GENERAL EXPRESSIONS OF DERIVATIVE-CONSTRAINED LINEAR-PHASE TYPE-I FIR FILTERS

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ABSTRACT

In this paper, we propose a novel structure of linear-phase type-I FIR filters. The structure consists of a linear combination of some basic filters, called the cardinal filters. The weighting coefficients are exactly the derivatives of the amplitude response at $\omega = 0$. We solve a closed-form recurrence relationship between the filter coefficients. Implementation of the cardinal filters is discussed.

Index Terms— FIR filters, linear phase filters, maximally flat filters

1. INTRODUCTION

To design an FIR filter, of which the derivatives of the frequency response at prescribed frequencies must meet some given values, is a classical problem in signal processing. In addition to its accuracy at around those prescribed frequencies, a derivative-constrained FIR filter often provides a flexible structure with closed-form coefficients that facilitates its implementation as will be later exhibited in this paper. Generally speaking, the concept of derivative-constrained analysis can be regarded as originating from Taylor expansion. Such technique is then widely applied to analog and digital filter design.

In analog filter design, for example, it is well-known that the derivatives of the squared magnitude response of Butterworth filters are zero at $\omega = 0$ [1]. In digital filter design, Herrmann obtained a closed-form expression for a low-pass linear-phase finite impulse response (FIR) filter, of which the derivatives of the frequency response are constrained at two frequencies $\omega = 0$ and π [2]. Another example in digital filter design is that Selesnick and Burrus generalized the Butterworth filters to their digital counterparts with derivatives of squared magnitude responses being zeros at $\omega = 0$ and π [3].

The derivative constraints can be set not only for frequency response but also for group delay response. For example, Thiran established the closed-form formulas for the coefficients of all-pole infinite impulse response (IIR) filters, where the derivatives of group delay responses are constrained at $\omega = 0$ [4]. Later, Fernandez-Vazquez and Jovanovic-Dolecek extended Thiran's work by setting constraints at two frequencies $\omega = 0$ and π and obtained the close-form expressions for coefficients of all-pole filters [5].

When the derivatives of frequency responses are relaxed to be constrained at arbitrary frequency, Hermanowicz gave a closed-form impulse response for the fractional delay FIR filters [6]. A good review together with some new results on derivative-constrained fractional delay filters can be found in [7].

Most results in the aforementioned papers are for the design of the so-called maximally flat (MF) filters. By the MF sense of optimality, it means that the number of filter coefficients is equal to the number of prescribed derivatives. A derivative-constrained filter is called *partially flat* if it is not MF. In this literature, Pei and Wang [8] have shown that some MF filters can be obtained by series expansion. In [9], frequency responses of MF FIR fractional delay filters are compared with other design methods, by which it is shown that the MF designs are highly close to the target one at the vicinity of the prescribed frequency $\omega = 0$. Such a feature is in general valid for most derivative-constrained filter design. An insightful survey and historical remarks on MF filter design are provided in [10].

In this paper, we are concerned with the general structure of the so-called *type-I* linear-phase FIR filters with derivative constraints, for which the definition will be given in Section 2. We seek for a universal solution that can be applied to any derivative-constrained linear-phase FIR filter design, hoping to provide a new realization structure for such filters. In literature, there are various approaches to design linear-phase FIR filters such as the windowing method and the Parks-McClellan algorithm [1] as well as those in [6, 11]. By means of the cardinal function conception in numerical analysis [12], we found that the derivative-constrained linear-phase FIR filters have similar cardinal-function structure and can be synthesized via a set of pre-specified cardinal filters in the form of linear combination, where the weighting coefficients are exactly the given derivatives. We therefore call these basic filters the *cardinal filters*. We will then provide the close-form formulas for these cardinal filters. Remarks on their realization structures as well as recursion formulas for the coefficients of the cardinal filters will follow.

The rest of the paper is organized as follows. In Section 2, we formulate the problem of designing linear-phase FIR filters and introduce the conception of cardinal filters. In Section 3, the closed-form expressions of cardinal filters are derived. Recursive formulas for the coefficients of cardinal filters are also presented. In Section 4, we address the realization structure of cardinal filters. Section 5 concludes the paper.

