EE641000 Quantum Information and Computation

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February 27, 2007

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Supplement – A Primer of Linear Algebra

A mathematical language for beginners in quantum computing and quantum information processing.

Complex Inner Product Spaces

- $\mathcal{H}$: a complex vector space.
- (,) : an inner product as a mapping from $\mathcal{H} \times \mathcal{H}$ to C such that
  - Non-negativity : $(|v\rangle, |v\rangle) \geq 0$ for all $|v\rangle \in \mathcal{H}$ and $(|v\rangle, |v\rangle) = 0$ if and only if $|v\rangle = 0$,
  - Hermitian symmetry : $(|v'\rangle, |v\rangle) = \overline{(|v\rangle, |v'\rangle)}$,
  - Linearity : $(|v'\rangle, \alpha|v_1\rangle + \beta|v_2\rangle) = \alpha(|v'\rangle, |v_1\rangle) + \beta(|v\rangle', |v_2\rangle)$.
- Dirac notation : $(|v'\rangle, |v\rangle) \equiv \langle v'|v\rangle$.

Matrix Representation of Linear Transformations

- $\mathcal{H}_1, \mathcal{H}_2$: finite-dimensional complex inner product spaces.
- T : a linear transformation from $\mathcal{H}_1$ to $\mathcal{H}_2$.
- $\mathcal{B} = \{|j\rangle\}$, $\mathcal{C} = \{|i\rangle\}$: orthonormal bases of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.
- $[a_{ij}] = A = [T]_{\mathcal{B},\mathcal{C}}$: matrix representation of T relative to the bases $\mathcal{B}$ and $\mathcal{C}$

$$a_{ij} = \langle i|T|j\rangle, \ \forall \ i, j.$$

The Vector Space $L(\mathcal{H}_1, \mathcal{H}_2)$

- $\mathcal{H}_1, \mathcal{H}_2$: finite-dimensional complex inner product spaces.
- $L(\mathcal{H}_1, \mathcal{H}_2)$: the complex vector space of all linear transformations from $\mathcal{H}_1$ to $\mathcal{H}_2$.
- $|v\rangle, |w\rangle$: vectors in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.
- $|w\rangle\langle v|$: *outer product* of $|w\rangle$ and $\langle v|$, which is defined as a linear transformations from $\mathcal{H}_1$ to $\mathcal{H}_2$

$$(|w\rangle\langle v|)(|v'\rangle) \stackrel{\triangle}{=} |w\rangle\langle v|v'\rangle, \ \forall \ |v'\rangle \in \mathcal{H}_1.$$

- $\mathcal{B} = \{|j\rangle\}, \mathcal{C} = \{|i\rangle\}$: orthonormal bases of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.
- $\{|i\rangle\langle j|\}$: a basis of $L(\mathcal{H}_1, \mathcal{H}_2)$.

For each $T \in L(H_1, H_2)$, we have

$$\boxed{T = \sum_{ij} \langle i|T|j\rangle |i\rangle\langle j|.}$$

The Dual Space $\mathcal{H}^*$

- $\mathcal{H}$: a finite-dimensional complex inner product space.
- $\mathcal{H}^* = L(\mathcal{H}, C)$: the complex vector space of all linear functional on $\mathcal{H}$, called the dual space of $\mathcal{H}$.
- $|v\rangle$: a vector in $\mathcal{H}$.
- $\langle v|$: a linear functional on $\mathcal{H}$ defined as

$$\langle v|(|v'\rangle) \stackrel{\triangle}{=} \langle v|v'\rangle.$$

- $W = \{\langle v| \mid |v\rangle \in \mathcal{H}\}$: a vector subspace of $\mathcal{H}^*$.

Since $\dim W = \dim \mathcal{H} = \dim \mathcal{H}^*$, we have $\mathcal{H}^* = W$, i.e., every linear functional on $\mathcal{H}$ is of the form $\langle v|$ for some $|v\rangle \in \mathcal{H}$.

The Adjoint of a Linear Transformation

- $\mathcal{H}_1, \mathcal{H}_2$: finite-dimensional complex inner product spaces.
- T : a linear transformation from $\mathcal{H}_1$ to $\mathcal{H}_2$.
- $|w\rangle$: a given vector in $\mathcal{H}_2$.
- $f(|v\rangle) \stackrel{\triangle}{=} (|w\rangle, T|v\rangle)$: a linear functional on $\mathcal{H}_1$.
- $|v^*\rangle$: the unique vector in $\mathcal{H}_1$ such that

$$f(|v\rangle) = (|w\rangle, T|v\rangle) = (|v^*\rangle, |v\rangle) \ \forall \ |v\rangle \in \mathcal{H}_1.$$

For each $|w\rangle \in \mathcal{H}_2$, we define

$$T^\dagger|w\rangle = |v^*\rangle.$$

- $T^\dagger$: a mapping from $\mathcal{H}_2$ to $\mathcal{H}_1$, called the *adjoint* of T.

$T^\dagger$ is a Linear Transformation from $\mathcal{H}_2$ to $\mathcal{H}_1$

- $|w\rangle, |z\rangle$: vectors in $\mathcal{H}_2$.
- α, β : complex numbers,

$$
\begin{aligned}
(T^\dagger(\alpha|w\rangle + \beta|z\rangle), |v\rangle) &= (\alpha|w\rangle + \beta|z\rangle, T|v\rangle) \\
&= \overline{\alpha}(|w\rangle, T|v\rangle) + \overline{\beta}(|z\rangle, T|v\rangle) \\
&= \overline{\alpha}(T^\dagger(|w\rangle), |v\rangle) + \overline{\beta}(T^\dagger(|z\rangle), |v\rangle) \\
&= (\alpha T^\dagger(|w\rangle) + \beta T^\dagger(|z\rangle), |v\rangle).
\end{aligned}
$$

$$\boxed{|v\rangle^{\dagger} = \langle v|}$$

- $\mathcal{H}$: a finite-dimensional complex inner product space.
- $|v\rangle$: a vector in $\mathcal{H}$ which can be regarded as a linear transformation from C to H as $|v\rangle(\alpha) \stackrel{\triangle}{=} \alpha|v\rangle$.

