\mathcal{E}_m}{N_0}} \int_0^{2\pi} e^{\frac{\mathbf{Re}\left[\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell} \cdot e^{\imath \phi}\right]}{N_0}} f_{\boldsymbol{\phi}}(\phi) d\phi \\
&= \arg \max_{1 \le m \le M} P_m e^{-\frac{\mathcal{E}_m}{N_0}} \int_0^{2\pi} e^{\frac{\mathbf{Re}\left[\left|\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell}\right| e^{\imath \alpha_m} \cdot e^{\imath \phi}\right]}{N_0}} f_{\boldsymbol{\phi}}(\phi)\, d\phi \\
&= \arg \max_{1 \le m \le M} P_m e^{-\frac{\mathcal{E}_m}{N_0}} \int_0^{2\pi} e^{\frac{\left|\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell}\right|}{N_0} \cos(\alpha_m + \phi)} f_{\boldsymbol{\phi}}(\phi)\, d\phi.
\end{aligned}$$

(b) For equal-energy and equiprobable signals and $\Pr[\phi = \pi/2] = \Pr[\phi = -\pi/2] = \frac{1}{2}$, the decision rule in (a) can be simplified to

$$
\begin{aligned}
\hat{m} &= \arg\max_{1\le m\le M} \int_0^{2\pi} e^{\frac{\left|\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell}\right|}{N_0}\cos(\alpha_m+\phi)} f_{\boldsymbol{\phi}}(\phi)\,d\phi \\
&= \arg\max_{1\le m\le M} \left\{ \frac{1}{2} e^{\frac{\left|\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell}\right|}{N_0}\cos(\alpha_m+\pi/2)} + \frac{1}{2} e^{\frac{\left|\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell}\right|}{N_0}\cos(\alpha_m-\pi/2)} \right\} \\
&= \arg\max_{1\le m\le M} \left\{ e^{-\frac{\left|\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell}\right|}{N_0}\sin(\alpha_m)} + e^{\frac{\left|\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell}\right|}{N_0}\sin(\alpha_m)} \right\} \\
&= \arg\max_{1\le m\le M} \left\{ e^{-\frac{\mathbf{Im}\left[\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell}\right]}{N_0}} + e^{\frac{\mathbf{Im}\left[\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell}\right]}{N_0}} \right\} \\
&= \arg\max_{1\le m\le M} \cosh\left(\frac{1}{N_0} \mathbf{Im}\left[\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell}\right] \right).
\end{aligned}
$$

Since $\cosh(x)$ is monotonically increasing with respect to $|x|$,

$$
\hat{m} = \arg\max_{1\le m\le M} \frac{1}{N_0} \left| \mathbf{Im}\left[\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell}\right] \right| = \arg\max_{1\le m\le M} \left| \mathbf{Im}\left[\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell}\right] \right|,
$$

which is in general not equal to $\arg\max_{1\le m\le M} \left|\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell}\right|$. Hence, only for certain distributions of ϕ, we have $\hat{m} = \arg\max_{1\le m\le M} \left|\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell}\right|$.

(c) The answer is "Not necessary." For certain distributions of ϕ, we may have

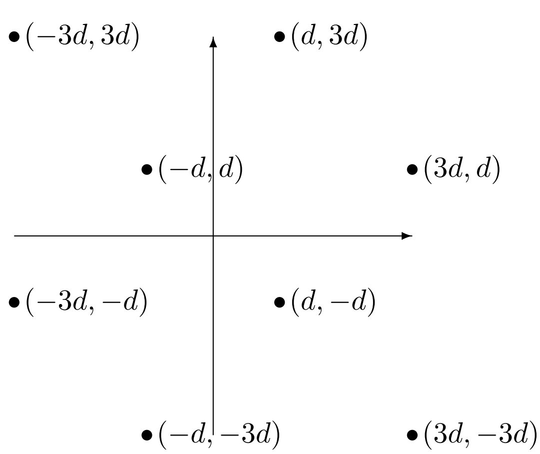
$$
\hat{m} = \arg\max_{1\le m\le M} \left| \mathbf{Im}\left[\vec{\boldsymbol{r}}_\ell^\dagger \vec{\boldsymbol{s}}_{m,\ell}\right] \right|
$$

instead of

$$
\hat{m} = \arg\max_{1\le m\le M} \left|\vec{\boldsymbol{r}}_\ell^\dagger \vec{\boldsymbol{s}}_{m,\ell}\right| = \arg\max_{1\le m\le M} \mathbf{Re}\left\{ \left(\boldsymbol{r}_\ell^{(k-1)}\right)^* \boldsymbol{r}_\ell^{(k)} e^{-\imath\theta_m} \right\}
$$

as demonstrated in (b).

