Csizsár’s Cutoff Rates for the General Hypothesis Testing Problem

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Abstract—In [6], Csizsár established the concept of forward \( \beta \)-cutoff rate for the error exponent hypothesis testing problem based on independent and identically distributed (i.i.d.) observations. Given \( \beta < 0 \), he defined the forward \( \beta \)-cutoff rate as the number \( R_0 \geq 0 \) that provides the best possible lower bound in the form \( \beta (E - R_0) \) to the type 1 error exponent function for hypothesis testing where \( 0 < E < R_0 \) is the rate of exponential convergence to 0 of the type 2 error probability. He then demonstrated that the forward \( \beta \)-cutoff rate is given by \( D_{\beta}(X||\bar{X}) \), where \( D_{\beta}(X||\bar{X}) \) denotes the Rényi \( \alpha \)-divergence [19], \( \alpha > 0, \alpha \neq 1 \). Similarly, for \( 0 \leq \beta < 1 \), Csizsár also established the concept of reverse \( \beta \)-cutoff rate for the correct exponent hypothesis testing problem.

In this work, we extend Csizsár’s results by investigating the forward and reverse \( \beta \)-cutoff rates for the hypothesis testing between two arbitrary sources with memory. We demonstrate that the \( \liminf \) Rényi \( \alpha \)-divergence rate provides the expression for the forward \( \beta \)-cutoff rate. We also show that if the log-likelihood large deviation spectrum admits a limit, then the reverse \( \beta \)-cutoff rate equals the \( \liminf \) Rényi \( \alpha \)-divergence rate, where \( \alpha = \frac{1}{\beta^2} \) and \( 0 < \beta < \beta_{\text{max}} \), where \( \beta_{\text{max}} \) is the largest \( \beta < 1 \) for which the \( \liminf \) Rényi \( \alpha \)-divergence rate is finite. For \( \beta_{\text{max}} \leq \beta < 1 \), we show that the reverse cutoff rate is in general only upper-bounded by the \( \liminf \) Rényi divergence rate. Unlike in [4], where the alphabet for the source coding cutoff rate problem was assumed to be finite, we assume arbitrary (countable or continuous) source alphabet. We also provide several examples to illustrate our forward and reverse \( \beta \)-cutoff rates results and the techniques employed to establish them.

Index Terms—\( \alpha \)-divergence rate, arbitrary sources with memory, forward and reverse cutoff rates, hypothesis testing error and correct exponents, information spectrum, large deviation theory.

I. INTRODUCTION

In [6], Csizsár established the concept of forward \( \beta \)-cutoff rate for the hypothesis testing problem based on independent and identically distributed (i.i.d.) observations. Given \( \beta < 0 \), he defined the forward \( \beta \)-cutoff rate as the number \( R_0 \geq 0 \) that provides the best possible lower bound in the form \( \beta (E - R_0) \) to the type 1 error exponent function for hypothesis testing where \( 0 < E < R_0 \) is the rate of exponential convergence to 0 of the type 2 error probability. He then demonstrated that the forward \( \beta \)-cutoff rate is given by \( D_{1/(1-\beta)}(X||\bar{X}) \), where \( D_{\beta}(X||\bar{X}) \) denotes Rényi’s \( \alpha \)-divergence, \( \alpha > 0, \alpha \neq 1 \) [19].

Csizsár also established the concept of reverse \( \beta \)-cutoff rate for the hypothesis testing problem based on i.i.d. observations. Given \( \beta > 0 \), he defined the reverse \( \beta \)-cutoff rate as the number \( R_0 \geq 0 \) that yields the best possible lower bound in the form \( \beta (E - R_0) \) to the hypothesis testing type 1 correct exponent (or reliability) function where \( 0 < R_0 < E \) is the rate of exponential convergence to 0 of the type 2 error probability. He then showed that the reverse \( \beta \)-cutoff rate is equal to \( D_{1/(1-\beta)}(X||\bar{X}) \). These results provide a new operational significance for the \( \alpha \)-divergence.

The error exponent for the binary hypothesis testing problem has been thoroughly studied for finite-state i.i.d. sources and Markov sources. The results for i.i.d. sources can be found in [7], [10], [12], and for irreducible Markov sources in [11], [15]. The error exponent for testing between ergodic Markov sources with continuous state–space under certain additional restrictions was established in [13]. In its full generality, i.e., for arbitrary sources (not necessarily, stationary, ergodic, etc.), the error exponent was studied in [5], [8], [9], [14], and [11]. Specifically, in [8], [9], Han established expressions for the type 2 error and correct exponents in terms of the spectral \( \inf \) / \( \sup \) divergence rates.

In the sequel, we extend Csizsár’s results by investigating the forward and reverse \( \beta \)-cutoff rates for the hypothesis testing between two arbitrary sources with memory. We demonstrate that the \( \liminf \) Rényi \( \alpha \)-divergence rate provides the expression for the forward \( \beta \)-cutoff rate. Our proof relies in part on the formulas established in [8], and considerable extensions of the techniques used in [4] that generalize Csizsár’s source coding cutoff rate results for arbitrary discrete sources with memory. Unlike in [4], where the source alphabet was assumed to be finite, we assume arbitrary (countable or continuous) source alphabet. The methods used in our proof are a mixture of the techniques used in deriving the forward and reverse \( \beta \)-cutoff rates for source coding [4]. However, some new techniques are also needed to obtain our result.

We also investigate the reverse \( \beta \)-cutoff rate problem for arbitrary sources with memory. We show that if the log-likelihood ratio large-deviation spectrum \( \rho(R) \) admits a limit, then the \( \liminf \) \( \alpha \)-divergence rate with \( \alpha = \frac{1}{\beta^2} \) provides the expression for the reverse \( \beta \)-cutoff rate for \( 0 < \beta < \beta_{\text{max}} \), where \( \beta_{\text{max}} \) is the largest \( \beta < 1 \) for which the \( \liminf \) \( \alpha \)-divergence rate is finite. We also observe (via an example), that if the limit of \( \rho(R) \) does not hold, the reverse cutoff rate may not equal the \( \liminf \) Rényi divergence rate in general. For \( \beta_{\text{max}} \leq \beta < 1 \), we show that the reverse \( \beta \)-cutoff rate is in general only upper-bounded by the \( \liminf \) Rényi divergence rate.

