

**A Short Course on Quantum Information and
Computation**

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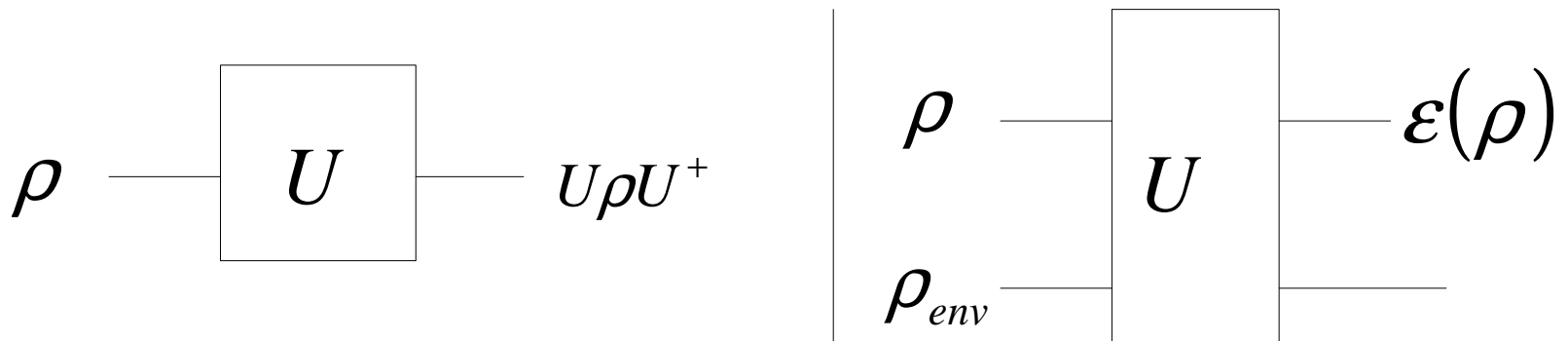
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Unit Three – Quantum Operations

Open Quantum Systems



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- Postulate 2 : the dynamics of a closed quantum system is described by a unitary transform
- An open quantum system together with its environment becomes a closed quantum system
- ρ : the density operator of the open quantum system, called the *principal* quantum system
- ρ_{env} : the density operator of the *environment*

- U : a unitary operator on the state space of the closed quantum system
- $\mathcal{E}(\rho)$: the density operator of the principal quantum system after the action of the unitary operator U

- Closed system

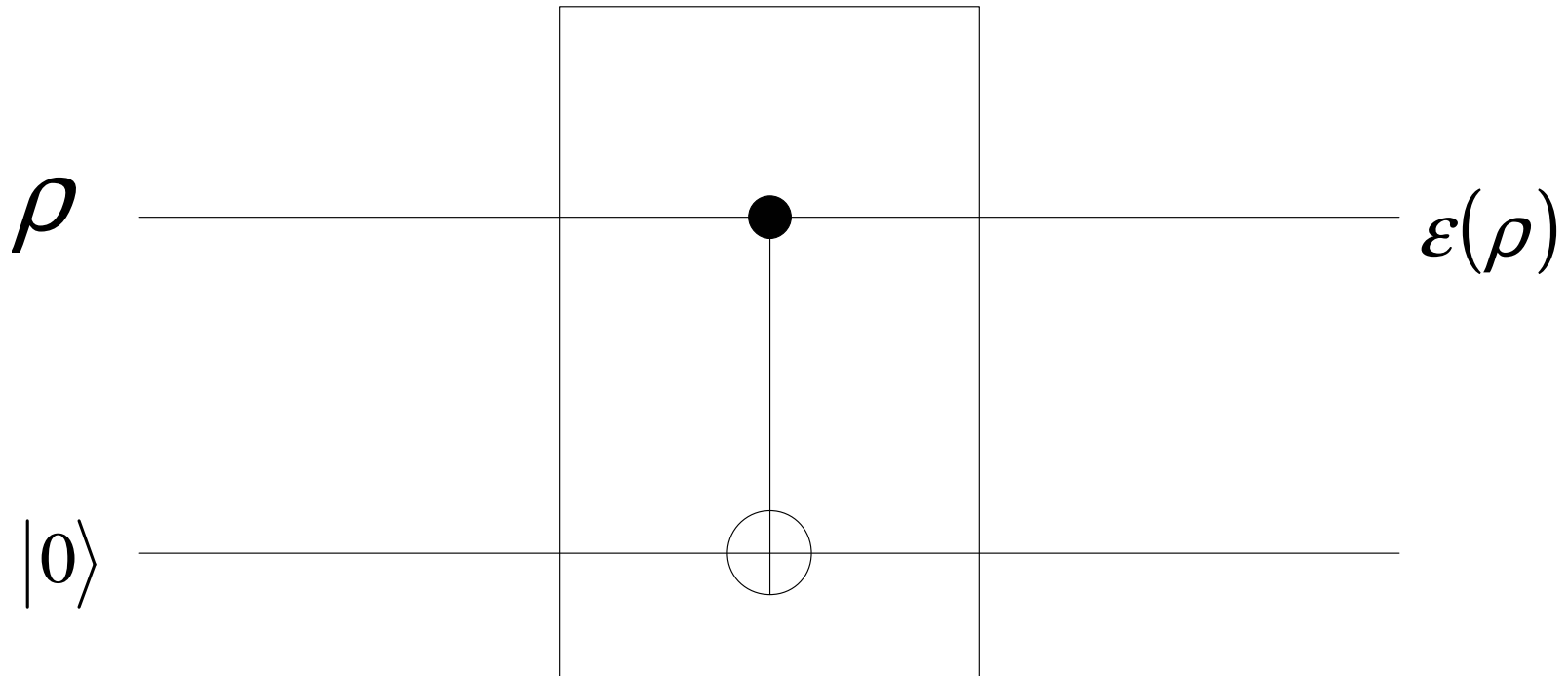
$$\mathcal{E}(\rho) = U\rho U^\dagger$$

- Open system

$$\mathcal{E}(\rho) = \text{tr}_{env} (U (\rho \otimes \rho_{env}) U^\dagger)$$

- * Assume that the principal quantum system is prepared such that its correlation with the environment can be completely destroyed (Correlated initial state of the principal-environmental system will be discussed later)
- * If the state space of the principal system has dimension d , we will show that it is sufficient to model the environment to have state space of dimension no greater than d^2

An Example



- $\rho = \sum_{ij} \alpha_{ij} |i\rangle\langle j|$: the density operator of the principal system
- $\rho_{env} = |0\rangle\langle 0|$: the environment is in the pure state $|0\rangle$
- $\mathcal{E}(\rho) = P_0\rho P_0 + P_1\rho P_1$: $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$ are projective operators

$$\begin{aligned}
\mathcal{E}(\rho) &= \text{tr}_2 \left(U \left(\sum_{ij} \alpha_{ij} |i\rangle\langle j| \otimes |0\rangle\langle 0| \right) U^\dagger \right) \\
&= \text{tr}_2 (\alpha_{00} |0\rangle\langle 0| \otimes |0\rangle\langle 0| + \alpha_{10} |1\rangle\langle 0| \otimes |1\rangle\langle 0| + \\
&\quad \alpha_{01} |0\rangle\langle 1| \otimes |0\rangle\langle 1| + \alpha_{11} |1\rangle\langle 1| \otimes |1\rangle\langle 1|) \\
&= \alpha_{00} |0\rangle\langle 0| + \alpha_{11} |1\rangle\langle 1| \\
P_0 \rho P_0 &= \alpha_{00} |0\rangle\langle 0| \\
P_1 \rho P_1 &= \alpha_{11} |1\rangle\langle 1| \\
\mathcal{E}(\rho) &= P_0 \rho P_0 + P_1 \rho P_1
\end{aligned}$$

Quantum Operations Formalism

The Input-Output Formalism

$$\mathcal{E}(\rho^A) = \text{tr}_B (U (\rho^A \otimes \rho^B) U^\dagger)$$

or

$$\mathcal{E}(\rho^A) = \text{tr}_A (U (\rho^A \otimes \rho^B) U^\dagger)$$

- ρ^A : the input density operator of quantum systems A
- ρ^B : the density operators of quantum system B
- $\mathcal{E}(\rho^A)$: the output density operator of quantum systems A or B

A General Definition of Triplet $\langle \psi|U|\varphi \rangle$

- H_1 and H_2 : complex inner product spaces
- U : a linear operator on the tensor product space $H_1 \otimes H_2$

$$U = \sum_i \alpha_i T_i^{H_1} \otimes T_i^{H_2},$$

where $T_i^{H_1}$ and $T_i^{H_2}$ are linear operators on H_1 and H_2 respectively

- $|\psi\rangle$ and $|\varphi\rangle$: two vectors in H_2
- $\langle \psi|U|\varphi \rangle$: a linear operator on H_1

$$\langle \psi|U|\varphi \rangle \triangleq \sum_i \alpha_i T_i^{H_1} \langle \psi|T_i^{H_2}|\varphi \rangle$$

- $\langle \psi| \cdot |\varphi \rangle$: a linear map from $L(H_1 \otimes H_2, H_1 \otimes H_2)$ to $L(H_1, H_1)$

Well-defined

- $\{|j\rangle\}$: an orthonormal basis of H_2
- $\{|j\rangle\langle k|\}$: a basis of $L(H_2, H_2)$
- Unique representation : with $T_i^{H_2} = \sum_{jk} \beta_{ijk} |j\rangle\langle k|$, we have

$$U = \sum_{jk} \left(\sum_i \alpha_i \beta_{ijk} T_i^{H_1} \right) \otimes |j\rangle\langle k|$$

Thus we have

$$\begin{aligned} \langle \psi | U | \varphi \rangle &\stackrel{\Delta}{=} \sum_i \alpha_i T_i^{H_1} \langle \psi | T_i^{H_2} | \varphi \rangle = \sum_i \alpha_i T_i^{H_1} \sum_{jk} \beta_{ijk} \langle \psi | j \rangle \langle k | \varphi \rangle \\ &= \sum_{jk} \left(\sum_i \alpha_i \beta_{ijk} T_i^{H_1} \right) \langle \psi | j \rangle \langle k | \varphi \rangle. \end{aligned}$$

A Theorem

$$\text{tr}_2(U) = \sum_i \langle i|U|i\rangle$$

- H_1 and H_2 : complex inner product spaces
- U : a linear operator on the tensor product space $H_1 \otimes H_2$
- $\{|i\rangle\}$: an orthonormal basis of H_2

Proof

$$\begin{aligned}\mathrm{tr}_2(U) &= \mathrm{tr}_2 \left(\sum_j \alpha_j T_j^{H_1} \otimes T_j^{H_2} \right) \\ &= \sum_j \alpha_j T_j^{H_1} \mathrm{tr}(T_j^{H_2}) \\ &= \sum_j \alpha_j T_j^{H_1} \sum_i \langle i | T_j^{H_2} | i \rangle \\ &= \sum_i \sum_j \alpha_j T_j^{H_1} \langle i | T_j^{H_2} | i \rangle \\ &= \sum_i \langle i | \left(\sum_j \alpha_j T_j^{H_1} \otimes T_j^{H_2} \right) | i \rangle \\ &= \sum_i \langle i | U | i \rangle\end{aligned}$$