2. PROBLEM FORMULATION

Let $H_N(z)$ be the transfer function of an Nth order FIR filter with impulse response $h_N[0]$, $h_N[1]$, $\cdots$, $h_N[N]$, i.e.,

$$H_N(z) = \sum_{n=0}^{N} h_N[n] z^{-n}. \tag{1}$$

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When N is an even number and the impulse response $h_N[0]$, $h_N[1]$, $\cdots$, $h_N[N]$ is symmetric with respect to $h[N/2]$, $H_N(z)$ is referred to as a *type-I (linear-phase) FIR filter* [1], and its corresponding frequency response $H_N(e^{j\omega})$ can be represented as

$$H_N(e^{j\omega}) = e^{-j\omega M} A_M(\omega), \tag{2}$$

where $M = N/2$ and

$$A_M(\omega) = \sum_{m=0}^{M} a_M[m] \cdot \cos(\omega m). \tag{3}$$

The coefficient $a_M[m]$ is related to the impulse response by

$$a_M[m] = \begin{cases} h_N[M], & m = 0 \\ 2h_N[M-m], & m = 1, 2, \cdots, M. \end{cases} \tag{4}$$

Based on the above setting, a problem of general interest is that given a desired amplitude response $D(\omega)$ over $-\pi < \omega < \pi$, find $A_M(\omega)$ that can well-approximate $D(\omega)$ in the sense that the *derivative constraints*, defined as

$$d_k \triangleq \frac{d^k}{d\omega^k} D(\omega)\bigg|_{\omega=0} = \frac{d^k}{d\omega^k} A_M(\omega)\bigg|_{\omega=0} \tag{5}$$

for $k = 0, 1, \cdots, K$, can be satisfied, where $K \leq N = 2M$. Equivalently, the problem is to find the coefficient $a_M[m]$ by solving

$$\frac{d^k}{d\omega^k} A_M(\omega)\bigg|_{\omega=0} = d_k, \; k = 0, 1, \cdots, K. \tag{6}$$

As such, both $A_M(\omega)$ and $D(\omega)$ can be practically associated with the K-th order Taylor polynomial P_K at the origin $\omega = 0$, where

$$P_K(\omega) = d_0 + d_2 \frac{\omega^2}{2!} + \cdots + d_K \frac{\omega^K}{K!}, \tag{7}$$

and hence are functionally close to each other near $\omega = 0$. Note that $D(\omega)$ must be an even function; otherwise, the desired approximation of $A_M(\omega)$ cannot be found.

Substituting (3) into (6), we get

$$\sum_{m=0}^{M} a_M[m] \cdot m^k \cos(k\pi/2) = d_k, \; k = 0, 1, \cdots, K. \tag{8}$$

Since $D(\omega)$ is an even function, $d_k = 0$ when k is odd; hence, (8) holds trivially for all odd k. We can thus reduce (8) into

$$\sum_{m=0}^{M} a_M[m] \cdot m^{2k} (-1)^k = d_{2k}, \; k = 0, 1, \cdots, M, \tag{9}$$

which, in matrix form, gives

$$\mathbb{M} \begin{bmatrix} a_M[0] \\ a_M[1] \\ \vdots \\ a_M[M] \end{bmatrix} = \begin{bmatrix} d_0 \\ d_2 \\ \vdots \\ d_{2M} \end{bmatrix}, \tag{10}$$

where

$$\mathbb{M} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & -1^2 & -2^2 & \cdots & -M^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & (-1^2)^M & (-2^2)^M & \cdots & (-M^2)^M \end{bmatrix}. \tag{11}$$

Now instead of solving (10) via matrix inversion, an alternative approach can be employed as follows.

Define an auxiliary type-I FIR filter of the form

$$A_{M,2j}(\omega) = \sum_{m=0}^{M} a_{M,2j}[m] \cdot \cos(\omega m) \tag{12}$$

that satisfies

$$\frac{d^{2\ell}}{d\omega^{2\ell}} A_{M,2j}(\omega)\bigg|_{\omega=0} = \delta_{j,\ell}, \; 0 \leq j, \ell \leq M, \tag{13}$$

where $\delta_{j,\ell}$ is the Kronecker delta symbol defined by $\delta_{j,\ell} = 0$ when $j \neq \ell$, and $\delta_{j,\ell} = 1$, otherwise. Then, it can be verified that $A_M(\omega)$ can be expressed as

$$A_M(\omega) = \sum_{k=0}^{M} d_{2k} \cdot A_{M,2k}(\omega). \tag{14}$$

Specifically, by substituting (12) into (14), we obtain

$$\frac{d^{2\ell}}{d\omega^{2\ell}} A_M(\omega)\bigg|_{\omega=0} = \frac{d^{2\ell}}{d\omega^{2\ell}} \left[\sum_{k=0}^{M} d_{2k} \cdot A_{M,2k}(\omega) \right]_{\omega=0}$$
$$= \sum_{k=0}^{M} d_{2k} \left[\frac{d^{2\ell}}{d\omega^{2\ell}} A_{M,2k}(\omega) \right]_{\omega=0} = \sum_{k=0}^{M} d_{2k} \cdot \delta_{\ell,k} = d_{2\ell}.$$