The rest of the paper is organized as follows. In Section II, we briefly recall previous results by Han [8] on the general expression for the Neyman–Pearson type 2 error subject to an exponential bound on the type 1 error. In Section III, we establish the formula for the forward \( \beta \)-cutoff rate, and we illustrate it via some examples in Section IV. In Section V, we recall the general expression for the reliability function of the type 2 probability of correct decoding [8] and formulate the reverse \( \beta \)-cutoff rate problem by carefully examining the inconsistency of definitions between [6] and [8]. In Section VI, we investigate the reverse \( \beta \)-cutoff rate, and in Section VII, we illustrate it with some examples. Finally, we conclude in Section VIII.
II. HYPOTHESIS TESTING ERROR EXPONENT

Let us first define the general source as an infinite sequence

$$X = \{X^n\}_{n=1}^{\infty} \triangleq (X_1^n, \ldots, X_n^n)^{\infty}_{n=1}$$

of $n$-dimensional random variables $X^n$ where each component random variable $X_i^n (1 \leq i \leq n)$ takes values in an arbitrary (countable or continuous) set $\mathcal{X}$ that we call the source alphabet. Given two arbitrary sources $X = \{X^n\}_{n=1}^{\infty}$ and $\tilde{X} = \{\tilde{X}^n\}_{n=1}^{\infty}$ taking values in the same source alphabet $\{X^n\}_{n=1}^{\infty}$, we may define the general hypothesis testing problem with $X = \{X^n\}_{n=1}^{\infty}$ as the null hypothesis and $\tilde{X} = \{\tilde{X}^n\}_{n=1}^{\infty}$ as the alternative hypothesis.

Let $\mathcal{A}_n$ be any subset of $\{X^n\}_{n=1}^{\infty}$, $n = 1, 2, \ldots$, that we call the acceptance region of the hypothesis testing, and define

$$\mu_n \triangleq \Pr\{X^n \not\in \mathcal{A}_n\} \quad \text{and} \quad \lambda_n \triangleq \Pr\{\tilde{X}^n \in \mathcal{A}_n\}$$

where $\mu_n$, $\lambda_n$ are called type 1 error probability and type 2 error probability, respectively.

**Definition 1 [6]:** Fix $E > 0$. A rate $r$ is called $E$-achievable if there exists a sequence of acceptance regions $\mathcal{A}_n$ such that

$$\liminf_{n \to \infty} -\frac{1}{n} \log \mu_n \geq r \quad \text{and} \quad \liminf_{n \to \infty} -\frac{1}{n} \log \lambda_n \geq E.$$

**Definition 2 [6]:** The supremum of all $E$-achievable rates is denoted by $D_\alpha(E|X||\tilde{X})$

$$D_\alpha(E|X||\tilde{X}) \triangleq \sup \left\{ r > 0 : r \text{ is } E\text{-achievable} \right\}$$

and $D_\alpha(E|X||\tilde{X}) = 0$ if the above set is empty (which is a degenerate uninteresting case). The dual of the function $D_\alpha(E|X||\tilde{X})$ is defined as

$$B_\alpha(r|X||\tilde{X}) \triangleq \sup \left\{ E > 0 : E \text{ is } r\text{-achievable} \right\}$$

and $B_\alpha(r|X||\tilde{X}) = 0$ if the above set is empty.

**Proposition 1 [8]:** Fix $r > 0$. For the general hypothesis testing problem, we have that

$$B_\alpha(r|X||\tilde{X}) = \inf_{R \in \mathbb{R}} \{ R + \eta(R) : \eta(R) < r \}$$

where

$$\eta(R) \triangleq \liminf_{n \to \infty} -\frac{1}{n} \log P_{X^n} \left\{ x^n \in X^n : \frac{1}{n} \log \frac{P_{X^n}(x^n)}{P_{\tilde{X}^n}(x^n)} \leq R \right\}$$

is the large deviation spectrum of the normalized log-likelihood ratio.

For the sake of simplicity, we assume throughout that the source alphabet is countable. However, we will point out the necessary modifications in the proofs for the case of a continuous alphabet. Proposition 1 is the main tool for our key lemma in Section III.

III. FORWARD $\beta$-CUTOFF RATE FOR ARBITRARY SOURCES

**Definition 3 [6]:** Fix $\beta < 0$. $R_\beta \geq 0$ is a forward $\beta$-achievable rate for the general hypothesis testing problem if

$$D_\alpha(E|X||\tilde{X}) \geq \beta (E - R_\beta)$$

for every $E > 0$, or equivalently

$$B_\alpha(r|X||\tilde{X}) \geq R_\beta + \frac{r}{\beta}$$

1If the source alphabet $\mathcal{X}$ is continuous, then $P_{X^n}(X^n)$ plays the role of the density function $f_{X^n}(X^n)$, when it exists.

![Fig. 1](image)

A graphical illustration of the forward $\beta$-cutoff rate, $R_\beta^{DF}(\beta|X||\tilde{X})$, for testing between two arbitrary sources $X$ and $\tilde{X}$.

for every $r > 0$. The forward $\beta$-cutoff rate is defined as the supremum of all forward $\beta$-achievable rates, and is denoted by $R_\beta^{DF}(\beta|X||\tilde{X})$.