A General Definition of Inner Product $\langle \psi | \varphi \rangle$

- H_1 and H_2 : complex inner product spaces
- $|\psi\rangle$: a vectors in H_1
- $|\varphi\rangle = \sum_i \alpha_i |v_i\rangle \otimes |w_i\rangle$: a vector in $H_1 \otimes H_2$
- $\langle \psi | \varphi \rangle$: a vector in H_2

$$\langle \psi | \varphi \rangle \triangleq \sum_i \alpha_i \langle \psi | v_i \rangle |w_i\rangle$$

- $\langle \psi | \cdot \rangle$: a linear transformation from $H_1 \otimes H_2$ to H_2

Well-defined

- $\{|j\rangle\}$: an orthonormal basis of H_2
- Unique representation : with $|w_i\rangle = \sum_j \beta_{ij}|j\rangle$, we have

$$|\varphi\rangle = \sum_j \left(\sum_i \alpha_i \beta_{ij} |v_i\rangle \right) \otimes |j\rangle$$

Thus we have

$$\begin{aligned} \langle \psi | \varphi \rangle &\triangleq \sum_i \alpha_i \langle \psi | v_i \rangle |w_i\rangle \\ &= \sum_i \alpha_i \langle \psi | v_i \rangle \sum_j \beta_{ij} |j\rangle \\ &= \sum_j \langle \psi | \left(\sum_i \alpha_i \beta_{ij} |v_i\rangle \right) |j\rangle \end{aligned}$$

The Operator-Sum Representation of \mathcal{E}

$$\mathcal{E}(\rho) = \sum_k \langle e_k | U(\rho \otimes \rho_{env}) U^\dagger | e_k \rangle = \sum_{km} E_{km} \rho E_{km}^\dagger$$

where $E_{km} = \langle e_k | U \sqrt{\lambda_m} | \psi_m \rangle$

- ρ : the density operator of the principal quantum system
- $\{|e_k\rangle\}$: an orthonormal basis of the (finite-dimensional) state space of the environment
- $\rho_{env} = \sum_m \lambda_m |\psi_m\rangle \langle \psi_m|$: the density operator of the environment with ensemble $\{\lambda_m, |\psi_m\rangle\}$
- $\{E_{km}\}$: operation elements for the quantum operation \mathcal{E}

Proof

- $U = \sum_i \alpha_i T_i^{pri} \otimes T_i^{env}$
- $\rho_{env} = \sum_m \lambda_m |\psi_m\rangle\langle\psi_m|$

$$\begin{aligned}
 & \mathcal{E}(\rho) \\
 = & \text{tr}_{env} (U (\rho \otimes \rho_{env}) U^\dagger) \\
 = & \sum_k \langle e_k | \left(\sum_i \alpha_i T_i^{pri} \otimes T_i^{env} \right) (\rho \otimes \rho_{env}) \left(\sum_j \alpha_j T_j^{pri} \otimes T_j^{env} \right)^\dagger | e_k \rangle \\
 = & \sum_k \sum_{ij} \alpha_i \alpha_j^* \langle e_k | \left(T_i^{pri} \rho T_j^{pri\dagger} \otimes T_i^{env} \rho_{env} T_j^{env\dagger} \right) | e_k \rangle \\
 = & \sum_k \sum_{ij} \alpha_i \alpha_j^* T_i^{pri} \rho T_j^{pri\dagger} \langle e_k | T_i^{env} \rho_{env} T_j^{env\dagger} | e_k \rangle \\
 = & \sum_k \sum_{ij} \alpha_i \alpha_j^* T_i^{pri} \rho T_j^{pri\dagger} \sum_m \lambda_m \langle e_k | T_i^{env} | \psi_m \rangle \langle \psi_m | T_j^{env\dagger} | e_k \rangle
 \end{aligned}$$

$$\begin{aligned}
&= \sum_k \sum_m \lambda_m \left(\sum_i \alpha_i T_i^{pri} \langle e_k | T_i^{env} | \psi_m \rangle \right) \rho \left(\sum_j \alpha_j T_j^{pri} \langle e_k | T_j^{env} | \psi_m \rangle \right)^\dagger \\
&= \sum_k \sum_m \lambda_m \langle e_k | \sum_i \alpha_i T_i^{pri} \otimes T_i^{env} | \psi_m \rangle \rho \left(\langle e_k | \sum_j \alpha_j T_j^{pri} \otimes T_j^{env} | \psi_m \rangle \right)^\dagger \\
&= \sum_k \sum_m \langle e_k | U \sqrt{\lambda_m} | \psi_m \rangle \rho \left(\langle e_k | U \sqrt{\lambda_m} | \psi_m \rangle \right)^\dagger \\
&= \sum_k \sum_m E_{km} \rho E_{km}^\dagger
\end{aligned}$$

where $E_{km} = \langle e_k | U \sqrt{\lambda_m} | \psi_m \rangle$

Lemma

Let T be a linear operator on a Hilbert space H . If

$$\text{tr}(T\rho) = 1$$

for any density operators ρ on H , then we have

$$T = I$$

Proof.

Note that $\text{tr}(T\rho) = 1$ for any density operator ρ on H if and only if $\text{tr}(T|\psi\rangle\langle\psi|) = 1$, i.e. $\langle\psi|T|\psi\rangle = 1$, for any unit vector $|\psi\rangle$ in H . Let $\{|e_k\rangle\}$ be an orthonormal basis of H . For any $i \neq j$ and any non-zero complex numbers a, b such that $|a|^2 + |b|^2 = 1$, we have

$$1 = (a|e_i\rangle + b|e_j\rangle)^\dagger T(a|e_i\rangle + b|e_j\rangle)$$

which implies that

$$a^*b\langle e_i|T|e_j\rangle + ab^*\langle e_j|T|e_i\rangle = 0.$$

By taking a, b both real, we have

$$\langle e_i|T|e_j\rangle + \langle e_j|T|e_i\rangle = 0.$$

But by taking a real and b pure imaginary, we have

$$\langle e_i|T|e_j\rangle - \langle e_j|T|e_i\rangle = 0.$$

Thus we conclude that $\langle e_i|T|e_j\rangle = \langle e_j|T|e_i\rangle = 0$ and the matrix representation of T relative to the orthonormal basis $\{|e_k\rangle\}$ is the identity matrix which implies that $T = I$.

Completeness Relation on Operation Elements

$$\sum_{km} E_{km}^\dagger E_{km} = I$$

- \mathcal{E} is trace-preserving : for any density operator ρ ,

$$\begin{aligned}\text{tr}(\mathcal{E}(\rho)) &= \text{tr}(\text{tr}_{env}(U(\rho \otimes \rho_{env})U^\dagger)) = \text{tr}(U(\rho \otimes \rho_{env})U^\dagger) \\ &= \text{tr}(U^\dagger U(\rho \otimes \rho_{env})) = \text{tr}(\rho \otimes \rho_{env}) = \text{tr}(\rho)\text{tr}(\rho_{env}) \\ &= 1\end{aligned}$$

- $1 = \text{tr}(\mathcal{E}(\rho)) = \sum_{km} \text{tr}(E_{km}\rho E_{km}^\dagger) = \sum_{km} \text{tr}(E_{km}^\dagger E_{km}\rho) = \text{tr}((\sum_{km} E_{km}^\dagger E_{km})\rho)$ for any density operator ρ

Purification of the Environment

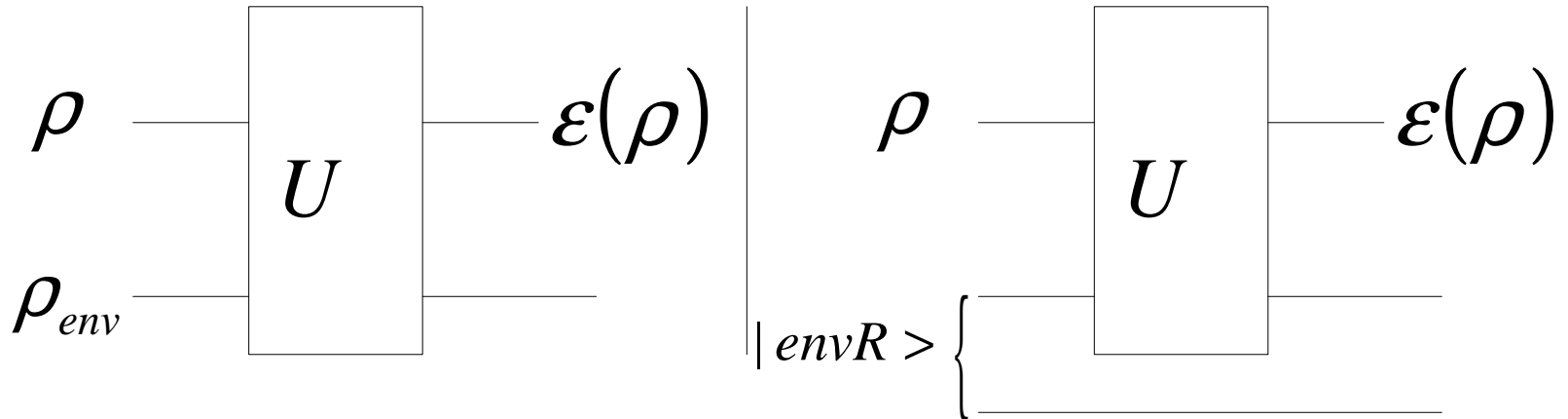
- ρ : the density operator of the principal quantum system
- $\rho_{env} = \sum_m \lambda_m |\psi_m\rangle\langle\psi_m|$: the density operator of the environment with ensemble $\{\lambda_m, |\psi_m\rangle\}$
- $\{|m_R\rangle\}$: an orthonormal basis of the state space of a reference system R , having the same cardinality as that of $\{|\psi_m\rangle\}$
- $|envR\rangle$: a pure state of the composite environment- R system

$$|envR\rangle = \sum_m \sqrt{\lambda_m} |\psi_m\rangle |m_R\rangle$$

such that

$$\rho_{env} = \text{tr}_R(|envR\rangle\langle envR|)$$

Purification of the Environment (Cont')



$$\begin{aligned} \mathcal{E}(\rho) &= \text{tr}_{env} (U (\rho \otimes \rho_{env}) U^\dagger) \\ &= \text{tr}_{envR} ((U \otimes I_R) (\rho \otimes |envR\rangle\langle envR|) (U \otimes I_R)^\dagger) \end{aligned}$$

Proof.