3. Below an equal-probable 8-ary QAM constellation is depicted for transmission over the AWGN channel, i.e., $\boldsymbol{r} = \boldsymbol{s}_m + \boldsymbol{n}$ for $1 \le m \le 8$ and $\boldsymbol{n}$ is a two-dimensional Gaussian vector with i.i.d components of mean zero and variance $N_0/2$.



(a) (8 pt) Determine the average transmission energy per information bit $\mathcal{E}_b$.

(b) (8 pt) Draw the partitions for optimal detection at the receiver.

(c) (8 pt) Determine the union bound for this constellation, where the union bound formula is given by

$$P_e \le \frac{1}{M} \sum_{m=1}^{M} \sum_{\substack{1 \le m' \le M \\ m' \ne m}} Q\left(\sqrt{\frac{d_{m,m'}^2}{2N_0}}\right).$$

Please express the bound as a function of $\mathcal{E}_b/N_0$.

Hint: Complete the $d_{m,m'}$ table below.

$d_{m,m'}$	$2\sqrt{2}d$	$4d$	$4\sqrt{2}d$	$2\sqrt{10}d$	$6\sqrt{2}d$
$(-3d, 3d)$, $(3d, -3d)$					
$(d, 3d)$, $(3d, d)$, $(-3d, -d)$, $(-d, -3d)$					
$(-d, d)$,$(d, -d)$					
Sum					

(d) (8 pt) One can actually improve the union bound in (b) by considering only those constellation points m' whose corresponding partition regions have shared borderlines with m. Determine the improved union bound.

Solutions.

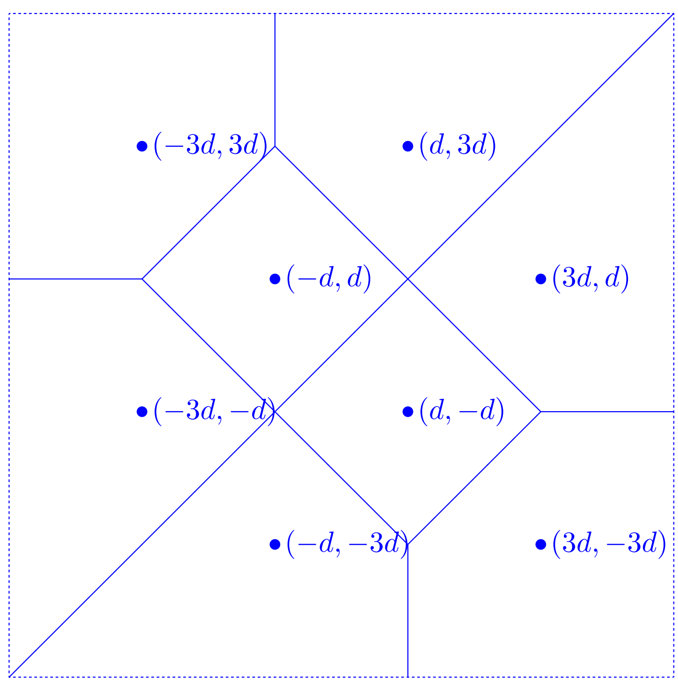
(a)

$$\mathcal{E}_s = E[|\boldsymbol{s}_m|^2] = \frac{2}{8}(9d^2 + 9d^2) + \frac{4}{8}(d^2 + 9d^2) + \frac{2}{8}(d^2 + d2) = \frac{9}{2}d^2 + 5d^2 + \frac{1}{2}d^2 = 10d^2$$

Hence

$$\mathcal{E}_b = \frac{\mathcal{E}_s}{3} = \frac{10}{3}d^2.$$

(b)



(c) The table of $d_{m,m'}$ for $m \neq m'$ can be given below.

$d_{m,m'}$	$2\sqrt{2}d$	$4d$	$4\sqrt{2}d$	$2\sqrt{10}d$	$6\sqrt{2}d$
$(-3d,3d)$, $(3d,-3d)$	1	2	1	2	1
$(d,3d)$, $(3d,d)$, $(-3d,-d)$, $(-d,-3d)$	2	2	1	2	0
$(-d,d)$,$(d,-d)$	4	2	1	0	0
sum	18	16	8	12	2

Hence

$$\begin{aligned}
P_e &\leq \frac{1}{M}\sum_{m=1}^{M}\sum_{\substack{1\leq m'\leq M\\ m'\neq m}} Q\left(\sqrt{\frac{d_{m,m'}^2}{2N_0}}\right)\\
&= \frac{1}{8}\left(18\,Q\left(\sqrt{\frac{4d^2}{N_0}}\right)+16\,Q\left(\sqrt{\frac{8d^2}{N_0}}\right)+8\,Q\left(\sqrt{\frac{16d^2}{N_0}}\right)\right.\\
&\quad \left.+12\,Q\left(\sqrt{\frac{20d^2}{N_0}}\right)+2\,Q\left(\sqrt{\frac{36d^2}{N_0}}\right)\right)\\
&= \frac{9}{4}Q\left(\sqrt{\frac{6\mathcal{E}_b}{5N_0}}\right)+2Q\left(\sqrt{\frac{12\mathcal{E}_b}{5N_0}}\right)+Q\left(\sqrt{\frac{24\mathcal{E}_b}{5N_0}}\right)\\
&\quad +\frac{3}{2}Q\left(\sqrt{\frac{6\mathcal{E}_b}{N_0}}\right)+\frac{1}{4}Q\left(\sqrt{\frac{54\mathcal{E}_b}{5N_0}}\right).
\end{aligned}$$