Note that in the degenerate case where $D_\alpha(E|X||\tilde{X})$ is identically 0, we have that $R_\beta^{DF}(\beta|X||\tilde{X}) = 0$. We herein assume that $D_\alpha(E|X||\tilde{X})$ is not 0 for all values of $E$. A graphical illustration of the forward $\beta$-cutoff rate $R_\beta^{DF}(\beta|X||\tilde{X})$ for testing between two arbitrary sources $X$ and $\tilde{X}$ is given in Fig. 1.

Before stating our main result, we first observe in the next lemma that the forward $\beta$-cutoff rate $R_\beta^{DF}(\beta|X||\tilde{X})$ is indeed the $R$-axis intercept of a support line of slope $\frac{\beta}{\beta - 1}$ to the large deviation spectrum $\eta(R)$.

**Lemma 1:** Fix $\beta < 0$. The following conditions are equivalent:

$$(\forall \ R \in \mathbb{R}) \quad \eta(R) \geq \frac{\beta}{\beta - 1} (R_\beta - R)$$

and

$$(\forall \ r > 0) \quad B_\alpha(r|X||\tilde{X}) \geq R_\beta + \frac{r}{\beta}.$$

**Proof:**

a) Equation (1) $\Rightarrow$ (2). For any $r > 0$, we obtain by Proposition 1 that

$$(\forall \delta > 0) \exists R_\delta \text{ with } \eta(R_\delta) < r \quad B_\alpha(r|X||\tilde{X}) + \delta \geq R_\delta + \eta(R_\delta).$$

Therefore,

$$B_\alpha(r|X||\tilde{X}) \geq R_\delta + \eta(R_\delta) - \delta \geq R_\delta + \frac{\beta}{\beta - 1} (R_\delta - R_\beta)$$

$$\geq - \delta + \frac{\beta}{\beta - 1} R_\delta - \frac{R_\delta}{\beta - 1} + \frac{r}{\beta}$$

$$\geq \frac{r}{\beta} + R_\delta - \delta$$

where (3) follows from (1), and (4) holds because

$$r > \eta(R_\delta) \geq \frac{\beta}{\beta - 1} (R_\delta - R_\delta).$$

Since $\delta$ can be made arbitrarily small, the proof of the forward part is completed.
b) Equation (2) \( \Rightarrow \) (1). Inequality (1) holds trivially for those \( R \) satisfying \( \eta(R) = \infty \). For any \( R \in \mathbb{R} \) with \( \eta(R) < \infty \), let \( r_{\delta} \triangleq \eta(R) + \delta \) for some \( \delta > 0 \). Then (by Proposition 1)

\[
B_\varepsilon(\{r_{\delta} \mid X\|\tilde{X}^n\}) \leq R + \eta(R).
\]

Therefore,

\[
\eta(R) \geq B_\varepsilon(\{r_{\delta} \mid X\|\tilde{X}^n\}) - R
\geq R_\varepsilon + \frac{r_{\delta}}{\beta} - R
= R_\varepsilon + \frac{\eta(R)}{\beta} + \frac{\delta}{\beta} = R
\]

where (6) follows by (2). Thus,

\[
\eta(R) \geq \frac{\beta}{\beta - 1} \left( R_\varepsilon - R + \frac{\delta}{\beta - 1} \right).
\]

Since \( \delta \) can be made arbitrarily small, the proof of the converse part is completed. \qed

**Theorem 1 (Forward \( \beta \)-Cutoff Rate Formula):** Fix \( \beta < 0 \). For the general hypothesis testing problem

\[
R_{0/f}^{(f)}(\beta \mid X\|\tilde{X}^n) = \lim_{n \to \infty} \min \left\{ \frac{1}{n} D_{\alpha \rightarrow 1} \left( X^n \mid \tilde{X}^n \right) \right\}
\]

where

\[
D_{\alpha}(X^n \mid \tilde{X}^n) \triangleq \frac{1}{\alpha - 1} \log \left( \sum_{x \in X^n} [P_X^n(x^n)]^{\alpha} [P_{\tilde{X}^n}(x^n)]^{1-\alpha} \right)
\]

is the \( n \)-dimensional \( \alpha \)-divergence.

*Proof:* Note that \( \eta(R) > 0 \) for some \( R \in \mathbb{R} \).

1) **Forward part:**

\[
R_{0/f}^{(f)}(\beta \mid X\|\tilde{X}^n) = \lim_{n \to \infty} \min \left\{ \frac{1}{n} D_{\alpha \rightarrow 1} \left( X^n \mid \tilde{X}^n \right) \right\}
\]

By the equivalence of conditions (1) and (2), it suffices to show that

\[
(\forall R \in \mathbb{R}) \eta(R) \geq \frac{\beta}{\beta - 1} \left( \lim_{n \to \infty} \min \left\{ \frac{1}{n} D_{\alpha \rightarrow 1} \left( X^n \mid \tilde{X}^n \right) - R \right\} \right).
\]

Indeed, we have the following:

\[
Pr \left[ \frac{1}{n} \log \frac{P_X^n(x^n)}{P_{\tilde{X}^n}(x^n)} \leq \frac{1}{\beta - 1} \left( \lim_{n \to \infty} \min \left\{ \frac{1}{n} D_{\alpha \rightarrow 1} \left( X^n \mid \tilde{X}^n \right) - R \right\} - R \right) \right]
\]

\[
\leq e^{-nR} \left( \sum_{x \in \mathbb{X}^n} [P_X^n(x^n)]^{\beta} [P_{\tilde{X}^n}(x^n)]^{1-\beta} \right) \exp \left\{ -n \left( \frac{1}{n} D_{\alpha \rightarrow 1} \left( X^n \mid \tilde{X}^n \right) - R \right) \right\}
\]

for \( 0 < t < 1 \), where (7) follows by Markov’s inequality. Therefore,

\[
\eta(R) \geq \frac{\beta}{\beta - 1} \left( \lim_{n \to \infty} \min \left\{ \frac{1}{n} D_{\alpha \rightarrow 1} \left( X^n \mid \tilde{X}^n \right) - R \right\} \right).
\]