Note that for an orthonormal basis $\{|e_k\rangle\}$ of the state space of the environment, $\{|e_k\rangle|m_R\rangle\}$ is an orthonormal basis of the state space

of the composite environment- R system. We have

$$\mathrm{tr}_{envR} \left((U \otimes I_R) (\rho \otimes |envR\rangle\langle envR|) (U \otimes I_R)^\dagger \right) = \sum_{km} F_{km} \rho F_{km}^\dagger$$

where

$$\begin{aligned} F_{km} &= \langle e_k | \langle m_R | (U \otimes I_R) | envR \rangle \\ &= \langle e_k | \langle m_R | (U \otimes I_R) \sum_j \sqrt{\lambda_j} |\psi_j\rangle |j_R\rangle \\ &= \sum_j \sqrt{\lambda_j} \langle e_k | \langle m_R | (U \otimes I_R) |\psi_j\rangle |j_R\rangle \\ &= \sum_j \sqrt{\lambda_j} \langle e_k | U |\psi_j\rangle \langle m_R | j_R \rangle \\ &= \langle e_k | U \sqrt{\lambda_m} |\psi_m\rangle \\ &= E_{km} \end{aligned}$$

Three Features of the Operator-Sum Representation

Physical Interpretation

- ρ : the density operator of the principal quantum system
- $\rho_{env} = \sum_m \lambda_m |\psi_m\rangle\langle\psi_m|$: the density operator of the environment
- $\{|e_k\rangle\}$: an orthonormal basis of the state space of the environment
- Principle of implicit measurement : the state of the principal system will not be affected if measurement is performed on the environment
- $\{|e_k\rangle\langle e_k|\}$: a projective measurement on the environment
- ρ_k : the state of the principal system given that outcome k occurs

$$\begin{aligned}
\rho_k &= \text{tr}_{env} \left(\frac{(I \otimes |e_k\rangle\langle e_k|)U(\rho \otimes \rho_{env})U^\dagger(I \otimes |e_k\rangle\langle e_k|)}{\text{tr}((I \otimes |e_k\rangle\langle e_k|)U(\rho \otimes \rho_{env})U^\dagger(I \otimes |e_k\rangle\langle e_k|))} \right) \\
&= \text{tr}_{env} \left(\frac{\langle e_k|U(\rho \otimes \rho_{env})U^\dagger|e_k\rangle \otimes |e_k\rangle\langle e_k|}{\text{tr}(\langle e_k|U(\rho \otimes \rho_{env})U^\dagger|e_k\rangle \otimes |e_k\rangle\langle e_k|)} \right) \\
&= \frac{\langle e_k|U(\rho \otimes \rho_{env})U^\dagger|e_k\rangle \text{tr}(|e_k\rangle\langle e_k|)}{\text{tr}(\langle e_k|U(\rho \otimes \rho_{env})U^\dagger|e_k\rangle) \text{tr}(|e_k\rangle\langle e_k|)} \\
&= \frac{\sum_m E_{km} \rho E_{km}^\dagger}{\text{tr}(\sum_m E_{km} \rho E_{km}^\dagger)}
\end{aligned}$$

where $E_{km} = \langle e_k|U\sqrt{\lambda_m}|\psi_m\rangle$

- $\mathcal{P}(k)$: the probability that outcome k occurs

$$\begin{aligned}
\mathcal{P}(k) &= \text{tr}((I \otimes |e_k\rangle\langle e_k|)U(\rho \otimes \rho_{env})U^\dagger(I \otimes |e_k\rangle\langle e_k|)) \\
&= \text{tr}\left(\sum_m E_{km} \rho E_{km}^\dagger\right)
\end{aligned}$$

$$\mathcal{E}(\rho) = \sum_k \mathcal{P}(k) \rho_k = \sum_{km} E_{km} \rho E_{km}^\dagger$$

- The action of the quantum operation \mathcal{E} is equivalent to taking the state ρ as input and randomly replacing it by

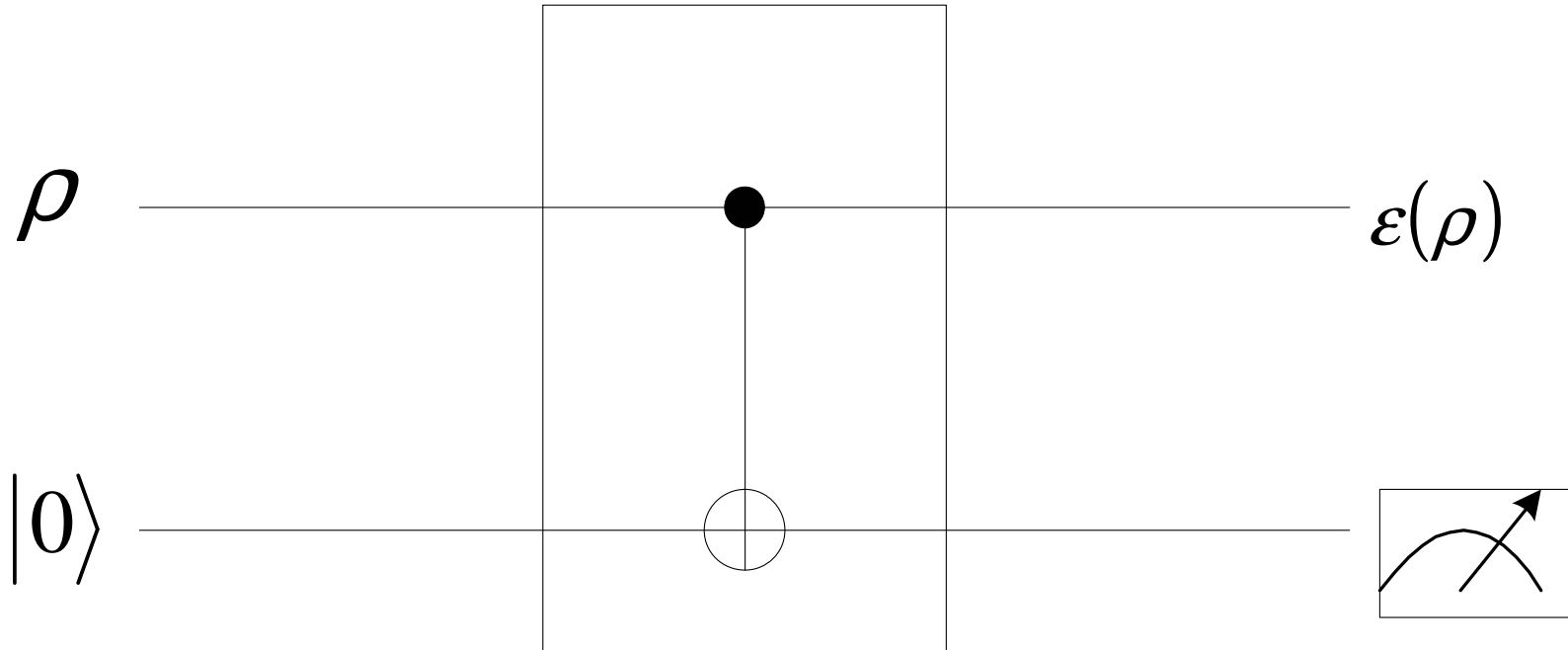
$$\frac{\sum_m E_{km} \rho E_{km}^\dagger}{\text{tr}(\sum_m E_{km} \rho E_{km}^\dagger)}$$

with probability

$$\text{tr}\left(\sum_m E_{km} \rho E_{km}^\dagger\right)$$

- A quantum operation which describes a quantum noise process will be referred to as a noisy quantum channel

An Example (Revisited)



- ρ : the density operator of the principal system
- $\rho_{env} = |0_E\rangle\langle 0_E|$: the environment is in the pure state $|0_E\rangle$
- $U =$
 $|0_P0_E\rangle\langle 0_P0_E| + |0_P1_E\rangle\langle 0_P1_E| + |1_P1_E\rangle\langle 1_P0_E| + |1_P0_E\rangle\langle 1_P1_E|$

- $\{|0_E\rangle, |1_E\rangle\}$: an orthonormal basis of the state space of the environment
- $E_0 = \langle 0_E | U | 0_E \rangle = |0_P\rangle \langle 0_P| = P_0$
- $E_1 = \langle 1_E | U | 0_E \rangle = |1_P\rangle \langle 1_P| = P_1$
- $\mathcal{E}(\rho) = P_0 \rho P_0 + P_1 \rho P_1$

Effect of Global Measurement

- ρ : the density operator of the principal quantum system
- $\rho_{env} = \sum_j \lambda_j |\psi_j\rangle\langle\psi_j|$: the density operator of the environment
- $\{|e_k\rangle\}$: an orthonormal basis of the state space of the environment
- $\{P_m\}$: projective measurement after the unitary operation U
- The state of the principal system given that outcome m occurs is

$$\text{tr}_{env} \left(\frac{P_m U (\rho \otimes \rho_{env}) U^\dagger P_m}{\text{tr}(P_m U (\rho \otimes \rho_{env}) U^\dagger P_m)} \right)$$

with probability

$$\text{tr}(P_m U (\rho \otimes \rho_{env}) U^\dagger P_m)$$

Define a map

$$\mathcal{E}_m(\rho) \triangleq \text{tr}_{env}(P_m U(\rho \otimes \rho_{env}) U^\dagger P_m) = \sum_{kj} E_{kj}^{(m)} \rho E_{kj}^{(m)\dagger}$$

where

$$E_{kj}^{(m)} = \langle e_k | P_m U \sqrt{\lambda_j} | \psi_j \rangle$$

- The state of the principal system given that outcome m occurs is

$$\frac{\mathcal{E}_m(\rho)}{\text{tr}(\mathcal{E}_m(\rho))}$$

with probability $\text{tr}(\mathcal{E}_m(\rho))$

- $\{\mathcal{E}_m(\rho)\}$: a kind of measurement process which generalizes the measurement described in Unit One where $\mathcal{E}_m(\rho) = E_m \rho E_m^\dagger$ for a quantum measurement $\{E_m\}$

The Converse Problem

- $\{E_k\}$: a given collection of operator elements acting on a *principal* quantum system and satisfying the completeness relation

$$\sum_k E_k^\dagger E_k = I$$

- The problem : find a system-environment model, i.e., an environment, a unitary operator U on the composite system-environment model such that $\{E_k\}$ is the operator elements in the operator-sum representation of the quantum operation \mathcal{E}

A System-Environment Model

- $\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$: the density operator of the principal system
- $\{|e_k\rangle\}$: an orthonormal basis of the state space of a chosen environment, having the same cardinality as that of $\{E_k\}$
- $|0\rangle$: an arbitrarily chosen state of the environment
- U : a linear operator acting on the states of the form $|\psi\rangle|0\rangle$ where $|\psi\rangle$ is any state of the principal system

$$U|\psi\rangle|0\rangle = \sum_k E_k |\psi\rangle|e_k\rangle$$

and for any states $|\psi\rangle$ and $|\varphi\rangle$ of the principal system,

$$\langle\psi|\langle 0|U^\dagger U|\varphi\rangle|0\rangle = \sum_{kj} (E_k |\psi\rangle|e_k\rangle)^\dagger (E_j |\varphi\rangle|e_j\rangle)$$

$$= \sum_{kj} \langle \psi | E_k^\dagger E_j | \varphi \rangle \langle e_k | e_j \rangle = \langle \psi | \sum_k E_k^\dagger E_k | \varphi \rangle = \langle \psi | \varphi \rangle$$