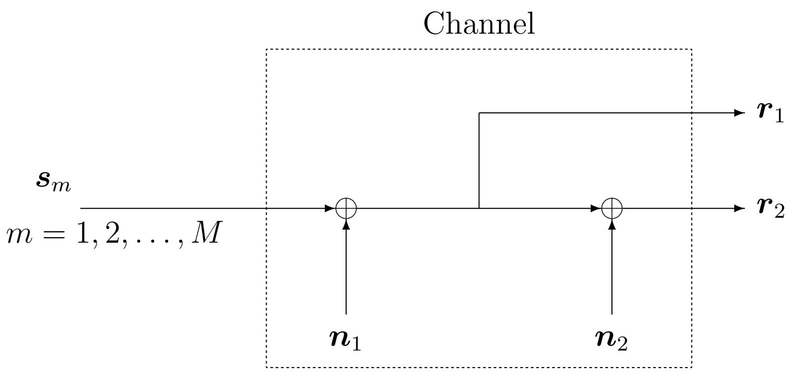
(d) The table of $d_{m,m'}$ for $m \neq m'$ can be refined below.

	$2\sqrt{2}d$	$4d$	$4\sqrt{2}d$	$2\sqrt{10}d$	$6\sqrt{2}d$
$(-3d,3d)$, $(3d,-3d)$	1	2	0	0	0
$(d,3d)$, $(3d,d)$, $(-3d,-d)$, $(-d,-3d)$	2	1	0	0	0
$(-d,d)$,$(d,-d)$	4	0	0	0	0
sum	18	8	0	0	0

Hence

$$\begin{aligned}
P_e &\leq \frac{1}{8}\left(18\,Q\left(\sqrt{\frac{4d^2}{N_0}}\right)+8\,Q\left(\sqrt{\frac{8d^2}{N_0}}\right)\right)\\
&\leq \frac{9}{4}\,Q\left(\sqrt{\frac{4d^2}{N_0}}\right)+Q\left(\sqrt{\frac{8d^2}{N_0}}\right)\\
&\leq \frac{9}{4}\,Q\left(\sqrt{\frac{6\mathcal{E}_b}{5N_0}}\right)+Q\left(\sqrt{\frac{12\mathcal{E}_b}{5N_0}}\right).
\end{aligned}$$

4. In the communication system below, the receiver receives two signals $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$.



(a) (8 pt) If the two additive noises $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are independent, is $\boldsymbol{r}_1$ a sufficient statistics for optimal (i.e., MAP) detection? If the answer is YES, prove it; otherwise, give a counterexample.

Hint: The MAP decision of this system is $\hat{m} = \arg\max_{1\le m\le M} \Pr\{\boldsymbol{s}_m | \boldsymbol{r}_1, \boldsymbol{r}_2\}$.

(b) (8 pt) If the two additive noises $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are dependent, is $\boldsymbol{r}_1$ a sufficient statistics for optimal detection? If the answer is YES, prove it; otherwise, give a counterexample.

Hint: Think of the extreme case such as that $\boldsymbol{n}_1$ is a function of $\boldsymbol{n}_2$.

Solutions.

(a) With

$$\begin{cases} \boldsymbol{r}_1 = \boldsymbol{s}_m + \boldsymbol{n}_1 \\ \boldsymbol{r}_2 = \boldsymbol{s}_m + \boldsymbol{n}_1 + \boldsymbol{n}_2 \end{cases} \quad \text{or equivalently} \quad \begin{cases} \boldsymbol{n}_1 = \boldsymbol{r}_1 - \boldsymbol{s}_m \\ \boldsymbol{n}_2 = \boldsymbol{r}_2 - \boldsymbol{r}_1 \end{cases}$$

the MAP decision of this system is

$$\begin{aligned} \hat{m} &= \arg\max_{1\le m\le M} \Pr\{\boldsymbol{s}_m | \boldsymbol{r}_1, \boldsymbol{r}_2\} \\ &= \arg\max_{1\le m\le M} P_m \, \mathtt{f}_{\boldsymbol{n}_1,\boldsymbol{n}_2}(\boldsymbol{r}_1 - \boldsymbol{s}_m, \boldsymbol{r}_2 - \boldsymbol{r}_1) \\ &= \arg\max_{1\le m\le M} P_m \, \mathtt{f}_{\boldsymbol{n}_1}(\boldsymbol{r}_1 - \boldsymbol{s}_m) \, \mathtt{f}_{\boldsymbol{n}_2}(\boldsymbol{r}_2 - \boldsymbol{r}_1) \\ &= \arg\max_{1\le m\le M} P_m \, \mathtt{f}_{\boldsymbol{n}_1}(\boldsymbol{r}_1 - \boldsymbol{s}_m), \end{aligned}$$