2) **Converse part:**

\[
R_{0/f}^{(f)}(\beta \mid X\|\tilde{X}^n) \leq \lim_{n \to \infty} \min \left\{ \frac{1}{n} D_{\alpha \rightarrow 1} \left( X^n \mid \tilde{X}^n \right) \right\}
\]

The converse holds trivially if

\[
\liminf_{n \to \infty} \frac{1}{n} D_{\alpha \rightarrow 1} \left( X^n \mid \tilde{X}^n \right)
\]

is infinite. Hence we can assume that

\[
\liminf_{n \to \infty} \frac{1}{n} D_{\alpha \rightarrow 1} \left( X^n \mid \tilde{X}^n \right) < K
\]

where \( K \) is some constant. By the equivalence of conditions (1) and (2), it suffices to show that for any \( \delta > 0 \) arbitrarily small, there exists \( R = R(\delta) \in \mathbb{R} \) such that

\[
\eta(R) \leq \frac{\beta}{\beta - 1} \left( 3\delta + \liminf_{n \to \infty} \frac{1}{n} D_{\alpha \rightarrow 1} \left( X^n \mid \tilde{X}^n \right) - R \right)
\]

Consider the twisted distribution defined as

\[
P_{X^n}(x^n) \triangleq \frac{1}{K} \sum_{x \in \mathbb{X}^n} [P_X^n(x^n)]^{t}[P_{\tilde{X}^n}(x^n)]^{1-t}
\]

\[
= \exp \left\{ t \left( \frac{\log P_{\tilde{X}^n}(x^n)}{\alpha} + \frac{1}{\alpha} \right) \right\} P_X^n(x^n)
\]

and define

\[
\tau \triangleq \sup \left\{ R \in \mathbb{R} : \eta^{(t)}(R) > 0 \right\}
\]

where

\[
\eta^{(t)}(R) \triangleq \liminf_{n \to \infty} \frac{1}{n} D_{\alpha \rightarrow 1(t)} \left( X^n \mid \tilde{X}^n \right)
\]

where \( t = \beta/(\beta - 1) \). Note that \( 0 < t < 1 \). Let \( \mathcal{X} \) be a set of positive integers such that

\[
\liminf_{n \to \infty} \frac{1}{n} D_{\alpha \rightarrow 1(t)} \left( X^n \mid \tilde{X}^n \right) = \liminf_{n \to \infty} \frac{1}{n} D_{\alpha \rightarrow 1(t)} \left( X^n \mid \tilde{X}^n \right)
\]

and define

\[
\tau \triangleq \sup \left\{ R \in \mathbb{R} : \eta^{(t)}(R) > 0 \right\}
\]

where

\[
\eta^{(t)}(R) \triangleq \liminf_{n \to \infty} \frac{1}{n} D_{\alpha \rightarrow 1(t)} \left( X^n \mid \tilde{X}^n \right)
\]

We then note by definition of \( \eta^{(t)}(\cdot) \) and the finiteness property of \( \tau \) that for any \( \delta > 0 \), there exists \( \varepsilon > 0 \) such that

\[
\eta^{(t)}(\tau - \delta) = \liminf_{n \to \infty} \frac{1}{n} \log P_{\tilde{X}^n}(x^n) \left\{ x \in \mathcal{X} : \frac{1}{n} \log \frac{P_X^n(x^n)}{P_{\tilde{X}^n}(x^n)} \leq R \right\} > \varepsilon > 0.
\]

As a result

\[
P_{\tilde{X}^n}(x^n) \left\{ x \in \mathcal{X} : \frac{1}{n} \log \frac{P_X^n(x^n)}{P_{\tilde{X}^n}(x^n)} > R - \delta \right\}
\]

\[
\geq 1 - e^{-n\varepsilon} \text{ for } n \in \mathcal{X} \text{ sufficiently large.}
\]

On the other hand, define

\[
\tilde{\eta}^{(t)}(R) \triangleq \liminf_{n \to \infty} \frac{1}{n} \log P_{\tilde{X}^n}(x^n) \left\{ x \in \mathcal{X} : \frac{1}{n} \log \frac{P_X^n(x^n)}{P_{\tilde{X}^n}(x^n)} \geq R \right\}
\]

and

\[
\tau \triangleq \inf \left\{ R \in \mathbb{R} : \tilde{\eta}^{(t)}(R) > 0 \right\}.
\]

Then by noting that

\[
\log \frac{P_X^n(x^n)}{P_{\tilde{X}^n}(x^n)} = D_{\alpha \rightarrow 1(t)} \left( X^n \mid \tilde{X}^n \right) - \frac{1}{t} \log \frac{P_X^n(x^n)}{P_{\tilde{X}^n}(x^n)}
\]
we have
\[ \varpi^{(t)}(R) = \sigma \left( -tR + \frac{t}{n} D_{1-t}(X^n||\tilde{X}^n) \right) \]
and
\[ \varpi = -\frac{1}{t} \sup \left\{ R \in \mathbb{R} : \sigma(R) > 0 \right\} + \frac{1}{n} D_{1-t}(X^n||\tilde{X}^n) \leq \frac{1}{n} D_{1-t}(X^n||\tilde{X}^n) < K \text{ for } n \in \mathcal{N} \text{ sufficiently large} \]
(10)