Thus U can be extended to a unitary operator

$U = \sum_i \alpha_i T_i^{pri} \otimes T_i^{env}$ on the composite system and we have for any state $|\psi\rangle$ of the principal system

$$\begin{aligned} (\langle e_k | U | 0 \rangle) |\psi\rangle &= \sum_i \alpha_i T_i^{pri} \langle e_k | T_i^{env} | 0 \rangle |\psi\rangle \\ &= \sum_i \alpha_i T_i^{pri} |\psi\rangle \langle e_k | T_i^{env} | 0 \rangle \\ &= \langle e_k | \left(\sum_i \alpha_i T_i^{pri} \otimes T_i^{env} \right) |\psi\rangle | 0 \rangle \\ &= \langle e_k | U |\psi\rangle | 0 \rangle = \langle e_k | \sum_j E_j |\psi\rangle | e_j \rangle = E_k |\psi\rangle \end{aligned}$$

which says that $E_k = \langle e_k | U | 0 \rangle$

Axiomatic Approach to Quantum Operations

Formal Definition of Quantum Operations

A quantum operation \mathcal{E} is a map from the set of density operators of the input space Q_1 to the set of positive operators of the output space Q_2 , satisfying the three axiomatic properties as follows.

Three Axioms

- Axiom I : $\text{tr}(\mathcal{E}(\rho))$ is the probability that the process represented by \mathcal{E} occurs, when ρ is the input density operator,

$$0 \leq \text{tr}(\mathcal{E}(\rho)) \leq 1$$

- Axiom II : \mathcal{E} is a convex-linear map on the set of density operators, i.e., for non-negative numbers λ_i with $\sum_i \lambda_i = 1$ and density operators ρ_i , $\mathcal{E}(\sum_i \lambda_i \rho_i) = \sum_i \lambda_i \mathcal{E}(\rho_i)$
- Axiom III : \mathcal{E} is a completely positive map, i.e., for an arbitrarily introduced system R of arbitrary dimension and the identity map \mathcal{I} on the set of all linear operators on R , $I \otimes \mathcal{E}$ is a well-defined map from the set of positive operators of the composite system RQ_1 to the set of positive operators of the composite system RQ_2

Axiom I

$\text{tr}(\mathcal{E}(\rho))$ is the probability that the process represented by \mathcal{E} occurs, when ρ is the input density operator,

$$0 \leq \text{tr}(\mathcal{E}(\rho)) \leq 1$$

- This is a mathematical convenience to include the case of measurement as quantum operation, where the trace may not be preserved and $\text{tr}(\mathcal{E}(\rho))$ is exactly the probability of the occurrence of a particular measurement outcome when the state before measurement is ρ . The state after measurement becomes $\mathcal{E}(\rho)/\text{tr}(\mathcal{E}(\rho))$

Axiom II

\mathcal{E} is a convex-linear map on the set of density operators, i.e., for non-negative numbers p_i with $\sum_i p_i = 1$ and density operators ρ_i ,

$$\mathcal{E}\left(\sum_i p_i \rho_i\right) = \sum_i p_i \mathcal{E}(\rho_i)$$

- $\rho = \sum_i p_i \rho_i$: the input quantum state as a random selection from the ensemble $\{p_i, \rho_i\}$ of quantum (mixed) states
- $\mathcal{E}(\rho)/\text{tr}(\mathcal{E}(\rho)) = \mathcal{E}(\rho)/p(\mathcal{E})$: the resulting state as a random selection from the ensemble $\{p(i|\mathcal{E}), \mathcal{E}(\rho_i)/\text{tr}(\mathcal{E}(\rho_i))\}$ of quantum (mixed) states, where $p(i|\mathcal{E})$ is the probability that the state prepared is ρ_i , given that the process described by \mathcal{E} occurs, i.e., we demand

$$\frac{\mathcal{E}(\rho)}{p(\mathcal{E})} = \sum_i p(i|\mathcal{E}) \frac{\mathcal{E}(\rho_i)}{\text{tr}(\mathcal{E}(\rho_i))}$$

- A Bayesian rule :

$$p(i|\mathcal{E}) = \frac{p(\mathcal{E}|i)p_i}{p(\mathcal{E})} = \frac{\text{tr}(\mathcal{E}(\rho_i))p_i}{p(\mathcal{E})}$$

- Justification :

$$\mathcal{E}(\rho) = p(\mathcal{E}) \sum_i p(i|\mathcal{E}) \frac{\mathcal{E}(\rho_i)}{\text{tr}(\mathcal{E}(\rho_i))} = \sum_i p_i \mathcal{E}(\rho_i)$$

Axiom III

\mathcal{E} is a completely positive map, i.e., for an arbitrarily introduced system R of arbitrary dimension and the identity map \mathcal{I} on the set of all linear operators on R , $\mathcal{I} \otimes \mathcal{E}$ is a well-defined map from the set of positive operators of the composite system RQ_1 to the set of positive operators of the composite system RQ_2

- It is required for a physical system that if ρ^{RQ_1} is a (mixed) state of a composite system RQ_1 and the quantum operation \mathcal{E} acts solely on the system Q_1 , then the result $\mathcal{I} \otimes \mathcal{E}(\rho^{RQ_1})$ must also be a state (up to a normalization factor) of the composite system RQ_2

Theorem

A map \mathcal{E} from the set of density operators of the input space Q_1 to the set of positive operators of the output space Q_2 satisfies the above three axiomatic properties if and only if

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger \quad (1)$$

for a collection of linear transformations E_k from the input space Q_1 to the output space Q_2 and

$$\sum_k E_k^\dagger E_k \leq I.$$

Furthermore, \mathcal{E} is trace-preserving, i.e., $\mathcal{E}(\rho)$ is a density operator for any density operator ρ if and only if $\sum_k E_k^\dagger E_k = I$.

Proof \Leftarrow

From (1), it is clear that $\mathcal{E}(\rho)$ is a positive operator on Q_2 for any density operator ρ on Q_1 and \mathcal{E} is a linear map.

- Axiom I : for a density operator $\rho = \sum_j \lambda_j |j\rangle\langle j|$ on Q_1 , we have

$$\begin{aligned} 0 \leq \text{tr}(\mathcal{E}(\rho)) &= \sum_k \text{tr}(E_k \rho E_k^\dagger) = \sum_k \sum_j \lambda_j \text{tr}(E_k |j\rangle\langle j| E_k^\dagger) \\ &= \sum_k \sum_j \lambda_j \langle j| E_k^\dagger E_k |j\rangle = \sum_j \lambda_j \langle j| (\sum_k E_k^\dagger E_k) |j\rangle \\ &\leq \sum_j \lambda_j \langle j|j\rangle = 1 \end{aligned}$$

since $\sum_k E_k^\dagger E_k \leq I$

- Axiom II : it is clear from the linearity of \mathcal{E}

- Axiom III : for any positive operator $B = \sum_i \alpha_i T_i^R \otimes T_i^{Q_1}$ on the composite system RQ_1 , we define

$$\begin{aligned}
& (\mathcal{I} \otimes \mathcal{E})(B) \\
& \triangleq \sum_i \alpha_i \mathcal{I}(T_i^R) \otimes \left(\sum_k E_k T_i^{Q_1} E_k^\dagger \right) \\
& = \sum_k \sum_i \alpha_i (I_R T_i^R I_R) \otimes (E_k T_i^{Q_1} E_k^\dagger) \\
& = \sum_k \sum_i \alpha_i (I_R \otimes E_k) (T_i^R \otimes T_i^{Q_1}) (I_R \otimes E_k^\dagger) \\
& = \sum_k (I_R \otimes E_k) B (I_R \otimes E_k^\dagger)
\end{aligned}$$

which is clearly well-defined .

Let $|\psi\rangle$ be a state of the composite system RQ_2 and let $|\varphi_k\rangle = (I_R \otimes E_k^\dagger)|\psi\rangle$ for all k , we have

$$\begin{aligned} \langle\psi|(\mathcal{I} \otimes \mathcal{E})(B)|\psi\rangle &= \sum_k \langle\psi|(I_R \otimes E_k)B(I_R \otimes E_k^\dagger)|\psi\rangle \\ &= \sum_k \langle\varphi_k|B|\varphi_k\rangle \geq 0 \end{aligned}$$

which implies that $(\mathcal{I} \otimes \mathcal{E})(B)$ is a positive operator on RQ_2

Proof \implies

- R : an arbitrarily introduced space, with the same dimension as Q_1
- $\{|i_R\rangle\}$ and $\{|i_{Q_1}\rangle\}$: orthonormal bases for R and Q_1 respectively
- $|u\rangle = \sum_i |i_R\rangle|i_{Q_1}\rangle$: a (maximally entangled) vector in the composite system RQ_1
- $\mathcal{I} \otimes \mathcal{E}$: a map from the set of positive operators on RQ_1 to the set of positive operators on RQ_2 by the complete positivity of \mathcal{E} from Axiom III
- $\sigma = (\mathcal{I} \otimes \mathcal{E})(|u\rangle\langle u|)$: a positive operator on RQ_2

$$\sigma = \sum_{ij} |i_R\rangle\langle j_R| \otimes \mathcal{E}(|i_{Q_1}\rangle\langle j_{Q_1}|)$$

- It will be seen that the positive operator σ completely specifies the quantum operation \mathcal{E}
- $|v\rangle = \sum_i \alpha_i |i_{Q_1}\rangle$: a vector in Q_1
- $|\tilde{v}\rangle = \sum_i \alpha_i^* |i_R\rangle$: a vector in R corresponding to the vector $|v\rangle$ in Q_1
- An identity : a strong connection between σ and \mathcal{E}

$$\begin{aligned}
\langle \tilde{v} | \sigma | \tilde{v} \rangle &= \langle \tilde{v} | \left(\sum_{ij} |i_R\rangle \langle j_R| \otimes \mathcal{E}(|i_{Q_1}\rangle \langle j_{Q_1}|) \right) | \tilde{v} \rangle \\
&= \sum_{ij} \langle \tilde{v} | i_R \rangle \langle j_R | \tilde{v} \rangle \mathcal{E}(|i_{Q_1}\rangle \langle j_{Q_1}|) \\
&= \sum_{ij} \alpha_i \alpha_j^* \mathcal{E}(|i_{Q_1}\rangle \langle j_{Q_1}|) \\
&= \mathcal{E}(|v\rangle \langle v|)
\end{aligned}$$