where the last equality holds because $\mathtt{f}_{\boldsymbol{n}_2}(\boldsymbol{r}_2 - \boldsymbol{r}_1)$ is irrelevant to m. Thus $\boldsymbol{r}_1$ is a sufficient statistic for optimal detection.

(b) The answer is NO. A quick counterexample is to have $\boldsymbol{n}_1 = -\boldsymbol{n}_2$, which gives $\boldsymbol{r}_2 = \boldsymbol{s}_m$ (while $\boldsymbol{r}_1 = \boldsymbol{s}_m + \boldsymbol{n}_1$). Another counterexample is to have $\boldsymbol{n}_1 = \boldsymbol{n}_2$, which gives $\boldsymbol{s}_m = 2\boldsymbol{r}_1 - \boldsymbol{r}_2$.

5. (a) (8 pt) Subject to a given channel transition probability density function $\mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_m) = \frac{1}{2} e^{-|\boldsymbol{r}-\boldsymbol{s}_m|}$ for 4-ary channel input $\boldsymbol{s}_m = -3, -1, 1, 3$ respectively for $m = 1, 2, 3, 4$, and channel output $\boldsymbol{r}$, derive the maximum-likelihood (ML) decision rule $g_{\text{ML}}(\boldsymbol{r})$.

(b) (8 pt) Continue from (a). Let the bit patterns $b_1 b_0$ corresponding to $m = 1, 2, 3, 4$ be respectively 00, 01, 11, 10. Derive the ML decision rule $g_{\text{ML},i}(\boldsymbol{r})$ for the ith bit, where $i = 0, 1$.

Hint: (i)

$$g_{\text{ML},i}(\boldsymbol{r}) \triangleq \arg\max_{b\in\{0,1\}} \sum_{\boldsymbol{s}_m : b_i = b} \mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_m)$$

and (ii) $\mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_1) + \mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_4) < \mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_2) + \mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_3)$ if, and only if,

$$|\boldsymbol{r}| < \lambda \triangleq \frac{1}{2}\log(e^4 + e^2 - 1) \approx 2.0553.$$

(c) (8 pt) The bit error probability $P_{\text{BER},i}$ of the ith bit b_i can be derived through either

symbol-based ML decision maker $g_{\text{ML}}(\boldsymbol{r})$ in (a)

or

bit-based ML decision maker $g_{\text{ML},i}(\boldsymbol{r})$ in (b).

The two bit error probabilities can be expressed as

$$P_{\text{BER},i}(g_{\text{ML}}) = \frac{1}{4}\sum_{b\in\{0,1\}}\sum_{\boldsymbol{s}_m : b_i = b}\int_{\{\boldsymbol{r} : g_{\text{ML}}(\boldsymbol{r}) \notin \mathcal{U}_i^{(b)}\}} \mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_m)\, d\boldsymbol{r}, \quad i = 0, 1$$

and

$$P_{\text{BER},i}(g_{\text{ML},i}) = \frac{1}{4}\sum_{b\in\{0,1\}}\sum_{\boldsymbol{s}_m : b_i = b}\int_{\{\boldsymbol{r} : g_{\text{ML},i}(\boldsymbol{r}) \neq b\}} \mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_m)\, d\boldsymbol{r}, \quad i = 0, 1$$

where $\mathcal{U}_i^{(b)}$ is the set of symbol indices, of which the ith bit is equal to b. Specifically, $\mathcal{U}_0^{(0)} = \{1,4\}$, $\mathcal{U}_0^{(1)} = \{2,3\}$, $\mathcal{U}_1^{(0)} = \{1,2\}$ and $\mathcal{U}_1^{(1)} = \{3,4\}$.

Show that the two bit error probabilities for bit b_0 are equal to

$$P_{\text{BER},0}(g_{\text{ML}}) = \frac{1}{4}e^{-3}(e^2 - e^{-2}) + \frac{1}{4}\left(e + e^{-1}\right)e^{-2}$$

and

$$P_{\text{BER},0}(g_{\text{ML},0}) = \frac{1}{4}e^{-3}(e^{\lambda} - e^{-\lambda}) + \frac{1}{4}\left(e + e^{-1}\right)e^{-\lambda},$$

respectively, where the constant λ is defined in the Hint of (b).

Solutions.