(10) follows from Lemma 5 in Appendix A, and (11) holds by definition of \( K \). This indicates the existence of \( \varpi > 0 \) such that \( \varpi^{(t)}(K) > \varpi \), which immediately gives that for \( n \in \mathcal{N} \) sufficiently large
\[ P_{X^n} \left\{ x^n \in \mathcal{X}^n : \frac{1}{n} \log P_{X^n}(x^n) \geq K \right\} \leq e^{-n\varpi}. \]
Therefore, for \( n \in \mathcal{N} \) sufficiently large
\begin{align}
P_{X^n} \left\{ x^n \in \mathcal{X}^n : K > \frac{1}{n} \log \frac{P_{X^n}(x^n)}{P_{X^n}(x^n)} > \tau - \delta \right\} \\
= P_{X^n} \left\{ x^n \in \mathcal{X}^n : \frac{1}{n} \log \frac{P_{X^n}(x^n)}{P_{X^n}(x^n)} > \tau - \delta \right\} \\
- P_{X^n} \left\{ x^n \in \mathcal{X}^n : \frac{1}{n} \log \frac{P_{X^n}(x^n)}{P_{X^n}(x^n)} \geq K \right\} \\
\geq 1 - e^{-n\varpi} - e^{-n\varpi}. \tag{12}
\end{align}

Let \( I_1 \triangleq (\tau - \delta, b_1) \), and
\[ I_k \triangleq [b_{k-1}, b_k) \text{ for } 2 \leq k \leq L \triangleq \left[ \frac{K - \tau + \delta}{2\delta} \right] \]
where \( b_k \triangleq (\tau - \delta) + 2k\delta \) for \( 1 \leq k < L \), and \( b_k \triangleq K \). By (12), there exists \( 1 \leq k(n) \leq L \) such that
\[ P_{X^n} \left\{ x^n \in \mathcal{X}^n : \frac{1}{n} \log \frac{P_{X^n}(x^n)}{P_{X^n}(x^n)} \in I_{k(n)} \right\} \geq 1 - e^{-n\varpi} - e^{-n\varpi} \frac{L}{L} \tag{13} \]
for \( n \in \mathcal{N} \) sufficiently large. Then, by letting
\[ R_1 \triangleq \lim \sup_{n \to \infty} b_{k(n)} + \delta \]
we obtain that for \( n \in \mathcal{N} \) sufficiently large
\begin{align}
P_{X^n} \left\{ x^n \in \mathcal{X}^n : \frac{1}{n} \log \frac{P_{X^n}(x^n)}{P_{X^n}(x^n)} \in I_{k(n)} \right\} \\
= \sum_{x^n \in \mathcal{X}^n} \frac{1}{n} \log \frac{P_{X^n}(x^n)}{P_{X^n}(x^n)} \in I_{k(n)} \right\} \\
= \exp \left\{ \log \frac{P_{X^n}(x^n)}{P_{X^n}(x^n)} + \frac{1}{n} D_{1-t}(X^n||\tilde{X}^n) \right\} \tag{14} \end{align}

\begin{align}
\leq e^{-n\varpi} \left\{ \frac{1}{n} \log \frac{P_{X^n}(x^n)}{P_{X^n}(x^n)} \right\} \sum_{x^n \in \mathcal{X}^n} \frac{1}{n} \log \frac{P_{X^n}(x^n)}{P_{X^n}(x^n)} \in I_{k(n)} \right\} \\
= e^{-n\varpi} \left\{ \frac{1}{n} \log \frac{P_{X^n}(x^n)}{P_{X^n}(x^n)} \right\} P_{X^n} \left\{ x^n \in \mathcal{X}^n : \frac{1}{n} \log \frac{P_{X^n}(x^n)}{P_{X^n}(x^n)} \in I_{k(n)} \right\} \\
\geq \frac{1 - e^{-n\varpi}}{L} - e^{-n\varpi} \frac{L}{L} \\
= \frac{1 - e^{-n\varpi}}{L} \tag{15} \\
\]

However, for sufficiently large \( n \in \mathcal{N} \), we have (14) and (15) at the bottom of the page, where (14) follows from (8), and (15) follows from (13). Consequently
\[ \eta(R_1) = \lim_{n \to \infty} \inf \frac{1}{n} \log P_{X^n} \left\{ x^n \in \mathcal{X}^n : \frac{1}{n} \log \frac{P_{X^n}(x^n)}{P_{X^n}(x^n)} \leq R_1 \right\} \]
\begin{align}
\leq \sup_{n \in \mathcal{N}} \frac{1}{n} \log P_{X^n} \left\{ x^n \in \mathcal{X}^n : \frac{1}{n} \log \frac{P_{X^n}(x^n)}{P_{X^n}(x^n)} \leq R_1 \right\} \\
- \frac{1}{n} \log P_{X^n} \left\{ x^n \in \mathcal{X}^n : \frac{1}{n} \log \frac{P_{X^n}(x^n)}{P_{X^n}(x^n)} \leq R_1 \right\} \leq t \left\{ - \sup_{n \in \mathcal{N}} \frac{1}{n} \log \frac{P_{X^n}(x^n)}{P_{X^n}(x^n)} \leq R_1 \right\} \\
\leq t \left\{ - \sup_{n \in \mathcal{N}} \frac{1}{n} \log \frac{P_{X^n}(x^n)}{P_{X^n}(x^n)} \leq R_1 \right\} \leq t \left\{ - \sup_{n \in \mathcal{N}} \frac{1}{n} \log \frac{P_{X^n}(x^n)}{P_{X^n}(x^n)} \leq R_1 \right\} \\
\end{align}

Since \( \delta \) can be made arbitrarily small, the proof is completed. \( \square \)

**Observations:**

1) While the proof of the forward part is straightforward, the proof of the converse part is considerably more complex. The objective of the converse part is to demonstrate that if \( \lim \inf_{n \to \infty} \frac{1}{n} \log P_{X^n} \left\{ x^n \in \mathcal{X}^n : \frac{1}{n} \log \frac{P_{X^n}(x^n)}{P_{X^n}(x^n)} \leq R_1 \right\} \)

lies above the curve of \( \eta(R) \) at \( R = R_1 \), thus violating its status of support line for \( \eta(R) \).