Since

$$(|v\rangle^{\dagger}|w\rangle, \alpha) = (|w\rangle, |v\rangle(\alpha)) = \alpha(|w\rangle, |v\rangle) = ((|v\rangle, |w\rangle), \alpha) = (\langle v|w\rangle, \alpha),$$

we have $|v\rangle^{\dagger} = \langle v|$.

Properties of Adjoint

- $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$: finite-dimensional complex inner product spaces.
- T, S : linear transformations from $\mathcal{H}_1$ to $\mathcal{H}_2$ and from $\mathcal{H}_2$ to $\mathcal{H}_3$, respectively.
- $(ST)^\dagger = T^\dagger S^\dagger$:

$$((ST)^\dagger|u\rangle, |v\rangle) = (|u\rangle, ST|v\rangle) = (S^\dagger|u\rangle, T|v\rangle) = (T^\dagger S^\dagger|u\rangle, |v\rangle).$$

- $(T^\dagger)^\dagger = T$:

$$\begin{aligned} (|w\rangle, T|v\rangle) &= (T^\dagger|w\rangle, |v\rangle) = \overline{(|v\rangle, T^\dagger|w\rangle)} = \overline{((T^\dagger)^\dagger|v\rangle, |w\rangle)} \\ &= (|w\rangle, (T^\dagger)^\dagger|v\rangle). \end{aligned}$$

- $(T|v\rangle)^\dagger = \langle v|T^\dagger$:

$$(T|v\rangle)^\dagger(|w\rangle) = (T|v\rangle, |w\rangle) = (|v\rangle, T^\dagger|w\rangle) = \langle v|T^\dagger|w\rangle.$$

Special Operators

Normal Operators

$$TT^{\dagger} = T^{\dagger}T$$

- T : a linear operator on a finite-dimensional complex inner product space.

Spectral Decomposition of Normal Operators

$$T = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|.$$

- T : a normal operator on a finite-dimensional complex inner product space $\mathcal{H}$.
- $\{|\psi_i\rangle\}$: an orthonormal eigenbasis of $\mathcal{H}$ associated with T.
- λ_i : eigenvalue of T corresponding to eigenvector $|\psi_i\rangle$.

Hermitian Operators

$$T = T^{\dagger}.$$

A linear operator T is Hermitian if and only if

- T is a normal operator,
- T has real eigenvalues only.

Projection Operator

- $\mathcal{H}$: a finite-dimensional complex inner product space space.
- W : a subspace of $\mathcal{H}$.
- $W^{\perp}$: the othogonal complement of W in $\mathcal{H}$.
- Othogonal decomposition : for each $|v\rangle \in H$, there exist unique $|w\rangle \in W$ and $|w^{\perp}\rangle \in W^{\perp}$ such that

$$|v\rangle = |w\rangle + |w^{\perp}\rangle.$$

- P : the projection of $\mathcal{H}$ onto W, i.e.

$$P(|v\rangle) = |w\rangle, \ \forall v \in V.$$

A Necessary and Sufficient Condition

P is a projector on $\mathcal{H}$ if and only if

- $P = P^\dagger$,
- $P^2 = P$.

Unitary Operators

$$U^\dagger U = UU^\dagger = I.$$

- Eigenvalues of a unitary operator have unit modulus.

A Representation of Unitary Operators

If

- $\{|i\rangle\}$: an orthonormal basis of $\mathcal{H}$,
- U : a unitary operator on $\mathcal{H}$,
- $|\psi_i\rangle$: $|\psi_i\rangle = U|i\rangle$,

then we have

- $\{|\psi_i\rangle\}$: another orthonormal basis of $\mathcal{H}$,
- $U = \sum_i |\psi_i\rangle\langle i|$.

Also given any two orthonormal bases $\{|\psi_i\rangle\}$ and $\{|\varphi_i\rangle\}$ of $\mathcal{H}$, the operator

$$U = \sum_i |\psi_i\rangle\langle\varphi_i|$$

is unitary.

Positive Operator

$$(|v\rangle, T|v\rangle) \geq 0, \ \forall \ |v\rangle.$$

Necessary and sufficient conditions are

- Normal,
- Having non-negative eigenvalues.

Positive Operators $T^\dagger T$ and $TT^\dagger$

- $\mathcal{H}_1, \mathcal{H}_2$: finite-dimensional complex inner product spaces.
- T : a linear transformation from $\mathcal{H}_1$ to $\mathcal{H}_2$.

Proof

$$\begin{aligned} (|v\rangle, T^\dagger T|v\rangle) &= (T|v\rangle, T|v\rangle) \geq 0, \\ (|w\rangle, TT^\dagger|w\rangle) &= (T^\dagger|w\rangle, T^\dagger|w\rangle) \geq 0. \end{aligned}$$

Positive Operators $|w\rangle\langle w|$

- $|w\rangle$: any vector in $\mathcal{H}$.

Proof

$$(|v\rangle, |w\rangle\langle w|v\rangle) = \langle v|w\rangle\langle w|v\rangle = |\langle v|w\rangle|^2 \geq 0.$$

Positive-Definite Operator

$$(|v\rangle, T|v\rangle) > 0, \ \forall \ |v\rangle \neq 0.$$

- Normal.
- Having positive eigenvalues.

Functions of Normal Operators

- $f(z)$: complex-values function on complex numbers.
- T : a normal operator with spectral decomposition

$$T = \sum_i \lambda_i \; |\psi_i\rangle\langle\psi_i|.$$

- $f(T)$ is defined as

$$f(T) = \sum_i f(\lambda_i) \; |\psi_i\rangle\langle\psi_i|.$$

- The above definition is well-defined, i.e., independent of the choice of orthonormal basis $\{|\psi_i\rangle\}$.