(a)

$$g_{\text{ML}}(\boldsymbol{r}) \triangleq \arg\max_{1\le m\le 4} \mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_m) = \arg\max_{1\le m\le 4} \frac{1}{2}e^{-|\boldsymbol{r}-\boldsymbol{s}_m|} = \begin{cases} 1, & \boldsymbol{r} < -2 \\ 2, & -2 \le \boldsymbol{r} < 0 \\ 3, & 0 \le \boldsymbol{r} < 2 \\ 4, & \boldsymbol{r} \ge 2 \end{cases}$$

(b)

$$\begin{aligned} g_{\text{ML},0}(\boldsymbol{r}) &\triangleq \arg\max_{b\in\{0,1\}} \sum_{\boldsymbol{s}_m : b_0 = b} \mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_m) \\ &= \arg\max_{b\in\{0,1\}} \Big\{ \underbrace{\mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_1) + \mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_4)}_{b=0}, \underbrace{\mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_2) + \mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_3)}_{b=1} \Big\} \\ &= \begin{cases} 0, & \boldsymbol{r} < -3 \text{ or } \boldsymbol{r} \ge c \\ 1, & -c \le \boldsymbol{r} < c \end{cases} \end{aligned}$$

and

$$\begin{aligned} g_{\mathrm{ML},1}(\boldsymbol{r}) &\triangleq \arg\max_{b\in\{0,1\}} \sum_{\boldsymbol{s}_m : b_1 = b} \mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_m) \\ &= \arg\max_{b\in\{0,1\}} \Big\{ \underbrace{\mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_1)+\mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_2)}_{b=0}, \underbrace{\mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_3)+\mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_4)}_{b=1} \Big\} \\ &= \begin{cases} 0, & \boldsymbol{r} < 0 \\ 1, & \boldsymbol{r} \geq 0 \end{cases} \end{aligned}$$

(c)

$$\begin{aligned} P_{\mathrm{BER},0}(g_{\mathrm{ML}}) &= \frac{1}{M} \sum_{b\in\{0,1\}} \sum_{\boldsymbol{s}_m : b_0 = b} \int_{\{\boldsymbol{r}\,:\, g_{\mathrm{ML}}(\boldsymbol{r}) \notin \mathcal{U}_0^{(b)}\}} \mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_m)\, d\boldsymbol{r} \\ &= \frac{1}{M} \sum_{m\in\mathcal{U}_0^{(0)}} \int_{\{\boldsymbol{r}\,:\, g_{\mathrm{ML}}(\boldsymbol{r}) \notin \mathcal{U}_0^{(0)}\}} \mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_m)\, d\boldsymbol{r} + \frac{1}{M} \sum_{m\in\mathcal{U}_0^{(1)}} \int_{\{\boldsymbol{r}\,:\, g_{\mathrm{ML}}(\boldsymbol{r}) \notin \mathcal{U}_0^{(1)}\}} \mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_m)\, d\boldsymbol{r} \\ &= \frac{1}{4} \int_{\{\boldsymbol{r}\,:\, g_{\mathrm{ML}}(\boldsymbol{r}) \notin \{1,4\}\}} [\mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_1)+\mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_4)]\, d\boldsymbol{r} + \frac{1}{4} \int_{\{\boldsymbol{r}\,:\, g_{\mathrm{ML}}(\boldsymbol{r}) \notin \{2,3\}\}} [\mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_2)+\mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_3)]\, d\boldsymbol{r} \\ &= \frac{1}{4} \int_{-2\leq \boldsymbol{r} < 2} \frac{1}{2} \left(e^{-|\boldsymbol{r}+3|} + e^{-|\boldsymbol{r}-3|}\right) d\boldsymbol{r} + \frac{1}{4} \int_{\boldsymbol{r}<-2 \text{ or } \boldsymbol{r}\geq 2} \frac{1}{2} \left(e^{-|\boldsymbol{r}+1|} + e^{-|\boldsymbol{r}-1|}\right) d\boldsymbol{r} \\ &= \frac{1}{4} \int_{-2}^{2} \frac{1}{2} e^{-3} \left(e^{-\boldsymbol{r}} + e^{\boldsymbol{r}}\right) d\boldsymbol{r} + \frac{1}{4} \int_{-\infty}^{-2} \frac{1}{2} \left(e + \frac{1}{e}\right) e^{\boldsymbol{r}}\, d\boldsymbol{r} + \frac{1}{4} \int_{2}^{\infty} \frac{1}{2} \left(\frac{1}{e} + e\right) e^{-\boldsymbol{r}}\, d\boldsymbol{r} \\ &= \frac{1}{4} \int_{-2}^{2} e^{-3} e^{\boldsymbol{r}}\, d\boldsymbol{r} + \frac{1}{4} \int_{-\infty}^{-2} \left(e + \frac{1}{e}\right) e^{\boldsymbol{r}}\, d\boldsymbol{r} \\ &= \frac{1}{4} e^{-3}(e^{2} - e^{-2}) + \frac{1}{4} \left(e + e^{-1}\right) e^{-2} \approx 0.1947 \end{aligned}$$