This proof is established by observing that the desired coordinate \( R \) lies in a small neighborhood of \( \tau \), where \( \tau \) is the smallest point for which \( \eta^{(t)}(R) \) vanishes. A key point is to choose the twisted parameter \( t \) to be equal to \( \beta - 1 \) which is the negative slope of the support line of \( \eta(R) \). We graphically illustrate this observation (based on a true example involving binary memoryless or i.i.d. sources) in Fig. 2. The computational details are described in Example 1 (cf. Section IV).
under the Gaussian distributions

\[ \inf_{\mu \in \mathcal{M}} \liminf_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} D_{\mu}(X_i^n || \bar{X}^n) - R \]

for testing two binary memoryless sources \( X = \{ X_i \}_{i=1}^{\infty} \) and \( \bar{X} = \{ \bar{X}_i \}_{i=1}^{\infty} \) under the distributions \( (1/2, 1/2) \) and \( (1/4, 3/4) \), respectively, and with \( \beta = -7 \). When \( R < -\log(3/2) \), \( \eta(R) = \eta^0(R) = \infty \).

\[ \eta(R) \]

\[ \eta^0(R) \]

\[ -\log(3/2) \]

\[ \tau \]

\[ O \]

\[ D(\bar{X} || \bar{X}) \]

\[ O \]

\[ D(X || \bar{X}) \]

\[ R \]

2) Note also that the proof holds if the alphabet is countable or continuous as opposed to the source-coding forward and reverse \( \beta \)-cutoff rates results \([4]\), where the finiteness property of the alphabet is necessary. The modifications in the proof for the continuous case are clear. Simply, replace the probability mass function by the proper probability function and the summation by integration. We graphically illustrate this observation (based on a true example involving memoryless Gaussian sources) in Fig. 3. The computational details are described in Example 2 (cf. Section IV).

3) The proof of the hypothesis testing forward \( \beta \)-cutoff rate is more involved than the proof of the source coding forward \( \beta \)-cutoff rate result given in \([4]\). The main difficulty arises from the formula in Proposition 1 where the infimum for \( R \) is taken over the entire real line contrary to \([4, \text{Proposition 1}]\) for source coding where \( R \) ranges from 0 to \( \infty \). This requires us to deal separately with the degenerate case \( \tau = -\infty \) (cf. Lemma 4 in Appendix A). Also, the technique used to prove the forward \( \beta \)-cutoff rate for hypothesis testing relies on the proofs of both the source coding forward and reverse \( \beta \)-cutoff rates, but in major parts though similar to the reverse source coding \( \beta \)-cutoff rate.

4) If the sources \( X \) and \( \bar{X} \) are arbitrary (not necessarily stationary, irreducible) time-invariant finite-alphabet Markov sources of arbitrary order, then we know that the \( \alpha \)-divergence rate exists and can be computed \([17], [18]\). Thus, in this case, the forward \( \beta \)-cutoff rate for testing between Markov sources can be obtained. Also, from the definition of \( R_0^{1/0}(\beta | X || \bar{X}) \), it follows directly that for all \( E > 0 \)

\[ D_1(E | X || \bar{X}) \geq \sup_{\beta > 0} \left[ \beta (E - R_0^{1/0}(\beta | X || \bar{X})) \right] \]

Note that this convex lower bound is computable for the entire class of Markov sources, while \( D_1(E | X || \bar{X}) \) is not necessarily computable in general (it is computable for irreducible Markov sources \([1], [15]\), see Fig. 4). We graphically illustrate this observation for testing between
irreducible Markov sources in Fig. 4 and arbitrary Markov sources (not necessarily stationary, irreducible) in Fig. 5. The computational details are described in Examples 3 and 4 (cf. Section IV).

IV. EXAMPLES FOR THE FORWARD $\beta$-CUTOFF RATE

Throughout this section, the natural logarithm is used.

Example 1 Finite-Alphabet Memoryless Sources: Naturally, in this case, the forward cutoff rate expression is readily known as Theorem 1 reduces to Csiszár’s result [6]. But this example is useful to illustrate the proof method discussed in Observation 1 of Section III since $\eta(R)$ and $\eta^{(i)}(R)$ can be explicitly derived.

Consider the binary hypothesis testing between two memoryless (i.i.d.) sources $X = \{X_i\}_{i=1}^{\infty}$ and $\tilde{X} = \{\tilde{X}_i\}_{i=1}^{\infty}$ under the distributions $(1/2, 1/2)$ and $(1/4, 3/4)$, respectively. Then the log-likelihood ratio $Z = \log \frac{f_X(x)}{f_{\tilde{X}}(x)}$ has the following distribution:

$$\Pr \{Z = \log(2)\} = 1 - \Pr \{Z = \log(2/3)\} = 1/2.$$  

By Cramer’s theorem [3, p. 9], we get that

$$\eta(R) = \begin{cases} 
\infty, & R < -\log(3/2) \\
\log(2), & R = -\log(3/2) \\
\log(\log(3/2) + R - \log(3/2)), & -\log(3/2) < R < -\log(3/2) \\
\log(2) - \log(\log(3/2) + R - \log(3/2)), & -\log(3/2) < R < -\log(3/2) \\
0, & \text{otherwise}
\end{cases}$$

where $E[Z]$ denotes the expectation of the random variable $Z$. Let $R'$ be the rate at which the line of slope $\beta/(1 - \beta)$ is tangent to $\eta(R)$. By straightforward calculations, we get that

$$R' = \log 2 - \frac{\log 3}{1 + 3^{1/\beta}}$$

and that the forward $\beta$-cutoff rate $R_0^{(i)}(\beta|X||\tilde{X})$, which is the $R$-axis intercept of the tangent line of slope $\beta/(1 - \beta)$ to $\eta(R)$, is given by

$$R_0^{(i)}(\beta|X||\tilde{X}) = \frac{2\beta - 1}{\beta} \log 2 - \frac{\beta - 1}{\beta} \log \left(1 + 3^{1/\beta}\right) - \log 3.$$  