An Example

$$e^{\theta Z} = \begin{bmatrix} e^{\theta} & 0 \\ 0 & e^{-\theta} \end{bmatrix}$$

where

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

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A Property

If T and S are two commuting normal operator on H, then

$$e^T e^S = e^{T+S}.$$

Tensor Product

Tensor Product of Vector Spaces

- V and W : two complex vector spaces.
- $\boldsymbol{v}$ and $\boldsymbol{w}$: vectors in V and W respectively.
- $\boldsymbol{v} \otimes \boldsymbol{w}$: the direct product of $\boldsymbol{v}$ and $\boldsymbol{w}$.
- $V \otimes W$: the tensor product of V and W, which is the set of all linear combinations of (finitely many) $\boldsymbol{v} \otimes \boldsymbol{w}$ with $\boldsymbol{v} \in V$ and $\boldsymbol{w} \in W$ satisfying
  1. $(\boldsymbol{v} + \boldsymbol{v}') \otimes \boldsymbol{w} = \boldsymbol{v} \otimes \boldsymbol{w} + \boldsymbol{v}' \otimes \boldsymbol{w}$,
  2. $\boldsymbol{v} \otimes (\boldsymbol{w} + \boldsymbol{w}') = \boldsymbol{v} \otimes \boldsymbol{w} + \boldsymbol{v} \otimes \boldsymbol{w}'$,
  3. $\alpha(\boldsymbol{v} \otimes \boldsymbol{w}) = (\alpha \boldsymbol{v}) \otimes \boldsymbol{w} = \boldsymbol{v} \otimes (\alpha \boldsymbol{w})$.
- $\sum_i \alpha_i (\boldsymbol{v}_i \otimes \boldsymbol{w}_i)$: a typical element in $V \otimes W$, where $\sum_i$ is a finite sum.

Tensor Product $V \otimes W$ as a Vector Space

- zero vector 0 : $0 \otimes 0 = \boldsymbol{v} \otimes 0 = 0 \otimes \boldsymbol{w}$.
  - By the third property, we have

$$0 \otimes 0 = \boldsymbol{v} \otimes 0 = 0 \otimes \boldsymbol{w}.$$

  - For any $\boldsymbol{v} \otimes \boldsymbol{w}$, we have

$$\boldsymbol{v} \otimes \boldsymbol{w} + \boldsymbol{v} \otimes 0 = \boldsymbol{v} \otimes (\boldsymbol{w} + 0) = \boldsymbol{v} \otimes \boldsymbol{w}.$$

- $-\boldsymbol{v} \otimes \boldsymbol{w}$: additive inverse of $\boldsymbol{v} \otimes \boldsymbol{w}$

$$\boldsymbol{v} \otimes \boldsymbol{w} - \boldsymbol{v} \otimes \boldsymbol{w} = \boldsymbol{v} \otimes (\boldsymbol{w} - \boldsymbol{w}) = \boldsymbol{v} \otimes 0 = 0.$$

Tensor Product $V \otimes W$ in a Reduced Representation

- $\{\boldsymbol{j}\}$: a basis of W.
- $\sum_i \alpha_i(\boldsymbol{v}_i \otimes \boldsymbol{w}_i) = \sum_i \alpha_i(\boldsymbol{v}_i \otimes (\sum_j \beta_{ij}\boldsymbol{j})) =$
  $\sum_j \sum_i \alpha_i\beta_{ij}(\boldsymbol{v}_i \otimes \boldsymbol{j}) = \sum_j (\sum_i \alpha_i\beta_{ij}\boldsymbol{v}_i) \otimes \boldsymbol{j} = \sum_j \boldsymbol{v}'_j \otimes \boldsymbol{j}$.

Then we have

$$V \otimes W = \{\sum_j \boldsymbol{v}_j \otimes \boldsymbol{j} \mid \ \boldsymbol{v}_j \in V\},$$

where all but finitely many $\boldsymbol{v}_j = 0$.

A Unique Representation Theorem

- V and W : complex vector spaces.
- $\{\boldsymbol{j}\}$: a basis of W.
- $\sum_j \boldsymbol{u}_j \otimes \boldsymbol{j}$ and $\sum_j \boldsymbol{v}_j \otimes \boldsymbol{j}$: two vectors in $V \otimes W$.

Then

$$\sum_j \boldsymbol{u}_j \otimes \boldsymbol{j} = \sum_j \boldsymbol{v}_j \otimes \boldsymbol{j}$$

if and only if

$$\boldsymbol{u}_j = \boldsymbol{v}_j, \quad \forall\ j.$$

Proof

- $V \times V \times \cdots \times V$: the direct sum of as many V's as $\boldsymbol{j}$'s

$$\begin{aligned} & V \times V \times \cdots \times V \\ \triangleq\ & \{(\ldots, \boldsymbol{v}_j, \ldots) \mid \boldsymbol{v}_j \in V, \text{all but finitely many } \boldsymbol{v}_j = 0\}. \end{aligned}$$

- $V \times V \times \cdots \times V$: a complex vector space,
  - $(\ldots, \boldsymbol{v}_j, \ldots) + (\ldots, \boldsymbol{v}'_j, \ldots) = (\ldots, \boldsymbol{v}_j + \boldsymbol{v}'_j, \ldots)$,
  - $\alpha(\ldots, \boldsymbol{v}_j, \ldots) = (\ldots, \alpha \boldsymbol{v}_j, \ldots)$,
  - zero element must be $(\ldots, 0, \ldots)$.