and

$$\begin{aligned} P_{\mathrm{BER},0}(g_{\mathrm{ML},0}) &= \frac{1}{M} \sum_{b\in\{0,1\}} \sum_{\boldsymbol{s}_m : b_0 = b} \int_{\{\boldsymbol{r}\,:\, g_{\mathrm{ML},0}(\boldsymbol{r}) \neq b\}} \mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_m)\, d\boldsymbol{r} \\ &= \frac{1}{M} \sum_{m\in\mathcal{U}_0^{(0)}} \int_{\{\boldsymbol{r}\,:\, g_{\mathrm{ML},0}(\boldsymbol{r}) \neq 0\}} \mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_m)\, d\boldsymbol{r} + \frac{1}{M} \sum_{m\in\mathcal{U}_0^{(1)}} \int_{\{\boldsymbol{r}\,:\, g_{\mathrm{ML},0}(\boldsymbol{r}) \neq 1\}} \mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_m)\, d\boldsymbol{r} \\ &= \frac{1}{M} \int_{\{\boldsymbol{r}\,:\, g_{\mathrm{ML},0}(\boldsymbol{r}) \neq 0\}} [\mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_1)+\mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_4)]\, d\boldsymbol{r} + \frac{1}{M} \int_{\{\boldsymbol{r}\,:\, g_{\mathrm{ML},0}(\boldsymbol{r}) \neq 1\}} [\mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_2)+\mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_3)]\, d\boldsymbol{r} \\ &= \frac{1}{4} \int_{-\lambda\leq \boldsymbol{r} < \lambda} \frac{1}{2} \left(e^{-|\boldsymbol{r}+3|} + e^{-|\boldsymbol{r}-3|}\right) d\boldsymbol{r} + \frac{1}{4} \int_{\boldsymbol{r}<-\lambda \text{ or } \boldsymbol{r}\geq \lambda} \frac{1}{2} \left(e^{-|\boldsymbol{r}+1|} + e^{-|\boldsymbol{r}-1|}\right) d\boldsymbol{r} \\ &= \frac{1}{4} \int_{-\lambda}^{\lambda} \frac{1}{2} e^{-3} \left(e^{-\boldsymbol{r}} + e^{\boldsymbol{r}}\right) d\boldsymbol{r} + \frac{1}{4} \int_{-\infty}^{-\lambda} \frac{1}{2} \left(e + \frac{1}{e}\right) e^{\boldsymbol{r}}\, d\boldsymbol{r} + \frac{1}{4} \int_{\lambda}^{\infty} \frac{1}{2} \left(\frac{1}{e} + e\right) e^{-\boldsymbol{r}}\, d\boldsymbol{r} \\ &= \frac{1}{4} \int_{-\lambda}^{\lambda} e^{-3} e^{\boldsymbol{r}}\, d\boldsymbol{r} + \frac{1}{4} \int_{-\infty}^{-\lambda} \left(e + \frac{1}{e}\right) e^{\boldsymbol{r}}\, d\boldsymbol{r} \\ &= \frac{1}{4} e^{-3}(e^{\lambda} - e^{-\lambda}) + \frac{1}{4} \left(e + e^{-1}\right) e^{-\lambda} \approx 0.1944 \end{aligned}$$

For your information, we can also obtain $\mathcal{U}_1^{(0)} = \{1,2\}$ and $\mathcal{U}_0^{(1)} = \{3,4\}$, and derive