On the other hand, the $\alpha$-divergence between $X$ under $\tilde{X}$ is given by

$$D_\alpha(X||\tilde{X}) = \frac{1}{\alpha - 1} \left((\alpha - 2) \log 2 + \log(1 + 3^{1-\alpha})\right)$$

which yields

$$D_\alpha^{1/\alpha}(X||\tilde{X}) = \frac{2\beta - 1}{\beta} \log 2 - \frac{\beta - 1}{\beta} \log \left(1 + 3^{1/\beta}\right) - \log 3.$$  

Note that the forward $\beta$-cutoff rate, $R_0^{(i)}(\beta|X||\tilde{X})$ and the lim inf $\alpha$-divergence rate (which is equal to the $\alpha$-divergence since the sources are memoryless) of order $\alpha = 1/(1 - \beta)$ are equal as expected from Theorem 1. Let us now derive $\tau$ in order to check that $\tau = R'$. First, we need to compute $\eta^{(i)}(R)$. The set $\mathcal{N}$ is equal to the set of natural numbers in this case. Note that the distribution of the random variable $Z^{(i)}$ under the twisted distribution with parameter $0 < t < 1$ is given by

$$P^{(i)} \{Z = \log 2\} = 1/(1 + 3^t)$$

and

$$P^{(i)} \{Z = \log(2/3)\} = 3^t/(1 + 3^t).$$
By Cramer’s theorem [3, p. 9], we get that

\[
\eta^{(t)}(R) = \begin{cases} 
\infty, & R < -\log(3/2) \\
\log(1+3^t), & t(R - \log(2)) + \log(\log(3/2) + R) - \log(R) \\
\log(3/2) - \log(\log(3/2) + R), & \log(2/R) - \log(2) \\
0, & \text{otherwise}
\end{cases}
\]

where \( E[Z^{(t)}] \) denotes the expectation of the random variable \( Z^{(t)} \). Therefore,

\[
\tau = \frac{\log 2}{1 + 3^t} + \log(2/3) \frac{3^t}{1 + 3^t}.
\]

It is easy to check that indeed we have \( \tau = R' \) when the twisted parameter \( t \) is chosen to be \( \beta/(\beta - 1) \). This example is illustrated in Fig. 2 for \( \beta = -7 \).

**Example 2 Continuous Alphabet Memoryless Sources:** Consider the hypothesis testing problem between two memoryless sources \( X = \{X_i\}_{i=1}^{\infty} \) and \( \bar{X} = \{\bar{X}_i\}_{i=1}^{\infty} \) under the Gaussian distributions \( N(\nu, 1) \) and \( N(-\nu, 1) \), respectively, where \( N(\alpha, \beta) \) represents a Gaussian distribution with mean \( \alpha \) and variance \( \beta \). It is easy to check that the log-likelihood ratio \( Z = \log \left( \frac{f_X(x)}{f_{\bar{X}}(\bar{x})} \right) \) is Gaussian distributed with mean \( 2\nu^2 \) and variance \( 4\nu^2 \), which gives that the moment generating function of \( Z \) is \( E[e^{\theta Z}] = e^{\theta^2\nu^2 + 2\nu^2\theta^2} \). By Cramer’s theorem, we get that

\[
\eta(R) = \begin{cases} 
\frac{1}{8\nu^2} (R - 2\nu^2)^2, & R < 2\nu^2 \\
0, & \text{otherwise}
\end{cases}
\]

Let \( R' \) be the rate at which the line of slope \( \beta/(1 - \beta) \) is tangent to \( \eta(R) \). We have that

\[
R' = 2\nu^2 \frac{1 + \beta}{1 - \beta}.
\]

Thus, the forward \( \beta \)-cutoff rate \( R^{(t)}_0(\beta | X || \bar{X}) \), which is the \( R \)-axis intercept of the tangent line of slope \( \beta/(1 - \beta) \) to \( \eta(R) \), is given by

\[
R^{(t)}_0(\beta | X || \bar{X}) = 2\nu^2 \frac{1}{1 - \beta}.
\]

On the other hand, the \( \alpha \)-divergence between \( X \) under \( \bar{X} \) is given by

\[
D_{\alpha}(X || \bar{X}) = 2\nu^2 \alpha, \quad \text{where yields}
\]

\[
D_{\alpha}^{(t)}(X || \bar{X}) = 2\nu^2 \frac{1}{1 - \beta}.
\]

Note that the forward \( \beta \)-cutoff rate \( R^{(t)}_0(\beta | X || \bar{X}) \), and the \( \liminf \) of \( \alpha \)-divergence rate (which is equal to the \( \alpha \)-divergence since the sources are memoryless) of order \( \alpha = 1/(\beta - 1) \) are equal, as expected from Theorem 1.

Now, let us compute \( \eta^{(t)}(R) \). The set \( \mathcal{N} \) in this case is equal to the set of natural numbers. For some normalization constant \( C \)

\[
P_X^{(t)}(x^n) = C \cdot \exp \left\{ -\frac{1}{2} \sum_{i=1}^{n} (x_i + \nu)^2 \right\} \exp \left\{ -\frac{1}{2} - \frac{1}{2} \sum_{i=1}^{n} (x_i - \nu)^2 \right\}
\]

\[
= C \cdot \exp \left\{ -\frac{1}{2} \sum_{i=1}^{n} t(x_i + \nu)^2 + (1 - t)(x_i - \nu)^2 \right\}
\]

\[
= C \cdot \exp \left\{ -\frac{1}{2} \sum_{i=1}^{n} (x_i^2 + 2t(2t - 1)x_i + \nu^2) \right\}
\]

which is a Gaussian distribution with mean \( (1 - 2t)\nu \) and unit variance. Similarly, by invoking Cramer’s theorem, we get that

\[
\eta^{(t)}(R) = \begin{cases} 
\frac{1}{8\nu^2} (R + (2t - 1)2\nu^2)^2, & R < (1 - 2t)2\nu^2 \\
0, & \text{otherwise}
\end{cases}
\]