As a result of (14), the coefficient $a_M[m]$ is given by

$$a_M[m] = \sum_{k=0}^{M} d_{2k} \cdot a_{M,2k}[m]. \tag{15}$$

It is easy to see that the alternative approach has the following advantages. First, $A_{M,2j}(\omega)$ is nothing to do with d_{2k}; thus, it can be explicitly derived beforehand. Secondly, (14) suggests a universal structure for the design of derivative-constrained type-I FIR filters; hence, all solutions satisfying (6) can be expressed as linear combinations of $A_{M,2j}(\omega)$ with weighting coefficient d_{2k}. Thirdly, coefficient $a_M[m]$ corresponding to each $A_M(\omega)$ can also be expressed as linear combinations of $a_{M,2j}[m]$ with exactly the same weighting coefficient d_{2k} as suggested by (15). For these reasons, the function $A_{M,2j}(\omega)$ has fundamental importance to derivative-constrained approximation problems, and is named the *cardinal function* for problems like (6).

3. A NEW APPROACH TO DERIVE THE CARDINAL FUNCTIONS

For given M and j, the cardinal function $A_{M,2j}(\omega)$ can of course be solved numerically through

$$\sum_{m=0}^{M} a_{M,2j}[m] \cdot m^{2\ell} (-1)^\ell = \delta_{j,\ell}, \; \ell = 0, 1, \cdots, M. \tag{16}$$

However, even if exhausting the numerical derivations of $A_{M,2j}(\omega)$ for every required M and j pair might be a feasible burden for small M, the genetic structure of $A_{M,2j}(\omega)$ can hardly be explored using such a numerical derivation.

As such, we propose a novel approach to determine the cardinal function $A_{M,2j}(\omega)$ in this section. This approach can result in a close-form expression for $A_{M,2j}(\omega)$ for every M and j pair, and can further suggest a realization structure in subsequent section.

Let $r_{2j}[m]$ be the coefficient of the Taylor series of even function $R_{2j}(x)$ defined as

$$R_{2j}(x) = \frac{[2\sin^{-1}(x)]^{2j}}{(2j)!}, \tag{17}$$

and denote by $R_{M,2j}(x)$ the polynomial formed by removing the terms in $R_{2j}(x) = \sum_{m=0}^{\infty} r_{2j}[2m]x^{2m}$ with power greater than $2M$, i.e.,

$$R_{M,2j}(x) = \sum_{m=0}^{M} r_{2j}[2m] \cdot x^{2m}. \tag{18}$$

Observe that

$$R_{M,2j}(x) - \frac{[2\sin^{-1}(x)]^{2j}}{(2j)!} = -r_{2j}[2M+2] \cdot x^{2M+2} - \cdots$$

Hence, taking $x = \sin(\omega/2)$ into the above equation yields

$$\begin{aligned} & R_{M,2j}(\sin(\omega/2)) - \frac{\omega^{2j}}{(2j)!} \\ = & -r_{2j}[2M+2] \cdot (\sin(\omega/2))^{2M+2} - \cdots \\ = & -r_{2j}[2M+2] \cdot (\omega \cdot p(\omega))^{2M+2} - \cdots \end{aligned}$$

where $p(\omega) = \frac{\sin(\omega/2)}{\omega} = \frac{1}{\omega}\left(\frac{\omega}{2} - \frac{(\omega/2)^3}{3!} + \cdots\right)$ with $p(0) = 1/2$. It can be verified that $R_{M,2j}(\sin(\omega/2))$ satisfies (13); accordingly, if $R_{M,2j}(\sin(\omega/2))$ can be expressed as a linear combination of $\cos(\omega m)$, we can immediately take $A_{M,2j}(\omega) = R_{M,2j}(\sin(\omega/2))$.

In order to show that $R_{M,2j}(\sin(\omega/2))$ is a linear combination of $\cos(\omega m)$, we first notice that

$$\begin{aligned} R_{M,2j}(\sin(\omega/2)) &= \sum_{m=0}^{M} r_{2j}[2m] \cdot (\sin(\omega/2))^{2m} \\ &= \sum_{m=0}^{M} r_{2j}[2m] \left(\frac{1-\cos(\omega)}{2}\right)^{m}. \end{aligned}$$

Now since $R_{M,2j}(\sin(\omega/2))$ is a linear combination of $(1 - \cos(\omega))^m$ for $0 \leq m \leq M$, it must be a linear combination of $\cos^m(\omega)$. Consequently, $R_{M,2j}(\sin(\omega/2))$ is a linear combination of $\cos(m\omega)$ because $\cos^m(\omega)$ can be expressed as a linear combination of $\cos(m\omega)$ for $0 \leq m \leq M$. For clarity, we summarize the above result in the following theorem.