- $V \otimes W \cong V \times V \times \cdots \times V$: a linear isomorphism f from $V \otimes W$ onto $V \times V \times \cdots \times V$,

$$
\begin{aligned}
f(\sum_j \boldsymbol{v}_j \otimes \boldsymbol{j}) &= (\ldots, \boldsymbol{v}_j, \ldots), \\
f^{-1}((\ldots, \boldsymbol{v}_j, \ldots)) &= \sum_j \boldsymbol{v}_j \otimes \boldsymbol{j}.
\end{aligned}
$$

Thus,

$$\sum_j \boldsymbol{u}_j \otimes \boldsymbol{j} = \sum_j \boldsymbol{v}_j \otimes \boldsymbol{j}$$

if and only if

$$(\ldots, \boldsymbol{u}_j, \ldots) = (\ldots, \boldsymbol{v}_j, \ldots).$$

The Zero Element 0 of the Tensor Product $V \otimes W$

A direct product $\boldsymbol{v} \otimes \boldsymbol{w}$ in $V \otimes W$ is equal to the zero element 0 of $V \otimes W$ if and only if

$$\boldsymbol{v} = 0 \text{ or } \boldsymbol{w} = 0.$$

Proof

Suppose $\boldsymbol{v} \otimes \boldsymbol{w} = 0$ and $\boldsymbol{w} \neq 0$. Let

$$\boldsymbol{w} = \sum_j \beta_j \boldsymbol{j}$$

where at least one β_j is non-zero. Thus

$$0 = \boldsymbol{v} \otimes \boldsymbol{w} = \sum_j \beta_j (\boldsymbol{v} \otimes \boldsymbol{j}) = \sum_j (\beta_j \boldsymbol{v}) \otimes \boldsymbol{j}$$

which implies that

$$\beta_j \boldsymbol{v} = 0$$

for all j by the unique representation theorem. Since at least one of β_j is non-zero, we have $\boldsymbol{v} = 0$.

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Bases of Tensor Product $V \otimes W$

- V and W : complex vector spaces.
- $\mathcal{B} = \{\boldsymbol{i}\}$ and $\mathcal{C} = \{\boldsymbol{j}\}$: bases of V and W respectively.
- $\mathcal{B} \otimes \mathcal{C} = \{\boldsymbol{i} \otimes \boldsymbol{j}\}$: a basis of $V \otimes W$.

Proof of Linear Independence

$$
\begin{aligned}
& \sum_{ij} \alpha_{ij} (\boldsymbol{i} \otimes \boldsymbol{j}) = 0 \\
\Leftrightarrow \quad & \sum_{j} (\sum_{i} \alpha_{ij} \boldsymbol{i}) \otimes \boldsymbol{j} = 0 \\
\Leftrightarrow \quad & \sum_{i} \alpha_{ij} \boldsymbol{i} = 0 \ \forall \ j \\
\Leftrightarrow \quad & \alpha_{ij} = 0 \ \forall \ i, j.
\end{aligned}
$$

Tensor Product $\mathcal{H}_1 \otimes \mathcal{H}_2$ as an Inner Product Space

- $\mathcal{H}_1$ and $\mathcal{H}_2$: inner product spaces.
- $\sum_i \alpha_i(|v_i\rangle \otimes |w_i\rangle)$ and $\sum_j \alpha'_j(|v'_j\rangle \otimes |w'_j\rangle)$: two vectors in $\mathcal{H}_1 \otimes \mathcal{H}_2$.

$$\left(\sum_i \alpha_i(|v_i\rangle \otimes |w_i\rangle), \sum_j \alpha'_j(|v'_j\rangle \otimes |w'_j\rangle)\right) \stackrel{\triangle}{=} \sum_{ij} \alpha_i^* \alpha'_j \langle v_i|v'_j\rangle\langle w_i|w'_j\rangle.$$

- $(|v\rangle \otimes |w\rangle, |v'\rangle \otimes |w'\rangle) = \langle v|v'\rangle\langle w|w'\rangle.$

Well-defined

- $\sum_i \alpha_i(|v_i\rangle \otimes |w_i\rangle) = \sum_j (\sum_i \alpha_i \beta_{ij} |v_i\rangle) \otimes |j\rangle$.
- $\sum_k \alpha'_k(|v'_k\rangle \otimes |w'_k\rangle) = \sum_{j'} (\sum_k \alpha'_k \beta'_{kj'} |v'_k\rangle) \otimes |j'\rangle$.

$$\begin{aligned}
& \left(\sum_i \alpha_i(|v_i\rangle \otimes |w_i\rangle), \sum_k \alpha'_k(|v'_k\rangle \otimes |w'_k\rangle) \right) \\
\triangleq\ & \sum_{ik} \alpha_i^* \alpha'_k \langle v_i|v'_k\rangle \langle w_i|w'_k\rangle = \sum_{ik} \alpha_i^* \alpha'_k \langle v_i|v'_k\rangle \sum_{jj'} \beta_{ij}^* \beta'_{kj'} \langle j|j'\rangle \\
=\ & \sum_{jj'} \left(\sum_i \alpha_i \beta_{ij} |v_i\rangle, \sum_k \alpha'_k \beta'_{kj'} |v'_k\rangle \right) \langle j|j'\rangle \\
\triangleq\ & \left(\sum_j \left(\sum_i \alpha_i \beta_{ij} |v_i\rangle \right) \otimes |j\rangle, \sum_{j'} \left(\sum_k \alpha'_k \beta'_{kj'} |v'_k\rangle \right) \otimes |j'\rangle \right).
\end{aligned}$$

A Useful Formula

- $\mathcal{H}_1$ and $\mathcal{H}_2$: complex inner product spaces.
- $\{|j\rangle\}$: an orthonormal basis of $\mathcal{H}_2$.
- $\sum_j |u_j\rangle \otimes |j\rangle$ and $\sum_j |v_j\rangle \otimes |j\rangle$: two vectors in $H_1 \otimes \mathcal{H}_2$.

$$\boxed{\left(\sum_j |u_j\rangle \otimes |j\rangle, \sum_j |v_j\rangle \otimes |j\rangle\right) = \sum_j \langle u_j | v_j \rangle.}$$

