Analysis of Decoding Complexity Using the Berry-Esseen Theorem

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Abstract — **This work presents a novel technique to analyze the computational efforts of an ordering-free variant of the generalized Dijkstras's algorithm (GDA) and the maximum-likelihood sequential decoding algorithm (MLSDA) based on the Berry-Esseen theorem. Different from the theoretical bounds determined by the conventional central limit theorem argument, which often holds only for sufficiently large blocklength, the new bounds are valid for any blocklength.**

I. Introduction

The Berry-Esseen theorem states that for every $a \in \Re$,

$$\left|\Pr\left\{\frac{1}{s_n}(X_1+\cdots+X_n)\le a\right\}-\Phi(a)\right|\le C\,\frac{\rho}{\sigma^2\sqrt{n}},\qquad (1)$$

where $\{X_i\}_{i=1}^n$ are i.i.d. zero-mean random variables with $\mathrm{Var}[X_1]=\sigma^2$ and $E[|X_1|^3]=\rho$, $\Phi(\cdot)$ is the unit Gaussian cdf, and C is an absolute constant. The best C yet obtained thus far is generally considered to be 0.7655 [3, 4]

In applying (1) to analyze the sequential decoding complexity for codes antipodally transmitted via additive white Gaussian noise (AWGN) channels, the original analytical problem is first transformed into one concerning the asymptotic probability mass of the sum of two i.i.d. random sequences, one of which is Gaussian distributed and the other is non-Gaussian distributed. The complexities of two sequential maximum-likelihood decoding algorithms are then analyzed, i.e., an ordering-free variant of the GDA [1] operated over a code tree of linear block codes, and the MLSDA [2] that searches for the codeword over a trellis of binary convolutional codes.

Simulation results reveals that the theoretical bound for the GDA applied to (24, 12) binary extended Golay code and (48, 24) binary extended quadratic residue code matches the simulations as the signal-to-noise ratio (SNR) per information bit (γ_b) is no less than 8dB; it, however, becomes markedly higher than the simulations when γ_b is small. For the MLSDA applied to (2,1,6) and (2,1,16) convolutional codes with generators 634, 564 and 1632044, 1145734 and input lengths 60 and 100, our bound is quite close to the simulations for $\gamma_b \ge 6$dB and $\gamma_b \le 2$dB. Even for moderate SNR, the simulations and the theoretical bound differ by at most 0.587 on a $\log_{10}$ scale.

II. Main Results

Define a function

$$\mathcal{B}(x,y,\gamma)=\begin{cases}\Phi(-\sqrt{2\gamma x}), & \text{if } y=0;\\ \Phi\left(-y\hat{\mu}/\sqrt{x}+\sqrt{2x\gamma}\right)+\tilde{A}e^{x(-\gamma+\lambda^2/2)} \\ \quad\times\left[\Phi(-\lambda)e^{-\gamma}e^{\lambda^2/2}+\Phi(\sqrt{2\gamma})\right]^y \\ \quad\times\Phi\left(\frac{y\hat{\mu}+\lambda x}{\sqrt{x}}\right), & \text{if } 0<\frac{x}{y}\le\frac{\sqrt{4\pi\gamma}e^{\gamma}}{1-\sqrt{4\pi\gamma}e^{\gamma}\Phi(-\sqrt{2\gamma})};\\ 1, & \text{otherwise,}\end{cases}$$

where $\lambda=\lambda(x,y,\gamma)$ is the unique solution (in $[0,\sqrt{2\gamma})$) of

$$(x+y)\sqrt{2\pi}\lambda e^{(1/2)\lambda^2}\Phi(-\lambda)=y-x\sqrt{2\pi}\lambda e^{\gamma}\Phi(\sqrt{2\gamma}),$$

$$\tilde{A}\triangleq\min\left(e^{y(\lambda-\sqrt{2\gamma})^2\tilde{\sigma}^2/2}\Phi\left((\lambda-\sqrt{2\gamma})\tilde{\sigma}\sqrt{y}\right)+\frac{1.531}{\sqrt{y}}\frac{\tilde{\rho}}{\tilde{\sigma}^3},1\right),$$

$$\hat{\mu}\triangleq-(1/\sqrt{2\pi})e^{-\gamma}+\sqrt{2\gamma}\Phi(-\sqrt{2\gamma}),$$

$$\tilde{\sigma}^2\triangleq-u-\lambda^2u(1+u)+\frac{1+u}{1+\sqrt{2\pi}\lambda e^{\gamma}\Phi(\sqrt{2\gamma})},$$

$$\begin{aligned}\tilde{\rho}\triangleq&\frac{\lambda(1+u)}{[1+\sqrt{2\pi}\lambda e^{\gamma}\Phi(\sqrt{2\gamma})]}\{1-\lambda^2u(1+2u)\\&+2\left[\lambda^2(1+u)^2+2\right]e^{-\lambda^2u(2+u)/2}\\&-u\left[\lambda^2(1+2u)+3\right]\sqrt{2\pi}\lambda e^{\gamma}\Phi(\sqrt{2\gamma})\\&-2(1+u)\left[\lambda^2(1+u)^2+3\right]\sqrt{2\pi}\lambda e^{\lambda^2/2}\Phi(-\lambda(1+u))\}\end{aligned}$$

and $u=x/y$.

Theorem 1 *For an (n,k) linear block code transmitted via an AWGN channel, the average number of branch metric computations evaluated by the ordering-free GDA satisfies:*

$$C_{\text{GDA}}(\gamma_b)\le 2\sum_{\ell=0}^{k-1}\sum_{d=0}^{\ell}\binom{\ell}{d}\mathcal{B}\left(d,n-\ell,\frac{k\gamma_b}{n}\right).$$

Theorem 2 *For a (n,k,m) binary convolutional code (of input length L) transmitted via an AWGN channel, the average number of branch metric computations evaluated by the MLSDA satisfies:*

$$C_{\text{MLSDA}}(\gamma_b)\le 2^k\sum_{\ell=0}^{L-1}\sum_{j=0}^{2^m-1}\mathcal{B}\left(d_j^*(\ell),n(L+m-\ell),\frac{kL\gamma_b}{n(L+m)}\right)$$

where $d_j^*(\ell)$ is the minimum Hamming weight of the elements in $\mathcal{S}_j(\ell)$, and $\mathcal{S}_j(\ell)$ is the set of paths, which end at the node located at level ℓ and corresponding to state j over a trellis. In any case $\mathcal{S}_j(\ell)$ is an empty set, we simply take $\mathcal{B}(\cdot,\cdot,\cdot)=0$.

References

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