Theorem 1 *Let $r_{2j}[2m]$ be the coefficient of power term x^{2m} in the Taylor series of $[2\sin^{-1}(x)]^{2j}/((2j)!)$. Then the cardinal function $A_{M,2j}(\omega)$ that satisfies (13) is given by*

$$A_{M,2j}(\omega) = \sum_{m=0}^{M} r_{2j}[2m] \left(\frac{1-\cos\omega}{2}\right)^{m}. \tag{19}$$

Figure 1 shows the curves of $A_{M,2j}(\omega)$ for $j = 1, 2$ and $M = 2, 3, 5$. Also plotted are the ideal curves $A_{\infty,2j}(\omega)$ for $j = 1$ and 2. From Fig. 1, we observe that $A_{M,2j}(\omega)$ well-approximates the ideal curves near $\omega = 0$ and deviates away from them at high frequency band.

For a better understanding of how to obtain the cardinal function $A_{M,2j}(\omega)$ that we just derived, an example is next provided.

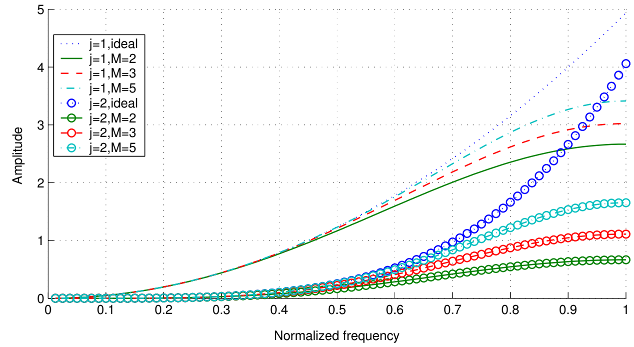

Fig. 1. Plots of $A_{M,2j}(\omega)$ for $j = 1, 2$ and $M = 2, 3, 5$. The ideal curves $A_{\infty,2j}(\omega)$ for $j = 1$ and 2 are also plotted, which are respectively $\omega^2/(2!)$ and $\omega^4/(4!)$.

Example 1 *Take $M = 4$. It is straightforward that $A_{4,0}(\omega) = 1$. It remains to determine $A_{4,2j}(\omega)$ for $j = 1, 2, 3, 4$. Respectively removing the power terms in the Taylor series of $R_2(x)$, $R_4(x)$, $R_6(x)$ and $R_8(x)$ with power greater than $2M = 8$ yields*

$$\begin{aligned} R_{4,2}(x) &= 2x^2 + \frac{2}{3}x^4 + \frac{16}{45}x^6 + \frac{8}{35}x^8 \\ R_{4,4}(x) &= \frac{2}{3}x^4 + \frac{4}{9}x^6 + \frac{14}{45}x^8 \\ R_{4,6}(x) &= \frac{4}{45}x^6 + \frac{4}{45}x^8 \\ R_{4,8}(x) &= \frac{2}{315}x^8 \end{aligned}$$

The desired cardinal functions can then be obtained by substituting $x = \sin(\omega/2)$ into the above functions. As an example,

$$\begin{aligned} A_{4,2}(\omega) &= R_{4,2}(\sin(\omega/2)) \\ &= \frac{205}{144} - \frac{8}{5}\cos(\omega) + \frac{1}{5}\cos(2\omega) \\ &\quad -\frac{8}{315}\cos(3\omega) + \frac{1}{560}\cos(4\omega) \\ &= \frac{1}{2}\omega^2 - \frac{1}{6300}\omega^{10} + \cdots \end{aligned}$$

where the Taylor series up to ω^{10} in the last equation confirm that $A_{4,2}(\omega)$ satisfies (13). ■

Theorem 1 as well as the follow-up Example 1 provides a systematic way to establish the cardinal functions. However, it may still be tedious to obtain every required $r_{2j}[2m]$ when M is moderately large. Along this observation, we proceed to explore the property of $A_{M,2j}(\omega)$ formula that we just derived and found that $r_{2j}[2m]$ can actually be efficiently obtained by a simple recursion. This finding is a consequence of the following two corollaries.