- $\sigma = \sum_k \lambda_k |s_k\rangle\langle s_k|$: spectral decomposition of σ

$$\langle \tilde{v} | \sigma | \tilde{v} \rangle = \sum_k \lambda_k \langle \tilde{v} | s_k \rangle \langle s_k | \tilde{v} \rangle$$

- E_k : a linear transformation from Q_1 to Q_2 defined as

$$E_k(|v\rangle) \triangleq \sqrt{\lambda_k} \langle \tilde{v} | s_k \rangle$$

for any vector $|v\rangle$ in Q_1

Thus for any state $|\psi\rangle$ in Q_1 , we have

$$\mathcal{E}(|\psi\rangle\langle\psi|) = \langle \tilde{\psi} | \sigma | \tilde{\psi} \rangle = \sum_k E_k |\psi\rangle\langle\psi| E_k^\dagger$$

and then for any density operator $\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$ with ensemble $\{\lambda_i, |\psi_i\rangle\}$, we have

$$\begin{aligned}\mathcal{E}(\rho) &= \mathcal{E}\left(\sum_i \lambda_i |\psi_i\rangle\langle\psi_i|\right) \\ &= \sum_i \lambda_i \mathcal{E}(|\psi_i\rangle\langle\psi_i|) \text{ by Axiom II} \\ &= \sum_i \lambda_i \sum_k E_k |\psi_i\rangle\langle\psi_i| E_k^\dagger \\ &= \sum_k E_k \left(\sum_i \lambda_i |\psi_i\rangle\langle\psi_i|\right) E_k^\dagger \\ &= \sum_k E_k \rho E_k^\dagger\end{aligned}$$

To show that

$$\sum_k E_k^\dagger E_k \leq I,$$

we need to show that

$$\langle \psi | (I - \sum_k E_k^\dagger E_k) | \psi \rangle \geq 0$$

for any state ψ in Q_1 . But

$$\begin{aligned} \langle \psi | (I - \sum_k E_k^\dagger E_k) | \psi \rangle &= \langle \psi | \psi \rangle - \langle \psi | (\sum_k E_k^\dagger E_k) | \psi \rangle \\ &= 1 - \sum_k \langle \psi | E_k^\dagger E_k | \psi \rangle = 1 - \sum_k \text{tr}(E_k | \psi \rangle \langle \psi | E_k^\dagger) \\ &= 1 - \text{tr}(\sum_k E_k | \psi \rangle \langle \psi | E_k^\dagger) = 1 - \text{tr}(\mathcal{E}(| \psi \rangle \langle \psi |)) \geq 0 \end{aligned}$$

by Axiom I

Freedom in the Operator-Sum Representation

Unitary Freedom in the Operator-Sum Representation

- $\{E_1, E_2, \dots, E_m\}$: operation elements of a quantum operation \mathcal{E}
- $\{F_1, F_2, \dots, F_n\}$: operation elements of a quantum operation \mathcal{F}
- $m = n$: by appending zero operators in the shorter list of operation elements

Then $\mathcal{E} = \mathcal{F}$ if and only if

$$E_i = \sum_j u_{ij} F_j$$

where $[u_{ij}]$ is an $m \times m$ unitary matrix

Proof \implies

Suppose that for any density operator ρ , we have

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger = \sum_k F_k \rho F_k^\dagger = \mathcal{F}(\rho)$$

- R : an introduced space, with the same dimension as Q_1
- $\{|i_R\rangle\}$ and $\{|i_{Q_1}\rangle\}$: orthonormal bases for R and Q_1 respectively
- $|u\rangle = \sum_i |i_R\rangle |i_{Q_1}\rangle$: a (maximally entangled) vector in the composite system RQ_1
- $\mathcal{I} \otimes \mathcal{E} = \mathcal{I} \otimes \mathcal{F}$: a map from the set of positive operators on RQ_1 to the set of positive operators on RQ_2 by the complete positivity of \mathcal{E} from the 3rd axiomatic property
- $\sigma = (\mathcal{I} \otimes \mathcal{E})(|u\rangle\langle u|) = (\mathcal{I} \otimes \mathcal{F})(|u\rangle\langle u|)$

Now,

$$\begin{aligned}
(\mathcal{I} \otimes \mathcal{E})(|u\rangle\langle u|) &= \sum_{ij} |i_R\rangle\langle j_R| \otimes \mathcal{E}(|i_{Q_1}\rangle\langle j_{Q_1}|) \\
&= \sum_{ij} |i_R\rangle\langle j_R| \otimes \left(\sum_k E_k |i_{Q_1}\rangle\langle j_{Q_1}| E_k^\dagger \right) \\
&= \sum_k \left(\sum_i |i_R\rangle\langle E_k |i_{Q_1}\rangle \right) \left(\sum_j |j_R\rangle\langle E_k |j_{Q_1}\rangle \right)^\dagger \\
&= \sum_k |e_k\rangle\langle e_k| \\
(\mathcal{I} \otimes \mathcal{F})(|u\rangle\langle u|) &= \sum_k |f_k\rangle\langle f_k|
\end{aligned}$$

where we define

$$|e_k\rangle \triangleq \sum_i |i_R\rangle\langle E_k |i_{Q_1}\rangle, \quad |f_k\rangle \triangleq \sum_i |i_R\rangle\langle F_k |i_{Q_1}\rangle.$$

Now we have

$$\sigma = \sum_k |e_k\rangle\langle e_k| = \sum_k |f_k\rangle\langle f_k|$$

and for any vector $|v\rangle = \sum_i \alpha_i |i_{Q_1}\rangle$ in Q_1 , we have

$$E_k |v\rangle = \langle \tilde{v} | e_k \rangle, \quad F_k |v\rangle = \langle \tilde{v} | f_k \rangle,$$

where $|\tilde{v}\rangle = \sum_i \alpha_i^* |i_R\rangle$ is the vector in R corresponding to the vector $|v\rangle$

- Unitary freedom in the ensemble for density operators : there exists an $m \times m$ unitary matrix $[u_{kl}]$ such that

$$|e_k\rangle = \sum_l u_{kl} |f_l\rangle$$

Thus we have $E_k |v\rangle = \langle \tilde{v} | e_k \rangle = \sum_l u_{kl} \langle \tilde{v} | f_l \rangle = \sum_l u_{kl} F_l |v\rangle$, i.e.,

$$E_k = \sum_l u_{kl} F_l$$

Two Examples of Quantum Operations

Trace as a Quantum Operation

- Q : a quantum systems with state space H_Q
- $\{|i\rangle\}$: an orthonormal basis of the state space H_Q of Q
- Q' : a quantum systems with one-diemnsional state space $H_{Q'}$
- $\{|0\rangle\}$: an orthonormal basis of the state space $H_{Q'}$ of Q'
- $E_i = |0\rangle\langle i|$: a linear transformation from H_Q to $H_{Q'}$
- A completeness relation : $\sum_i E_i^\dagger E_i = I$
- \mathcal{E} : a quantum operation defined as

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger = \sum_i |0\rangle\langle i| \rho |i\rangle\langle 0|$$

It is clear that

$$\mathcal{E}(\rho) = \text{tr}(\rho)|0\rangle\langle 0|$$

Partial Trace as a Quantum Operation

- Q and R : two quantum systems with state spaces H_Q and H_R
- $\{|j\rangle\}$: an orthonormal basis of the state space H_R of R
- $\sum_j |v_j\rangle|j\rangle$: a vector (in a unique representation format) in $H_Q \otimes H_R$
- E_i : a linear transformation from $H_Q \otimes H_R$ to H_Q defined as

$$E_i \left(\sum_j |v_j\rangle|j\rangle \right) = |v_i\rangle$$

- E_i^\dagger : the adjoint of E_i , which is a linear transformation from H_Q to $H_Q \otimes H_R$ and can be shown to be

$$E_i^\dagger(|v\rangle) = |v\rangle|i\rangle$$

- A completeness relation : $\sum_i E_i^\dagger E_i = I$
- \mathcal{E} : a quantum operation can be defined as

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger$$

for all density operators ρ on the composite quantum QR . In fact, \mathcal{E} is a linear map from $L(H_Q \otimes H_R, H_Q \otimes H_R)$ to $L(H_Q, H_Q)$

- T^Q : a linear operator on H_Q , we have

$$\mathcal{E}(T^Q \otimes |j\rangle\langle j'|) = T^Q \delta_{jj'} = \text{tr}_R(T^Q \otimes |j\rangle\langle j'|)$$

Proof. Let $|v\rangle \in H_Q$. Then

$$\begin{aligned}
 \mathcal{E}(T^Q \otimes |j\rangle\langle j'|)(|v\rangle) &= \sum_i E_i(T^Q \otimes |j\rangle\langle j'|)E_i^\dagger(|v\rangle) \\
 &= \sum_i E_i(T^Q \otimes |j\rangle\langle j'|)(|v\rangle|i\rangle) = \sum_i E_i(T^Q|v\rangle \otimes \delta_{ij'}|j\rangle) \\
 &= \sum_i \delta_{ij}\delta_{ij'}T^Q|v\rangle = \delta_{jj'}T^Q|v\rangle = \text{tr}_R(T^Q \otimes |j\rangle\langle j'|)(|v\rangle)
 \end{aligned}$$

For each linear operator $T^{QR} = \sum_{jj'} T_{jj'}^Q \otimes |j\rangle\langle j'|$ on the composite system QR , we have

$$\mathcal{E}(T^{QR}) = \sum_i E_i T^{QR} E_i^\dagger = \text{tr}_R(T^{QR})$$

by linearity of \mathcal{E} and tr_R .