$$\begin{aligned}
P_{\text{BER},1}(g_{\text{ML}}) &= \frac{1}{M} \sum_{b\in\{0,1\}} \sum_{\boldsymbol{s}_m\,:\,b_1=b} \int_{\{\boldsymbol{r}\,:\,g_{\text{ML}}(\boldsymbol{r})\notin\mathcal{U}_1^{(b)}\}} \mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_m)\,d\boldsymbol{r} \\
&= \frac{1}{M} \sum_{m\in\mathcal{U}_1^{(0)}} \int_{\{\boldsymbol{r}\,:\,g_{\text{ML}}(\boldsymbol{r})\notin\mathcal{U}_1^{(0)}\}} \mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_m)\,d\boldsymbol{r} + \frac{1}{M} \sum_{m\in\mathcal{U}_1^{(1)}} \int_{\{\boldsymbol{r}\,:\,g_{\text{ML}}(\boldsymbol{r})\notin\mathcal{U}_1^{(1)}\}} \mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_m)\,d\boldsymbol{r} \\
&= \frac{1}{4} \int_{\{\boldsymbol{r}\,:\,g_{\text{ML}}(\boldsymbol{r})\notin\{1,2\}\}} [\mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_1) + \mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_2)]\,d\boldsymbol{r} + \frac{1}{4} \int_{\{\boldsymbol{r}\,:\,g_{\text{ML}}(\boldsymbol{r})\notin\{3,4\}\}} [\mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_3) + \mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_4)]\,d\boldsymbol{r} \\
&= \frac{1}{4} \int_0^{\infty} \frac{1}{2} \left(e^{-|\boldsymbol{r}+3|} + e^{-|\boldsymbol{r}+1|}\right) d\boldsymbol{r} + \frac{1}{4} \int_{-\infty}^{0} \frac{1}{2} \left(e^{-|\boldsymbol{r}-1|} + e^{-|\boldsymbol{r}-3|}\right) d\boldsymbol{r} \\
&= \frac{1}{4} \int_0^{\infty} \left(e^{-|\boldsymbol{r}+3|} + e^{-|\boldsymbol{r}+1|}\right) d\boldsymbol{r} \\
&= \frac{1}{4} \int_0^{\infty} e^{-|\boldsymbol{r}+3|}\,d\boldsymbol{r} + \frac{1}{4} \int_0^{\infty} e^{-|\boldsymbol{r}+1|}\,d\boldsymbol{r} \\
&= \frac{1}{4} \int_3^{\infty} e^{-|\boldsymbol{r}|}\,d\boldsymbol{r} + \frac{1}{4} \int_1^{\infty} e^{-|\boldsymbol{r}|}\,d\boldsymbol{r} \\
&= \frac{1}{4}e^{-3} + \frac{1}{4}e^{-1} \approx 0.1044
\end{aligned}$$

and

$$\begin{aligned}
P_{\text{BER},1}(g_{\text{ML},1}) &= \frac{1}{M} \sum_{b\in\{0,1\}} \sum_{\boldsymbol{s}_m\,:\,b_1=b} \int_{\{\boldsymbol{r}\,:\,g_{\text{ML},1}(\boldsymbol{r})\neq b\}} \mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_m)\,d\boldsymbol{r} \\
&= \frac{1}{M} \sum_{m\in\mathcal{U}_1^{(0)}} \int_{\{\boldsymbol{r}\,:\,g_{\text{ML},1}(\boldsymbol{r})\neq 0\}} \mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_m)\,d\boldsymbol{r} + \frac{1}{M} \sum_{m\in\mathcal{U}_1^{(1)}} \int_{\{\boldsymbol{r}\,:\,g_{\text{ML},1}(\boldsymbol{r})\neq 1\}} \mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_m)\,d\boldsymbol{r} \\
&= \frac{1}{M} \int_{\{\boldsymbol{r}\,:\,g_{\text{ML},1}(\boldsymbol{r})\neq 0\}} [\mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_1) + \mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_2)]\,d\boldsymbol{r} + \frac{1}{M} \int_{\{\boldsymbol{r}\,:\,g_{\text{ML},1}(\boldsymbol{r})\neq 1\}} [\mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_3) + \mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_4)]\,d\boldsymbol{r} \\
&= \frac{1}{4} \int_0^{\infty} \frac{1}{2} \left(e^{-|\boldsymbol{r}+3|} + e^{-|\boldsymbol{r}+1|}\right) d\boldsymbol{r} + \frac{1}{4} \int_{-\infty}^{0} \frac{1}{2} \left(e^{-|\boldsymbol{r}-1|} + e^{-|\boldsymbol{r}-3|}\right) d\boldsymbol{r} \\
&= P_{\text{BER},1}(g_{\text{ML}}).
\end{aligned}$$

Note that one can recover the transmitted channel symbol through the two bit-based ML decision rules as $(\hat{b}_1, \hat{b}_0) = (g_{\text{ML},1}(\boldsymbol{r}), g_{\text{ML},0}(\boldsymbol{r}))$. Hence, two kinds of symbol errors, $P_{\text{SER}}(g_{\text{ML}})$ and $P_{\text{SER}}(g_{\text{ML},1}, g_{\text{ML},0})$, will be resulted. Below we derive the symbol correction rates (SCR) instead: i.e., $P_{\rm SCR}(g_{\rm ML}) = 1 - P_{\rm SER}(g_{\rm ML})$ and $P_{\rm SCR}(g_{{\rm ML},1}, g_{{\rm ML},0}) = 1 - P_{\rm SER}(g_{{\rm ML},1}, g_{{\rm ML},0})$.