Corollary 1 *For $j \geq 1$, $R_{2j}(x)$ and $R_{2j-2}(x)$ satisfy the differential equation below:*

$$(1-x^2)\frac{d^2}{dx^2}R_{2j}(x) - x\frac{d}{dx}R_{2j}(x) - 4R_{2j-2}(x) = 0 \tag{20}$$

with initial condition $R_0(x) = 1$.
Proof. The proof follows by substituting (17) into (20). ■

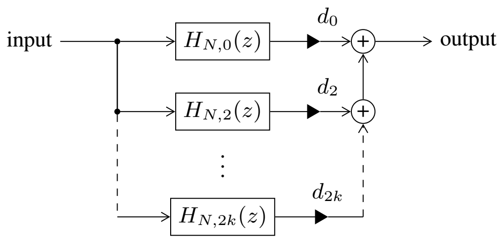

Fig. 2. A generic structure of type-I FIR filters synthesized using cardinal filters. The coefficients d_{2k} are exactly the derivatives of the amplitude response at $\omega = 0$.

Corollary 2 *$r_{2j}[2m]$ satisfies the following recursion:*

$$r_{2j}[2m+2] = \frac{m^2 \cdot r_{2j}[2m] + r_{2j-2}[2m]}{(m+1)\,(m+1/2)} \tag{21}$$

for $j \geq 1$ and $m \geq 0$ with initial conditions: $r_0[0] = 1, r_0[2m] = 0$ for $m \geq 1$, and $r_{2j}[0] = 0$ for $j \geq 1$.

Proof. *Substitute the Tailor series of $R_{2j}(x)$ defined in (17) into (20); then, the coefficient of every power term x^{2m} should be zero, which immediately gives (21). The initial conditions follow from $R_0(x) = 1$ and $R_{2j}(0) = 0$ for $j \geq 1$.* ∎

4. REALIZATION OF CARDINAL FUNCTIONS

In this section, we realize the type-I FIR filters from the aspect of the cardinal function $A_{M,2j}(\omega)$. By following (14), we can rewrite the transfer function of an Nth order type-I FIR filter in (1) as

$$H_N(z) = \sum_{k=0}^{M} d_{2k} \cdot H_{N,2k}(z), \tag{22}$$

where the cardinal filter $H_{N,2k}(z)$ is obtained by substituting $\cos\omega = (z + z^{-1})/2$ into (19) and is equal to

$$H_{N,2k}(z) = z^{-M} \sum_{m=0}^{M} r_{2k}[2m] \left(\frac{1 - (z+z^{-1})/2}{2} \right)^m .$$

Figure 2 depicts the structure of type-I FIR filters that are based on cardinal filters.

The cardinal filter $H_{N,2k}(z)$ can be realized by any standard linear-phase FIR structure. Alternatively, we can implement $H_{N,2k}(z)$ by re-expressing it as

$$H_{N,2k}(z) = \sum_{m=0}^{M} r_{2k}[2m] \cdot z^{-(M-m)} (E(z))^m . \tag{23}$$

where

$$E(z) = \frac{2z^{-1} - 1 - z^{-2}}{4} = -\left(\frac{1 - z^{-1}}{2} \right)^2 . \tag{24}$$

The block diagram of this alternative implementation is shown in Fig. 3.

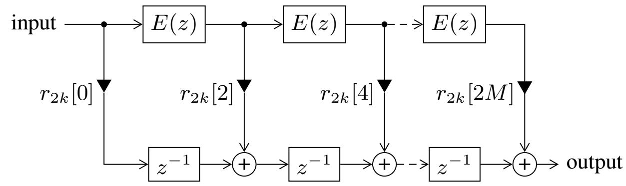

Fig. 3. A generic structure of cardinal filters $H_{N,2k}(z)$.

5. CONCLUSION

In this paper, a new systematic approach to construct type-I FIR filters subject to derivative constraints is proposed. The new approach indicates that the type-I FIR filters can be represented as a linear combination of cardinal filters that are independent of the desired frequency response, where the weighting coefficients are exactly the derivatives of the desired amplitude response at $\omega = 0$. Consequently, this new structure is especially suitable for synthesizing derivative-constrained filters such as the classical MF filters. We further found that the coefficients of the cardinal filters satisfy a simple recursive formula; hence, the establishment of the cardinal filters is further simplified.

Generally speaking, when $H_N(z)$ is implemented using (22), the complexity is $(M + 1)$ times larger than that of the direct form [1]. The benefit of our proposal actually resides on a rapid redesign of the whole filter without involving a heavy optimization process when a new set of derivative constrains is considered. Due to page limitations, synthesis of specific frequency responses by means of the proposed structure and comparisons with existing ones are not presented in this paper.

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