Geometric Visualization of Quantum Operations on a Qubit

Bloch Vector Representation of Density Operators of a Qubit

- $\{|0\rangle, |1\rangle\}$: a computational basis of the state space H of a qubit
- ρ : a density operator of the qubit with matrix representation relative to the computational basis

$$[\rho] = \begin{bmatrix} \frac{1+r_z}{2} & \frac{r_x - ir_y}{2} \\ \frac{r_x + ir_y}{2} & \frac{1-r_z}{2} \end{bmatrix}$$

- $\sigma_x, \sigma_y, \sigma_z$: Pauli operators
- Bloch vector representation :

$$\rho = \frac{I + r_x\sigma_x + r_y\sigma_y + r_z\sigma_z}{2} = \frac{I + \vec{r} \cdot \vec{\sigma}}{2}$$

– ρ is a density operator $\implies \|\vec{r}\|^2 = r_x^2 + r_y^2 + r_z^2 \leq 1$

- $\rho = I/2 \iff \vec{r} = \vec{0}$

- $\text{tr}(\rho^2) = (1 + \|\vec{r}\|^2)/2$
- $\rho = |\psi\rangle\langle\psi|$ is a pure state if and only if $\|\vec{r}\| = 1$
 - $|\psi\rangle = \cos \frac{\theta}{2}|0\rangle + e^{i\varphi} \sin \frac{\theta}{2}|1\rangle$ with the visualizing representation

$$(\cos \varphi \sin \theta, \sin \varphi \sin \theta, \cos \theta)$$

on the Bloch sphere in Unit Two, which is equal to the Bloch vector \vec{r} in above, i.e.,

$$[\rho] = \begin{bmatrix} \frac{1+\cos \theta}{2} & \frac{\cos \varphi \sin \theta - i \sin \varphi \sin \theta}{2} \\ \frac{\cos \varphi \sin \theta + i \sin \varphi \sin \theta}{2} & \frac{1-\cos \theta}{2} \end{bmatrix}$$

The Vector Space $L(H)$ with Trace Inner Product

- $L(H)$: the vector space of all linear operators on H
- $(T, S) \triangleq \text{tr}(T^\dagger S)$: the Hilbert-Schmidt or trace inner product of T and S in $L(H)$
 - $(T, T) = \text{tr}(T^\dagger T) \geq 0$: $T^\dagger T$ is a positive operator
 - * $(T, T) = \text{tr}(T^\dagger T) = 0 \iff$ all singular values of T are zeros $\iff \text{rank}(T)=0 \iff T = 0$
 - $(T, S) = \text{tr}(T^\dagger S) = \text{tr}((S^\dagger T)^\dagger) = \overline{\text{tr}(S^\dagger T)} = \overline{(S, T)}$: Hermitian symmetry
 - $(T, \alpha_1 S_1 + \alpha_2 S_2) = \text{tr}(T^\dagger(\alpha_1 S_1 + \alpha_2 S_2)) = \alpha_1 \text{tr}(T, S_1) + \alpha_2 \text{tr}(T, S_2) = \alpha_1 (T, S_1) + \alpha_2 (T, S_2)$: linearity
- $L(H)$: a 4-dimensional complex inner product space with trace inner product

- $\{I/\sqrt{2}, \sigma_x/\sqrt{2}, \sigma_y/\sqrt{2}, \sigma_z/\sqrt{2}\}$: an orthonormal basis of $L(H)$

Trace-Preserving Quantum Operations as Affine Maps from Bloch Sphere to Itself

- $\sigma_1\sigma_2 = i\sigma_3, \sigma_2\sigma_3 = i\sigma_1, \sigma_3\sigma_1 = i\sigma_2$
- $\mathcal{E} = \sum_k E_k \rho E_k^\dagger$: a trace-preserving quantum operation on a qubit (where input system = output system) with

$$E_k = \alpha_{k0}I + \sum_{i=1}^3 \alpha_{ki}\sigma_i$$

where $\sum_k E_k^\dagger E_k = I$ which implies that

$$\sum_k \sum_{i=0}^3 |\alpha_{ki}|^2 = 1, \sum_k \Re\{\alpha_{k0}\alpha_{k1}^*\} + \Im\{\alpha_{k2}\alpha_{k3}^*\} = 0,$$

$$\sum_k \Re\{\alpha_{k0}\alpha_{k2}^*\} + \Im\{\alpha_{k3}\alpha_{k1}^*\} = 0, \sum_k \Re\{\alpha_{k0}\alpha_{k3}^*\} + \Im\{\alpha_{k1}\alpha_{k2}^*\} = 0$$

- $\rho = \frac{I + \vec{r} \cdot \vec{\sigma}}{2}$ and $\mathcal{E}(\rho) = \frac{I + \vec{r}' \cdot \vec{\sigma}}{2}$

$$\vec{r} \xrightarrow{\mathcal{E}} \vec{r}' = M\vec{r} + \vec{c}$$

where

$$M_{ij} = \sum_k \left(2\Re\{\alpha_{ki}\alpha_{kj}^*\} + 2\Im\{\alpha_{k0}^* \sum_{p=1}^3 \epsilon_{ijp} \alpha_{kp}\} + \left(|\alpha_{k0}|^2 - \sum_{p=1}^3 |\alpha_{kp}|^2 \right) \delta_{ij} \right)$$

and

$$c_j = 2i \sum_k \sum_{mn} \epsilon_{mnj} \alpha_{kn} \alpha_{km}^*$$

Examples of Single Qubit Noisy Quantum Channels

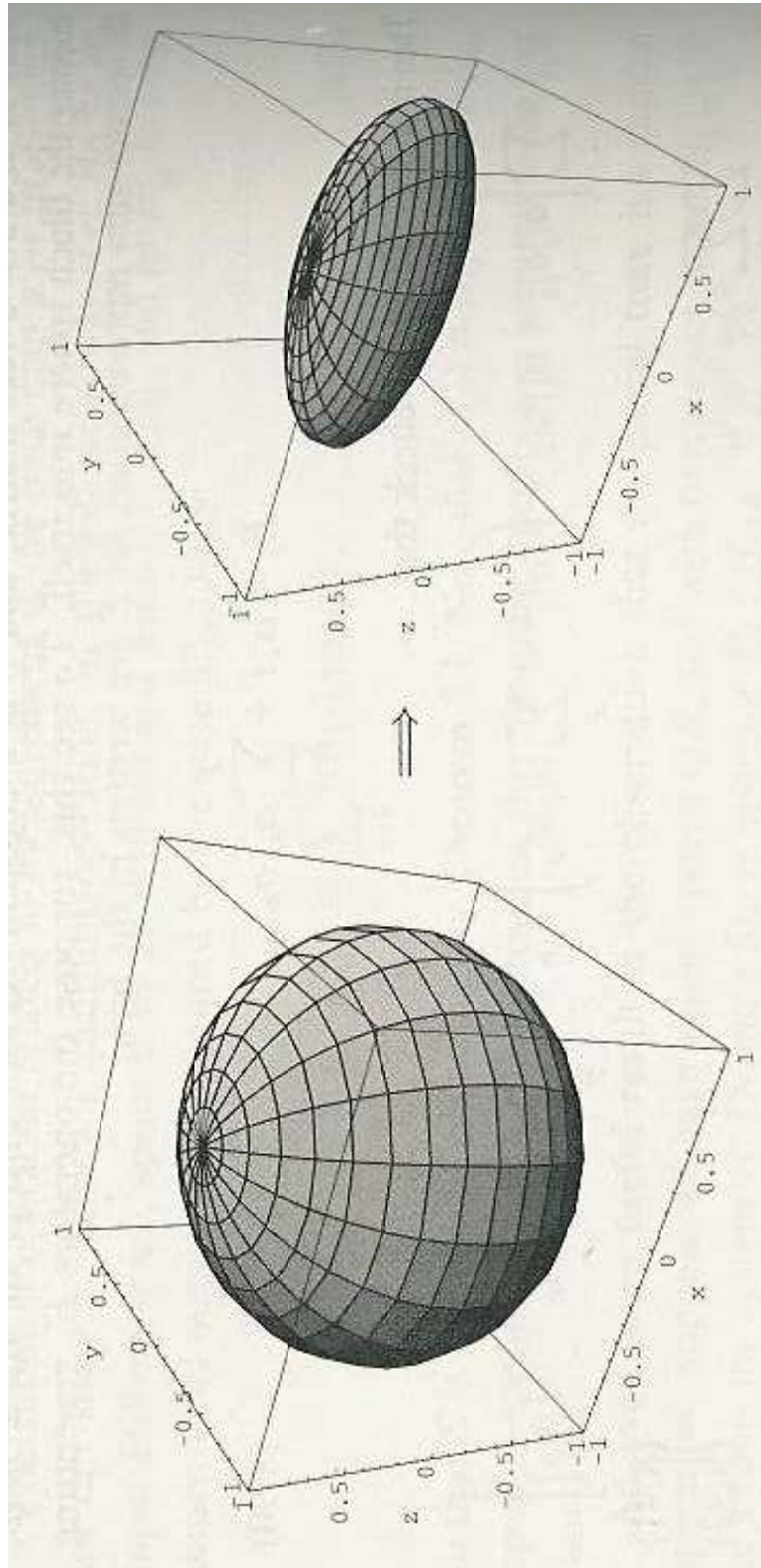
Bit Flip Channel

- The bit flip channel flips the state of a qubit from $|0\rangle$ to $|1\rangle$ (and vice versa) with probability p
- $\mathcal{E}(\rho) = E_0\rho E_0^\dagger + E_1\rho E_1^\dagger$
 - ρ : the density operator of single qubit
 - $E_0 = \sqrt{1-p}I = \sqrt{1-p} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
 - $E_1 = \sqrt{p}\sigma_x = \sqrt{p} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- Trace-preserving : $E_0^\dagger E_0 + E_1^\dagger E_1 = I$

Visualization of the Bit Flip Channel

- $\rho = (I + \vec{r} \cdot \vec{\sigma})/2$ and $\mathcal{E}(\rho) = (I + \vec{r}' \cdot \vec{\sigma})/2$

$$\vec{r}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 - 2p & 0 \\ 0 & 0 & 1 - 2p \end{bmatrix} \vec{r}$$



Phase Flip Channel

- The phase flip channel flips the phase of the state $|1\rangle$ of a qubit with probability p
- $\mathcal{E}(\rho) = E_0\rho E_0^\dagger + E_1\rho E_1^\dagger$
 - ρ : the density operator of single qubit
 - $E_0 = \sqrt{1-p}I = \sqrt{1-p} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
 - $E_1 = \sqrt{p}\sigma_z = \sqrt{p} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
- Trace-preserving : $E_0^\dagger E_0 + E_1^\dagger E_1 = I$

Visualization of the Phase Flip Channel

- $\rho = (I + \vec{r} \cdot \vec{\sigma})/2$ and $\mathcal{E}(\rho) = (I + \vec{r}' \cdot \vec{\sigma})/2$

$$\vec{r}' = \begin{bmatrix} 1 - 2p & 0 & 0 \\ 0 & 1 - 2p & 0 \\ 0 & 0 & 1 \end{bmatrix} \vec{r}$$