Orthonormal Bases of Tensor Product $\mathcal{H}_1 \otimes \mathcal{H}_2$

- $\mathcal{H}_1$ and $\mathcal{H}_2$: complex inner product spaces.
- $\mathcal{B} = \{|i\rangle\}$ and $\mathcal{C} = \{|j\rangle\}$: orthonormal bases of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.
- $\mathcal{B} \otimes \mathcal{C} = \{|i\rangle \otimes |j\rangle\}$: an orthonormal basis of $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Direct Product of Linear Transformations

- V, V', W, W' : complex vector spaces.
- $L(V, V')$ and $L(W, W')$: the vector space of all linear transformations from V to V' and from W to W', respectively.
- T and S : linear transformations in $L(V, V')$ and in $L(W, W')$, respectively.
- $\sum_i \alpha_i(\boldsymbol{v}_i \otimes \boldsymbol{w}_i)$: a vector in $V \bigotimes W$.
- $T \bigotimes S$: the direct product of T and S, which is a linear transformation from $V \otimes W$ to $V' \otimes W'$,

$$(T \otimes S)(\sum_i \alpha_i(\boldsymbol{v}_i \otimes \boldsymbol{w}_i)) \triangleq \sum_i \alpha_i(T\boldsymbol{v}_i \otimes S\boldsymbol{w}_i).$$

  – Well-defined.

Matrix Version

- $\mathcal{B} = \{\boldsymbol{i}\}, \mathcal{B}' = \{\boldsymbol{i}'\}, \mathcal{C} = \{\boldsymbol{j}\}, \mathcal{C}' = \{\boldsymbol{j}'\}$: bases of V, V', W, W', respectively.
- $\mathcal{B} \otimes \mathcal{C}, \mathcal{B}' \otimes \mathcal{C}'$: bases of $V \otimes W, V' \otimes W'$, respectively.
- $A = [T]_{\mathcal{B} \to \mathcal{B}'}$ and $B = [S]_{\mathcal{C} \to \mathcal{C}'}$: matrix representations of T and S, respectively.
- $G = [T \otimes S]_{\mathcal{B} \otimes \mathcal{C} \to \mathcal{B}' \otimes \mathcal{C}'}$: matrix representations of $T \otimes S$.

$$\boxed{g_{i'j',ij} = a_{i',i} b_{j',j}.}$$

- $G \stackrel{\triangle}{=} A \otimes B$.

Proof

$$T \otimes S(\boldsymbol{i} \otimes \boldsymbol{j}) = \sum_{i'j'} g_{i'j',ij}(\boldsymbol{i'} \otimes \boldsymbol{j'})$$

and

$$\begin{aligned} T \otimes S(\boldsymbol{i} \otimes \boldsymbol{j}) &= T(\boldsymbol{i}) \otimes S(\boldsymbol{j}) \\ &= \left(\sum_{i'} a_{i',i}\boldsymbol{i'}\right) \otimes \left(\sum_{j'} b_{j',j}\boldsymbol{j'}\right) \\ &= \sum_{i'j'} a_{i',i}b_{j',j}(\boldsymbol{i'} \otimes \boldsymbol{j'}). \end{aligned}$$

Tensor Product of $L(V, V')$ and $L(W, W')$

$L(V, V') \otimes L(W, W')$, called the tensor product of $L(V, V')$ and $L(W, W')$, is the set of all linear combinations of $T \otimes S$, $T \in L(V, V')$ and $S \in L(W, W')$, which satisfies

1. $(T + T') \otimes S = T \otimes S + T' \otimes S$,
2. $T \otimes (S + S') = T \otimes S + T \otimes S'$,
3. $\alpha(T \otimes S) = (\alpha T) \otimes S = T \otimes (\alpha S)$.

The tensor product $L(V, V') \otimes L(W, W')$ is a vector space.

Bases of Tensor Product space $L(\mathcal{H}_1, \mathcal{H}_2) \otimes L(\mathcal{H}_3, \mathcal{H}_4)$

- $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4$: finite-dimensional complex inner product spaces.
- $\{|i\rangle\}, \{|i'\rangle\}, \{|j\rangle\}, \{|j'\rangle\}$: orthonormal bases of $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4$, respectively.
- $\{|i'\rangle\langle i|\}$: a basis of $L(\mathcal{H}_1, \mathcal{H}_2)$.
- $\{|j'\rangle\langle j|\}$: a basis of $L(\mathcal{H}_3, \mathcal{H}_4)$.
- $\{|i'\rangle\langle i| \otimes |j'\rangle\langle j|\}$: a basis of $L(\mathcal{H}_1, \mathcal{H}_2) \otimes L(\mathcal{H}_3, \mathcal{H}_4)$.

Bases of Vector Space $L(\mathcal{H}_1 \otimes \mathcal{H}_3, \mathcal{H}_2 \otimes \mathcal{H}_4)$

- $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4$: finite-dimensional complex inner product spaces.
- $\{|i\rangle\}, \{|i'\rangle\}, \{|j\rangle\}, \{|j'\rangle\}$: orthonormal bases of $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4$, respectively.
- $\{|i\rangle \otimes |j\rangle\}$: an orthonormal basis of $\mathcal{H}_1 \otimes \mathcal{H}_3$.
- $\{|i'\rangle \otimes |j'\rangle\}$: an orthonormal basis of $\mathcal{H}_2 \otimes \mathcal{H}_4$.
- $\{(|i'\rangle \otimes |j'\rangle)(|i\rangle \otimes |j\rangle)^\dagger\}$: a basis of $L(\mathcal{H}_1 \otimes \mathcal{H}_3, \mathcal{H}_2 \otimes \mathcal{H}_4)$.