$$\begin{aligned}
P_{\rm SCR}(g_{\rm ML}) &= \frac{1}{4}\int_{-\infty}^{-2} \mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_1)d\boldsymbol{r} + \frac{1}{4}\int_{-2}^{0} \mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_2)d\boldsymbol{r} + \frac{1}{4}\int_{0}^{2} \mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_3)d\boldsymbol{r} + \frac{1}{4}\int_{2}^{\infty} \mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_4)d\boldsymbol{r} \\
&= \frac{1}{4}\int_{-\infty}^{-2} \frac{1}{2}e^{-|\boldsymbol{r}+3|}d\boldsymbol{r} + \frac{1}{4}\int_{-2}^{0} \frac{1}{2}e^{-|\boldsymbol{r}+1|}d\boldsymbol{r} + \frac{1}{4}\int_{0}^{2} \frac{1}{2}e^{-|\boldsymbol{r}-1|}d\boldsymbol{r} + \frac{1}{4}\int_{2}^{\infty} \frac{1}{2}e^{-|\boldsymbol{r}-3|}d\boldsymbol{r} \\
&= \frac{1}{4}\int_{-\infty}^{1} \frac{1}{2}e^{-|\boldsymbol{r}|}d\boldsymbol{r} + \frac{1}{4}\int_{-1}^{1} \frac{1}{2}e^{-|\boldsymbol{r}|}d\boldsymbol{r} + \frac{1}{4}\int_{-1}^{1} \frac{1}{2}e^{-|\boldsymbol{r}|}d\boldsymbol{r} + \frac{1}{4}\int_{-1}^{\infty} \frac{1}{2}e^{-|\boldsymbol{r}|}d\boldsymbol{r} \\
&= \frac{1}{4}\int_{-\infty}^{1} e^{-|\boldsymbol{r}|}d\boldsymbol{r} + \frac{1}{4}\int_{-1}^{1} e^{-|\boldsymbol{r}|}d\boldsymbol{r} \\
&= \frac{1}{4}\left(2 - e^{-1}\right) + \frac{1}{4}\left(2 - 2e^{-1}\right) \\
&= 1 - \frac{3}{4}e^{-1} \approx 0.7241
\end{aligned}$$

and

$$\begin{aligned}
P_{\rm SCR}(g_{{\rm ML},1}, g_{{\rm ML},0}) &= \frac{1}{4}\int_{-\infty}^{-c} \mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_1)d\boldsymbol{r} + \frac{1}{4}\int_{-c}^{0} \mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_2)d\boldsymbol{r} + \frac{1}{4}\int_{0}^{c} \mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_3)d\boldsymbol{r} + \frac{1}{4}\int_{c}^{\infty} \mathtt{f}(\boldsymbol{r}|\boldsymbol{s}_4)d\boldsymbol{r} \\
&= \frac{1}{4}\int_{-\infty}^{-c} \frac{1}{2}e^{-|\boldsymbol{r}+3|}d\boldsymbol{r} + \frac{1}{4}\int_{-c}^{0} \frac{1}{2}e^{-|\boldsymbol{r}+1|}d\boldsymbol{r} + \frac{1}{4}\int_{0}^{c} \frac{1}{2}e^{-|\boldsymbol{r}-1|}d\boldsymbol{r} + \frac{1}{4}\int_{c}^{\infty} \frac{1}{2}e^{-|\boldsymbol{r}-3|}d\boldsymbol{r} \\
&= \frac{1}{4}\int_{-\infty}^{-c+3} \frac{1}{2}e^{-|\boldsymbol{r}|}d\boldsymbol{r} + \frac{1}{4}\int_{-c+1}^{1} \frac{1}{2}e^{-|\boldsymbol{r}|}d\boldsymbol{r} + \frac{1}{4}\int_{-1}^{c-1} \frac{1}{2}e^{-|\boldsymbol{r}|}d\boldsymbol{r} + \frac{1}{4}\int_{c-3}^{\infty} \frac{1}{2}e^{-|\boldsymbol{r}|}d\boldsymbol{r} \\
&= \frac{1}{4}\int_{-\infty}^{-c+3} e^{-|\boldsymbol{r}|}d\boldsymbol{r} + \frac{1}{4}\int_{-c+1}^{1} e^{-|\boldsymbol{r}|}d\boldsymbol{r} \\
&= \frac{1}{4}\left(2 - e^{-3+c}\right) + \frac{1}{4}\left(2 - e^{-1} - e^{1-c}\right) \\
&= 1 - \frac{1}{4}e^{-1} - \frac{1}{4}e \cdot e^{-c} - \frac{1}{4}e^{-3}e^{c} \\
&= 1 - \frac{1}{4}e^{-1} - \frac{1}{4}\frac{e}{\sqrt{e^4+e^2-1}} - \frac{1}{4}e^{-3}\sqrt{e^4+e^2-1} \approx 0.7238
\end{aligned}$$

In summary, we conclude that for the Gray mapping,

$$\begin{cases}
P_{{\rm BER},0}(g_{\rm ML}) > P_{{\rm BER},0}(g_{{\rm ML},0}); \\
P_{{\rm BER},1}(g_{\rm ML}) = P_{{\rm BER},1}(g_{{\rm ML},1}); \\
P_{\rm SER}(g_{\rm ML}) < P_{\rm SER}(g_{{\rm ML},1}, g_{{\rm ML},0}).
\end{cases}$$