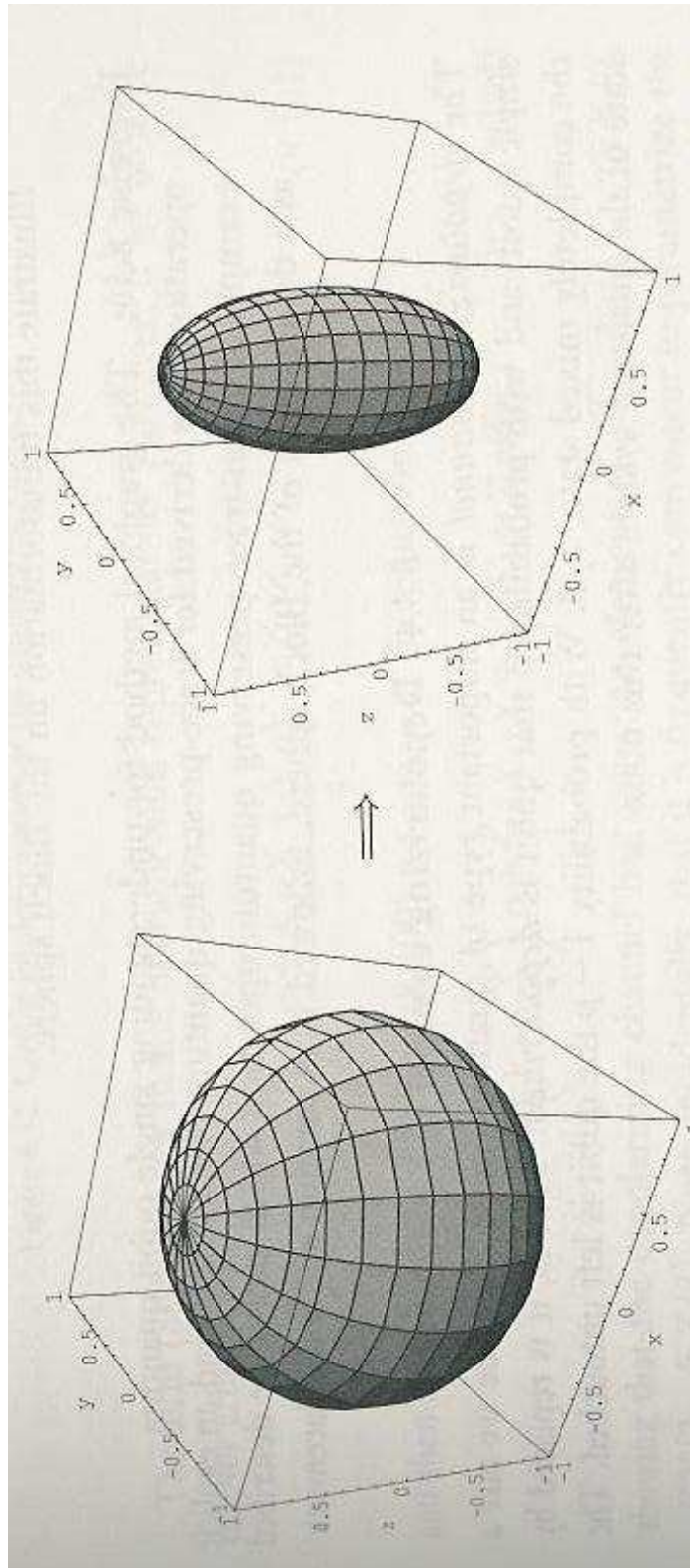
- $p = 1/2$: $\vec{r}' = (0, 0, r_z)$ and the collection of operator elements

$$E_0 = \frac{1}{\sqrt{2}}I, \quad E_1 = \frac{1}{\sqrt{2}}\sigma_z,$$

is unitarily equivalent to the collection of operator elements

$$P_0 = |0\rangle\langle 0| = \frac{1}{\sqrt{2}}E_0 + \frac{1}{\sqrt{2}}E_1, \quad P_1 = |1\rangle\langle 1| = \frac{1}{\sqrt{2}}E_0 - \frac{1}{\sqrt{2}}E_1$$

which is the projective measurement on the basis $\{|0\rangle, |1\rangle\}$



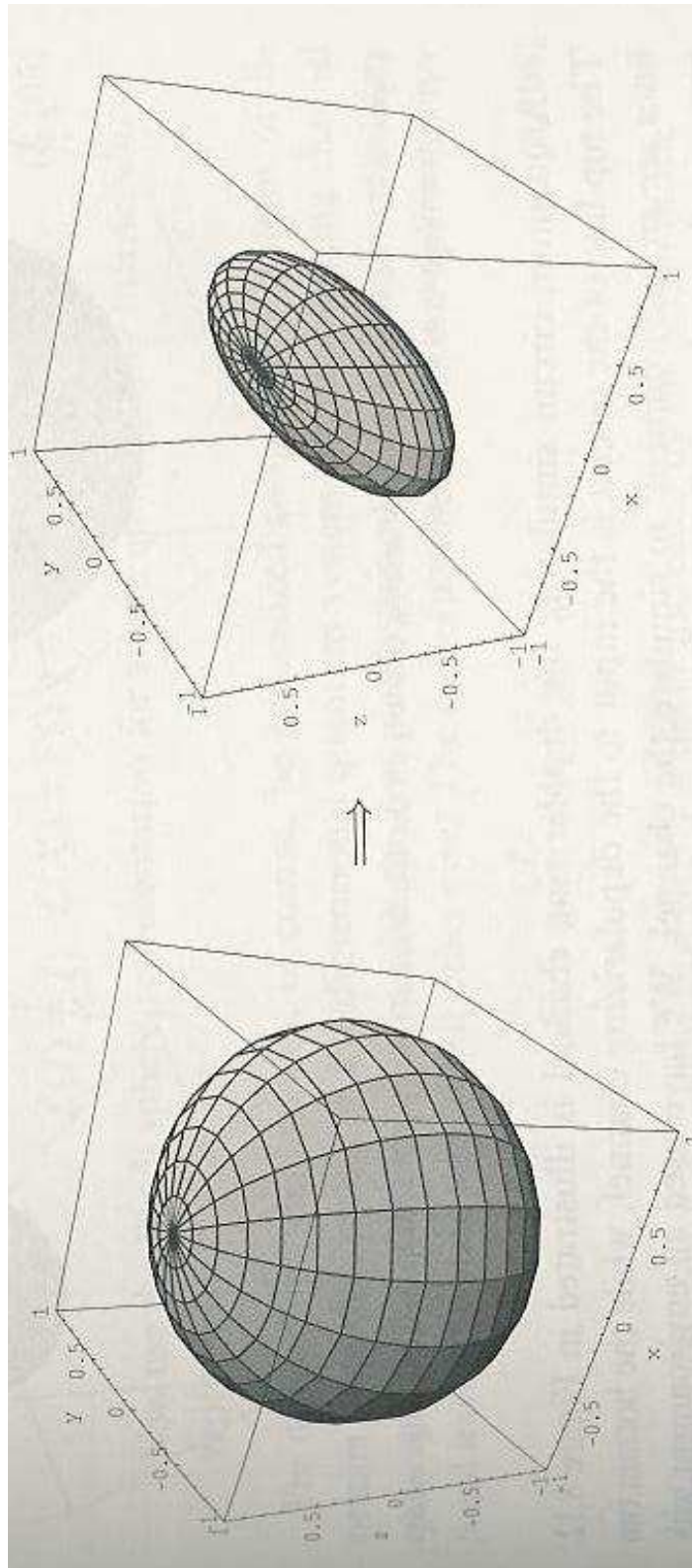
Bit-Phase Flip Channel

- The bit-phase flip channel flips the state of the qubit from $|0\rangle$ to $|1\rangle$ (and vice versa) and then flips the phase of the state $|1\rangle$ of the qubit with probability p
- $\sigma_y = -i\sigma_z\sigma_x$
- $\mathcal{E}(\rho) = E_0\rho E_0^\dagger + E_1\rho E_1^\dagger$
 - ρ : the density operator of single qubit
 - $E_0 = \sqrt{1-p}I = \sqrt{1-p} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
 - $E_1 = \sqrt{p}\sigma_y = \sqrt{p} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$
- Trace-preserving : $E_0^\dagger E_0 + E_1^\dagger E_1 = I$

Visualization of the Bit-Phase Flip Channel

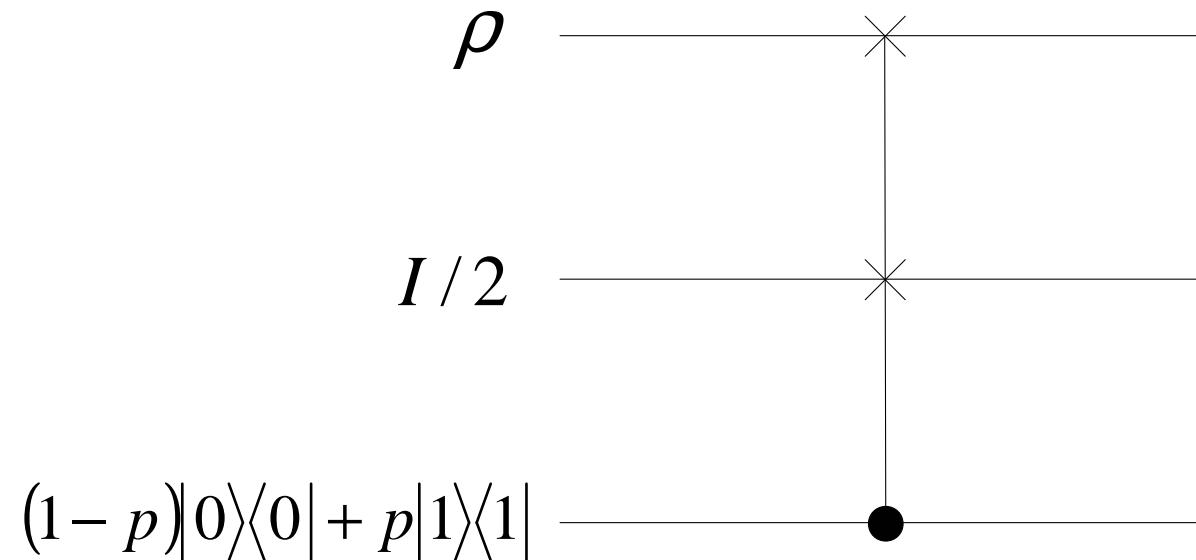
- $\rho = (I + \vec{r} \cdot \vec{\sigma})/2$ and $\mathcal{E}(\rho) = (I + \vec{r}' \cdot \vec{\sigma})/2$

$$\vec{r}' = \begin{bmatrix} 1 - 2p & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 - 2p \end{bmatrix} \vec{r}$$