An Identity

$$(|i'\rangle \otimes |j'\rangle)(|i\rangle \otimes |j\rangle)^\dagger = |i'\rangle\langle i| \otimes |j'\rangle\langle j|.$$

Proof

$$
\begin{aligned}
(|i'\rangle \otimes |j'\rangle)(|i\rangle \otimes |j\rangle)^\dagger(|v\rangle \otimes |w\rangle) &= \langle i|v\rangle\langle j|w\rangle(|i'\rangle \otimes |j'\rangle) \\
(|i'\rangle\langle i| \otimes |j'\rangle\langle j|)(|v\rangle \otimes |w\rangle) &= (|i'\rangle\langle i|v\rangle) \otimes (|j'\rangle\langle j|w\rangle) \\
&= \langle i|v\rangle\langle j|w\rangle(|i'\rangle \otimes |j'\rangle).
\end{aligned}
$$

A Theorem

$$L(\mathcal{H}_1 \otimes \mathcal{H}_3, \mathcal{H}_2 \otimes \mathcal{H}_4) = L(\mathcal{H}_1, \mathcal{H}_2) \otimes L(\mathcal{H}_3, \mathcal{H}_4).$$

$(|v\rangle \otimes |w\rangle)^\dagger$ **v.s.** $|v\rangle^\dagger \otimes |w\rangle^\dagger$

- $(|v\rangle \otimes |w\rangle)^\dagger$: a linear transformation from $\mathcal{H}_1 \otimes \mathcal{H}_2$ into C.
- $|v\rangle^\dagger \otimes |w\rangle^\dagger$: a linear transformation from $\mathcal{H}_1 \otimes \mathcal{H}_2$ into $C \otimes C$.
- $C \otimes C \cong C$: a linear isomorphism f from $C \otimes C$ onto C

$$f(\alpha \otimes \beta) = \alpha\beta, \quad f^{-1}(\gamma) = \gamma \otimes 1.$$

Then

$$(|v\rangle \otimes |w\rangle)^\dagger = f \circ (|v\rangle^\dagger \otimes |w\rangle^\dagger).$$

Proof.

$$\begin{aligned}
(|v\rangle \otimes |w\rangle)^\dagger(|x\rangle \otimes |y\rangle) &= (|v\rangle \otimes |w\rangle, |x\rangle \otimes |y\rangle) = \langle v|x\rangle\langle w|y\rangle \\
&= f(\langle v|x\rangle \otimes \langle w|y\rangle) = f((\langle v| \otimes \langle w|)(|x\rangle \otimes |y\rangle)) \\
&= f \circ (|v\rangle^\dagger \otimes |w\rangle^\dagger)(|x\rangle \otimes |y\rangle).
\end{aligned}$$

Adjoint and Tensor Product

$(T \otimes S)^\dagger = T^\dagger \otimes S^\dagger$

- $\mathcal{H}_1, \mathcal{H}_2$: finite-dimensional complex inner product spaces.
- $\{|v\rangle\}, \{|v'\rangle\}$: vectors in $\mathcal{H}_1$.
- $\{|w\rangle\}, \{|w'\rangle\}$: vectors in $\mathcal{H}_2$.
- T, S : linear operators on $\mathcal{H}_1$ and on $\mathcal{H}_2$, respectively.

$$
\begin{aligned}
& ((T \otimes S)^\dagger |v'\rangle \otimes |w'\rangle, |v\rangle \otimes |w\rangle) \\
= \quad & (|v'\rangle \otimes |w'\rangle, (T \otimes S)|v\rangle \otimes |w\rangle) = \langle v'|T|v\rangle \langle w'|S|w\rangle \\
= \quad & (T^\dagger |v'\rangle, |v\rangle)(S^\dagger |w'\rangle, |w\rangle) = (T^\dagger |v'\rangle \otimes S^\dagger |w'\rangle, |v\rangle \otimes |w\rangle) \\
= \quad & ((T^\dagger \otimes S^\dagger)|v'\rangle \otimes |w'\rangle, |v\rangle \otimes |w\rangle).
\end{aligned}
$$

Inherent Properties

- $\mathcal{H}_1, \mathcal{H}_2$: finite-dimensional complex inner product spaces.
- T, S : operators on $\mathcal{H}_1$ and on $\mathcal{H}_2$, respectively.

1. T, S are normal $\Rightarrow T \otimes S$ is normal.

$$\begin{aligned} & (T \otimes S)^\dagger (T \otimes S) = (T^\dagger \otimes S^\dagger)(T \otimes S) = (T^\dagger T) \otimes (S^\dagger S) \\ = \;& (TT^\dagger) \otimes (SS^\dagger) = (T \otimes S)(T^\dagger \otimes S^\dagger) = (T \otimes S)(T \otimes S)^\dagger. \end{aligned}$$

2. T, S are Hermitian $\Rightarrow T \otimes S$ is Hermitian.
3. T, S are unitary $\Rightarrow T \otimes S$ is unitary.
4. T, S are projectors $\Rightarrow T \otimes S$ is a projector.

Spectral Decomposition

- $\mathcal{H}_1, \mathcal{H}_2$: finite-dimensional complex inner product spaces.
- T, S : normal operators on $\mathcal{H}_1$ and on $\mathcal{H}_2$, respectively, with spectral decomposition

$$T = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|, \quad S = \sum_j \kappa_j |\varphi_j\rangle\langle\varphi_j|.$$

- $T \otimes S$: normal operators on $\mathcal{H}_1 \otimes \mathcal{H}_2$ such that

$$(T \otimes S)(|\psi_i\rangle \otimes |\varphi_j\rangle) = T|\psi_i\rangle \otimes S|\varphi_j\rangle = \lambda_i \kappa_j (|\psi_i\rangle \otimes |\varphi_j\rangle).$$

we have spectral decomposition

$$\boxed{T \otimes S = \sum_{ij} \lambda_i \kappa_j (|\psi_i\rangle \otimes |\varphi_j\rangle)(|\psi_i\rangle \otimes |\varphi_j\rangle)^\dagger.}$$

The Trace Function

Trace Function on Matrices

- A : an $n \times n$ complex matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

- $\mathrm{tr}(A)$: the trace of matrix A

$$\mathrm{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}.$$

Properties of Trace Function

- A, B : $n \times n$ complex matrices.
- α, β : complex numbers.
- Linearity : $\mathrm{tr}(\alpha A + \beta B) = \alpha\ \mathrm{tr}(A) + \beta\ \mathrm{tr}(B)$.
- Cyclicity : $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.
- Invariance under similarity transformation : if $B = MAM^{-1}$, where M is an invertible $n \times n$ complex matrix, then

$$\mathrm{tr}(B) = \mathrm{tr}(MAM^{-1}) = \mathrm{tr}(M^{-1}MA) = \mathrm{tr}(A).$$

Trace Function on Operators

- T : an operator on an n-dimensional complex vector space V.
- A : a matrix representation of T relative to a basis $\mathcal{B}$ of V.
- $\mathrm{tr}(T)$: the trace of operator T,

$$\mathrm{tr}(T) \equiv \mathrm{tr}(A).$$

- Well-defined : if B is the matrix representation of T relative to another basis $\mathcal{B}'$ of V, then $B = MAM^{-1}$ with $M = [\mathcal{B} \to \mathcal{B}']$, i.e., B is similar to A and

$$\mathrm{tr}(T) \equiv \mathrm{tr}(B) = \mathrm{tr}(A) \equiv \mathrm{tr}(T).$$

Computation of tr(T)

$$\mathrm{tr}(T) = \sum_i \langle \psi_i | T | \psi_i \rangle .$$

- T : an operator on a finite dimensional complex inner product space $\mathcal{H}$.
- $\{|\psi_i\rangle\}$: an orthonormal basis of $\mathcal{H}$.

A Useful Formula

$$\mathrm{tr}(T|v\rangle\langle w|) = \langle w|T|v\rangle.$$

- $|v\rangle, |w\rangle$: vectors in a finite-dimensional complex inner product space $\mathcal{H}$.
- T : an operator on $\mathcal{H}$.

Proof

- $\{|\psi_i\rangle\}$: an orthonormal basis of $\mathcal{H}$.

Then we have

$$\mathrm{tr}(T|v\rangle\langle w|) = \sum_i \langle\psi_i|T|v\rangle\langle w|\psi_i\rangle = \sum_i \langle w|\psi_i\rangle\langle\psi_i|T|v\rangle = \langle w|T|v\rangle$$

since

$$\sum_i |\psi_i\rangle\langle\psi_i| = I.$$

Trace Function and Tensor Product

Trace of $T \otimes S$

$$\mathrm{tr}(T \otimes S) = \mathrm{tr}(T)\ \mathrm{tr}(S).$$

- T and S : linear operators on finite-dimensional vector spaces V and W respectively.
- $T \otimes S$: linear operator on $V \otimes W$.

Proof

- $A = [T], B = [S]$: matrix representations of T, S, respectively.
- $G = A \otimes B$: matrix representation of $T \otimes S$.

$$
\begin{aligned}
\mathrm{tr}(T \otimes S) &= \mathrm{tr}(G) = \sum_{ij} g_{ij,ij} \\
&= \sum_{ij} a_{ii} b_{jj} = \sum_{i} a_{ii} \sum_{jj} b_{jj} \\
&= \mathrm{tr}(A)\ \mathrm{tr}(B) \\
&= \mathrm{tr}(T)\ \mathrm{tr}(S).
\end{aligned}
$$

Partial Trace Function

- V and W : vector spaces V and W.
- T^{VW} : a linear operator on the tensor product space $V \otimes W$,

$$T^{VW} = \sum_i \alpha_i T_i^V \otimes T_i^W,$$

where T_i^V and T_i^W are linear operators on V and on W, respectively.

The partial trace of T^{VW} over W is defined as

$$\mathrm{tr}_W(T^{VW}) \triangleq \sum_i \alpha_i T_i^V \ \mathrm{tr}(T_i^W),$$

which is an operator on V.

Well-defined

- $\{E_{jj'}\}$: a basis of $L(W)$ $(=L(W,W))$.
- Unique representation : with $T_i^W = \sum_{jj'} \beta_{ijj'} E_{jj'}$, we have

$$T^{VW} = \sum_{jj'} \left(\sum_i \alpha_i \beta_{ijj'} T_i^V \right) \otimes E_{jj'}.$$

$$\begin{aligned} \mathrm{tr}_W(T^{VW}) &\stackrel{\triangle}{=} \sum_i \alpha_i T_i^V \ \mathrm{tr}(T_i^W) = \sum_i \alpha_i T_i^V \ \mathrm{tr} \left(\sum_{jj'} \beta_{ijj'} E_{jj'} \right) \\ &= \sum_{jj'} \left(\sum_i \alpha_i \beta_{ijj'} T_i^V \right) \ \mathrm{tr}(E_{jj'}) \end{aligned}$$

$$\triangleq \quad \mathrm{tr}_W \left(\sum_{jj'} \left(\sum_i \alpha_i \beta_{ijj'} T_i^V \right) \otimes E_{jj'} \right).$$

$\mathbf{tr}(T^{VW}) = \mathbf{tr}(\mathbf{tr}_W(T^{VW}))$

- V and W : vector spaces V and W.
- T^{VW} : a linear operator on the tensor product space $V \otimes W$

$$T^{VW} = \sum_i \alpha_i T_i^V \otimes T_i^W,$$

where T_i^V and T_i^W are linear operators on V and on W, respectively.