The Depolarizing Channel

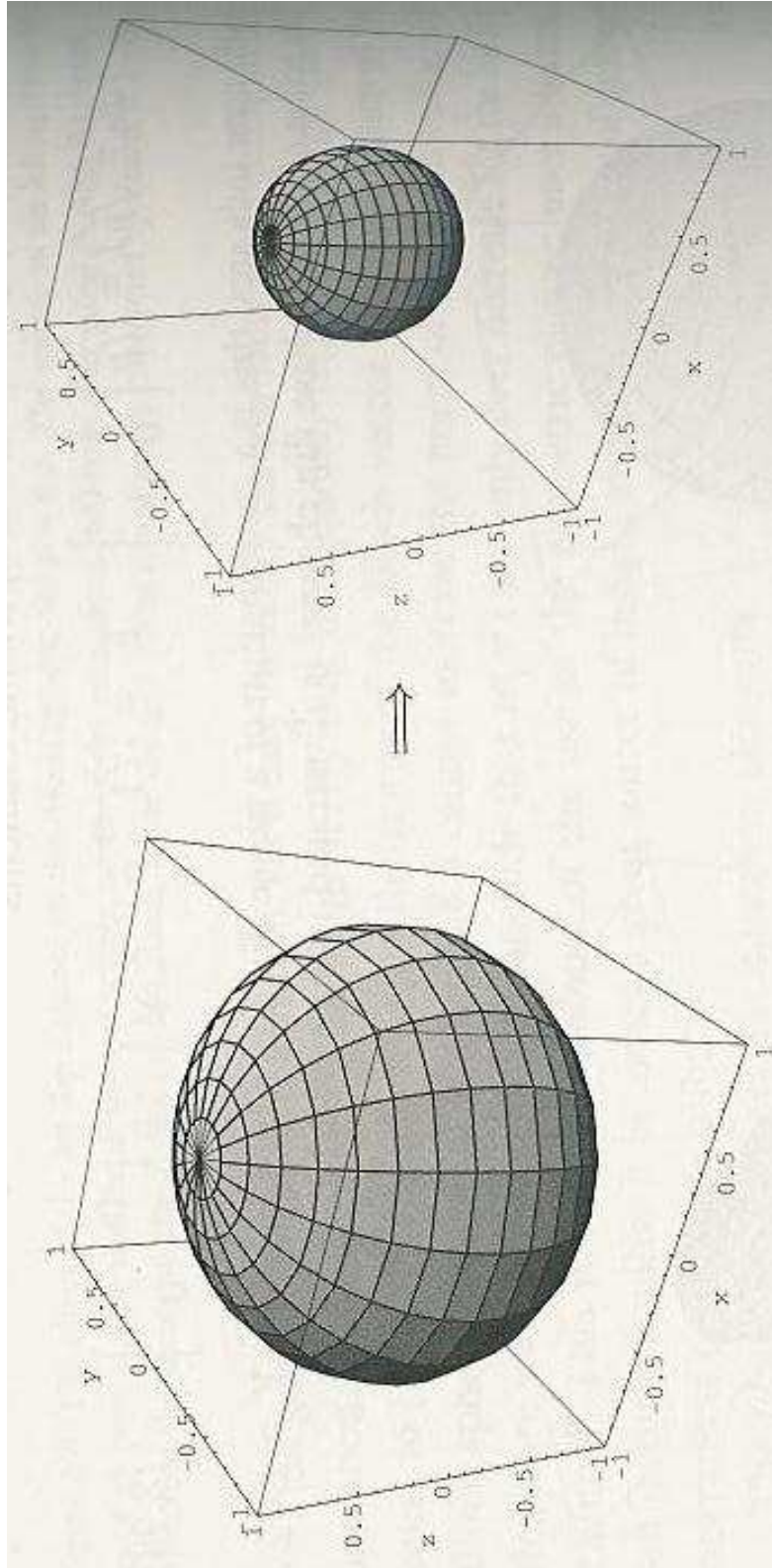
- With probability p , the single qubit is depolarized, i.e., its state is replaced by the completely mixed state $I/2$
- $\mathcal{E}(\rho) = p(I/2) + (1 - p)\rho$
- A circuit implementation of the depolarizing channel



Visualization of the Depolarizing Channel

- $\rho = (I + \vec{r} \cdot \vec{\sigma})/2$ and $\mathcal{E}(\rho) = (I + \vec{r}' \cdot \vec{\sigma})/2$

$$\vec{r}' = \begin{bmatrix} 1-p & 0 & 0 \\ 0 & 1-p & 0 \\ 0 & 0 & 1-p \end{bmatrix} \vec{r}$$



The Operator-Sum Representation of Depolarizing Channels

- An identity :

$$\frac{I}{2} = \frac{\rho + \sigma_x \rho \sigma_x + \sigma_y \rho \sigma_y + \sigma_z \rho \sigma_z}{4}$$

- $\mathcal{E}(\rho) = (1 - (3/4)p)\rho + (p/4)(\sigma_x \rho \sigma_x + \sigma_y \rho \sigma_y + \sigma_z \rho \sigma_z)$

- $E_0 = \sqrt{1 - 3p/4}I$

- $E_1 = \sqrt{p}\sigma_x/2$

- $E_2 = \sqrt{p}\sigma_y/2$

- $E_3 = \sqrt{p}\sigma_z/2$

- Trace-preserving : $\sum_{k=0}^3 E_k^\dagger E_k = I$

- Another expression :

$$\mathcal{E}(\rho) = (1 - p')\rho + (p'/3)(\sigma_x \rho \sigma_x + \sigma_y \rho \sigma_y + \sigma_z \rho \sigma_z)$$

The Depolarizing Channel for n -Qubit Systems

- With probability p , the n -qubit system is depolarized, i.e., its state is replaced by the completely mixed state $I/(2^n)$
- $\mathcal{E}(\rho) = pI/(2^n) + (1 - p)\rho$

Amplitude Damping Channel

- $|0\rangle$: the ground state without a quantum of energy
- $|1\rangle$: the state with a quantum of energy
- $\mathcal{E}(\rho) = E_0\rho E_0^\dagger + E_1\rho E_1^\dagger$
 - ρ : the density operator of a single qubit
 - $E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix}$: a quantum of energy was not lost to the environment such that the qubit must be more probably in the state $|0\rangle$ than in the state $|1\rangle$
 - $E_1 = \sqrt{\gamma} \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix}$: a quantum of energy was lost to the environment with probability γ such that the the qubit must be in the state $|1\rangle$

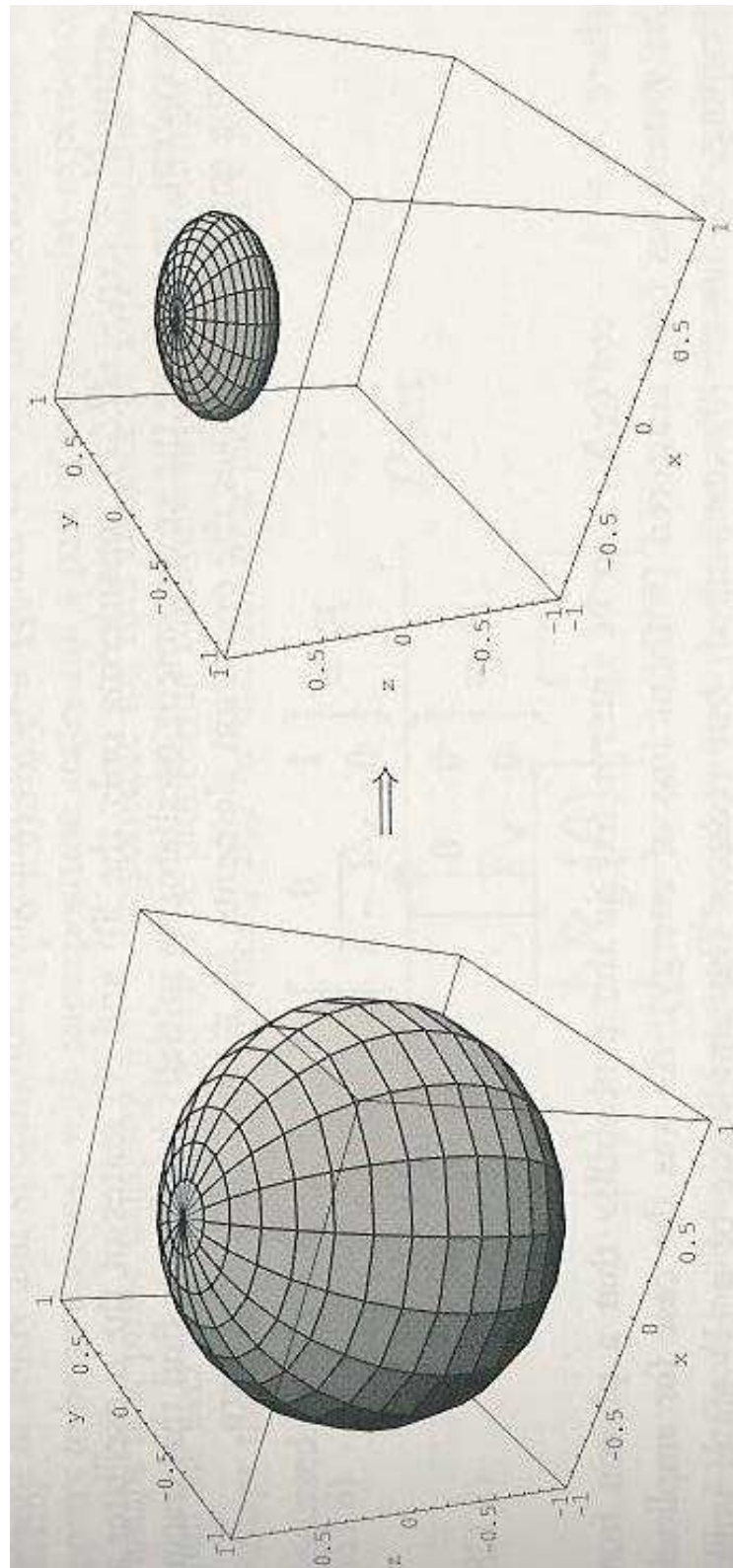
- Trace-preserving : $E_0^\dagger E_0 + E_1^\dagger E_1 = I$

Visualization of the Amplitude Damping Channel

- $\rho = (I + \vec{r} \cdot \vec{\sigma})/2$ and $\mathcal{E}(\rho) = (I + \vec{r}' \cdot \vec{\sigma})/2$

$$\vec{r}' = \begin{bmatrix} \sqrt{1-\gamma} & 0 & 0 \\ 0 & \sqrt{1-\gamma} & 0 \\ 0 & 0 & 1-\gamma \end{bmatrix} \vec{r} + \begin{bmatrix} 0 \\ 0 \\ \gamma \end{bmatrix}$$

- When describing $\gamma = 1 - e^{-t/T_1}$ as a time-varying function, the effect of amplitude damping is as a flow on the Bloch sphere, which moves every points in the unit sphere to the north pole of the unit sphere on which the state $|0\rangle$ resides



Phase Damping Channel

- The phase damping channel is the same as the phase flip channel
- $\mathcal{E}(\rho) = E_0\rho E_0^\dagger + E_1\rho E_1^\dagger$
 - ρ : the density operator of single qubit
 - $E_0 = \sqrt{1-p}I = \sqrt{1-p} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
 - $E_1 = \sqrt{p}\sigma_z = \sqrt{p} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
- Trace-preserving : $E_0^\dagger E_0 + E_1^\dagger E_1 = I$

Visualization of the Phase Damping Channel

- $\rho = (I + \vec{r} \cdot \vec{\sigma})/2$ and $\mathcal{E}(\rho) = (I + \vec{r}' \cdot \vec{\sigma})/2$

$$\vec{r}' = \begin{bmatrix} 1 - 2p & 0 & 0 \\ 0 & 1 - 2p & 0 \\ 0 & 0 & 1 \end{bmatrix} \vec{r}$$

- When describing $p = (1 - e^{-t/T_1})/2$ as a time-varying function, the effect of phase damping is as a flow on the Bloch sphere, which perpendicularly moves every points in the unit sphere to the z -axis of the unit sphere