We have

$$\begin{aligned} \mathrm{tr}(T^{VW}) &= \sum_i \alpha_i \ \mathrm{tr}(T_i^V \otimes T_i^W) = \sum_i \alpha_i \ \mathrm{tr}(T_i^V) \ \mathrm{tr}(T_i^W) \\ &= \mathrm{tr}\left(\sum_i \alpha_i T_i^V \ \mathrm{tr}(T_i^W)\right) = \mathrm{tr}\left(\mathrm{tr}_W(T^{VW})\right). \end{aligned}$$

Trace as an Inner Product

Hilbert-Schmidt or Trace Inner Product

- $\mathcal{H}$: a finite-dimensional complex inner product space.
- $L(\mathcal{H})$: the vector space of all linear operators on $\mathcal{H}$.
- $(T, S) \stackrel{\triangle}{=} \operatorname{tr}(T^\dagger S)$: the Hilbert-Schmidt or trace inner product of T and S in $L(\mathcal{H})$.
  - $(T, T) = \operatorname{tr}(T^\dagger T) \geq 0$: $T^\dagger T$ is a positive operator.
    - $(T, T) = \operatorname{tr}(T^\dagger T) = 0 \iff$ all sigular values of T are zeros $\iff \operatorname{rank}(T)=0 \iff T = 0$.
  - $(T, S) = \operatorname{tr}(T^\dagger S) = \operatorname{tr}((S^\dagger T)^\dagger) = \overline{\operatorname{tr}(S^\dagger T)} = \overline{(S, T)}$: Hermitian symmetry.
  - $(T, \alpha_1 S_1 + \alpha_2 S_2) = \operatorname{tr}(T^\dagger(\alpha_1 S_1 + \alpha_2 S_2)) =$ $\alpha_1 \operatorname{tr}(T, S_1) + \alpha_2 \operatorname{tr}(T, S_2) = \alpha(T, S_1) + \alpha_2(T, S_2)$: linearity.
- $L(\mathcal{H})$: a complex inner product space with trace inner product.

$L^H(\mathcal{H})$ – the Set of all Hermitian Operators on $\mathcal{H}$

- $\mathcal{H}$: a finite-dimensional complex inner product space.
- $L^H(\mathcal{H})$: a real vector space.
- $\mathrm{tr}(TS)$: the trace inner product of T and S in $L^H(\mathcal{H})$.
- $L^H(\mathcal{H})$: a real inner product space.

Example

- $\mathcal{H}$: a two-dimensional complex inner product space.
- $\mathcal{B} = \{|0\rangle, |1\rangle\}$: an orthonormal basis of $\mathcal{H}$.
- $L^H(\mathcal{H})$: the real inner product space of all Hermitian operators on $\mathcal{H}$ with trace inner product.
- $\{\sigma_0, \sigma_x, \sigma_y, \sigma_z\}$: the Pauli matrices, which form an orthonormal basis of $L^H(\mathcal{H})$.

$$I = [\sigma_0]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$X = [\sigma_x]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$
\begin{aligned}
Y &= [\sigma_y]_{\mathcal{B}} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \\
Z &= [\sigma_z]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
\end{aligned}
$$

Singular Value Decomposition

Singular Values

- H_1, H_2 : finite-dimensional complex inner product spaces.
- T : a linear transformation from H_1 to H_2.
- $T^\dagger T$: a positive operator on H_1.
- $T^\dagger T = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$: spectral decomposition
  - λ_i : non-negative eigenvalues of $T^\dagger T$.
  - $\{|\psi_i\rangle\}$: an orthonormal eigenbasis of V.
- $\sqrt{\lambda_i}$: singular values of T.

The Action of T on the Eigenbasis $\{|\psi_i\rangle\}$ of $T^\dagger T$

$$(T|\psi_i\rangle, T|\psi_j\rangle) = \langle\psi_i|T^\dagger T|\psi_j\rangle = \lambda_j\langle\psi_i|\psi_j\rangle = \lambda_j\delta_{ij}.$$

- $\{T|\psi_i\rangle\}$: orthogonal vectors in H_2.
- $||T|\psi_i\rangle||^2 = \lambda_i$.
- Number of non-zero λ_i = the rank of T.

Singular Value Decomposition of T

- r : the rank of T.
- $\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_r}$: non-zero singular values of T.
- $\{T|\psi_1\rangle/\sqrt{\lambda_1}, \ldots, T|\psi_r\rangle/\sqrt{\lambda_r}\}$: an orthonormal set in H_2.
- $\{|\varphi_j\rangle\}$: an orthonormal basis of H_2 with

$$|\varphi_j\rangle = T|\psi_j\rangle/\sqrt{\lambda_j}, \forall\ 1 \leq j \leq r.$$

$$T = \sum_{j=1}^{r} \sqrt{\lambda_j}|\varphi_j\rangle\langle\psi_j|.$$

Matrix Version

- $\mathcal{B} = \{|i\rangle\}, \mathcal{B}' = \{|j\rangle\}$: orthonormal bases of H_1 and H_2.

- $A = [T]_{\mathcal{B}\to\mathcal{B}'}$: matrix representation of T relative to the bases $\mathcal{B}$ and $\mathcal{B}'$.

- $D = [T]_{\{|\psi_i\rangle\}\to\{|\varphi_j\rangle\}}$: matrix representation of T relative to the bases $\{|\psi_i\rangle\}$ and $\{|\varphi_j\rangle\}$

$$D = \mathrm{diag}(\sqrt{\lambda_i}).$$

- $M = [\{|\psi_i\rangle\} \to \mathcal{B}]$: coordinate transformation, a unitary matrix.

- $N = [\{|\varphi_j\rangle \to \mathcal{B}'\}]$: coordinate transformation, a unitary matrix.

$$[T]_{\mathcal{B}\to\mathcal{B}'} = [\{|\varphi_j\rangle \to \mathcal{B}'\}][T]_{\{|\psi_i\rangle\}\to\{|\varphi_j\rangle\}}[\mathcal{B} \to \{|\psi_i\rangle\}]$$

$$A = NDM^\dagger.$$