Hence, \( \tau = (1 - 2t)2\nu^2 \). It is straightforward to check that \( \tau = R' \) when the twisted parameter \( t \) is chosen to be \( \beta/(\beta - 1) \). This example is depicted in Fig. 3 for \( \beta = -0.5 \).
Example 3 Irreducible Finite-Alphabet Markov Sources: Suppose that X and \(\tilde{X}\) are two irreducible finite-alphabet sources with arbitrary initial distributions and probability transition matrices P and Q defined as follows:

\[
P = \begin{pmatrix} 1/3 & 2/3 \\ 1/4 & 3/4 \end{pmatrix}, \quad Q = \begin{pmatrix} 1/5 & 4/5 \\ 5/6 & 1/6 \end{pmatrix}.
\]

Define a new matrix R = \((r_{ij})\) by

\[
r_{ij} = p_{ij}^{\alpha} q_{ij}^{\beta}, \quad i, j = 0, 1.
\]

The \(\alpha\)-divergence rate between X and \(\tilde{X}\) exists and is given by

\[
\lim_{n \to \infty} \frac{1}{n} D_n(\mathcal{X}^n | \tilde{\mathcal{X}}^n) = \frac{1}{\alpha - 1} \log \lambda
\]

where \(\lambda\) is the largest positive real eigenvalue of \(R\) [17, 18]. Hence, the computation of the convex lower bound for \(D_n(E|X||\tilde{X})\) is easily obtained as shown in Fig. 4 for the values

\[
\beta = -5, -3, -2, -1/3, -1, -2/3, -1/2, -2/5
\]

(proceeding from left to right), where \(\alpha = \frac{1}{1-\beta}\). Note that in this case the convex lower bound is tight [1], [15].

Example 4 Arbitrary Finite-Alphabet Markov Sources: Suppose that X and \(\tilde{X}\) are two arbitrary Markov sources with arbitrary initial distributions and probability transition matrices P and Q defined as follows:

\[
P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 & 0 \\ 0 & 0 & 3/5 & 2/5 & 0 \\ 0 & 1/6 & 5/6 & 0 & 0 \\ 1/4 & 0 & 1/4 & 0 & 1/2 \\ 1/5 & 4/5 & 0 & 0 & 0 \\ 2/3 & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 1/6 & 5/6 & 0 & 0 \\ 1/8 & 0 & 1/2 & 0 & 3/8 \end{pmatrix}.
\]

Define a new matrix R = \((r_{ij})\) by

\[
r_{ij} = p_{ij}^{\alpha} q_{ij}^{\beta}, \quad i, j = 0, 1, 2, 3, 4.
\]

The \(\alpha\)-divergence rate between X and \(\tilde{X}\) can be computed [17, 18]. Hence, the convex lower bound for \(D_n(E|X||\tilde{X})\) can be easily derived as shown in Fig. 4 for the values

\[
\beta = -5, -3, -2, -1/3, -1/2, -2/5, -1/6
\]

(proceeding from left to right), where \(\alpha = \frac{1}{1-\beta}\).

V. HYPOTHESIS TESTING CORRECT EXPONENT AND PROBLEM FORMULATION

In [6], Csiszár investigated the hypothesis testing problem between i.i.d. observations by considering the \(\beta\)-cutoff rate for the exponent of the best correct probability of type 1 with exponential constraint on the probability of type 2 error. More formally, he used the following definitions.

Definition 4 [6]: Fix \(E > 0\). A rate \(r\) is called \(E\)-achievable if there exists a sequence of acceptance regions \(\mathcal{A}_n\) such that

\[
\limsup_{n \to \infty} \frac{-1}{n} \log (1 - \mu_n) \leq r \quad \text{and} \quad \liminf_{n \to \infty} \frac{-1}{n} \log \lambda_n \geq E
\]

where \(\mu_n\) and \(\lambda_n\) are type 1 and type 2 error probabilities, respectively. The infimum of all \(E\)-achievable rates is defined as

\[
D^*_r(E|X||\tilde{X}) = \inf \{ r > 0 : r \text{ is } E\text{-achievable} \}
\]

and \(D^*_r(E|X||\tilde{X}) = \infty\) if the above set is empty.

For \(0 < r < D^*_r(E|X||\tilde{X})\), every acceptable region \(\mathcal{A}_n\) with

\[
\liminf_{n \to \infty} \frac{-1}{n} \log \lambda_n \geq E
\]

satisfies \(\mu_n > 1 - e^{-nr}\) for \(n\) infinitely often.

Definition 5 [6]: Fix \(\beta > 0\). \(R_0 > 0\) is a reverse \(\beta\)-achievable rate for the general hypothesis testing problem if

\[
D^*_r(E|X||\tilde{X}) \geq \beta (E - R_0)
\]

for every \(E > 0\). The reverse \(\beta\)-cutoff rate is defined as the infimum of all reverse \(\beta\)-achievable rates, and is denoted by \(R_0^{\beta}(\beta|X||\tilde{X})\).

However, in [14, Sec. 7], Nagaoka and Hayashi investigated (using Han’s definitions [8]) the general hypothesis testing problem between arbitrary sources with memory by considering the exponent of the best correct probability of type 2 with exponential constraint on the probability of type 1 error. More formally, they used the following definition.

Definition 6 [8], [14]: Fix \(r > 0\). A rate \(E\) is called \(r\)-unachievable if there exists a sequence of acceptance regions \(\mathcal{A}_n\) such that

\[
\liminf_{n \to \infty} \frac{-1}{n} \log \mu_n \geq r \quad \text{and} \quad \limsup_{n \to \infty} \frac{-1}{n} \log (1 - \mu_n) \leq E.
\]

The infimum of all \(r\)-unachievable rates is denoted by \(B^*_r(r|X||\tilde{X})\)

\[
B^*_r(r|X||\tilde{X}) = \inf \{ E > 0 : E \text{ is } r\text{-unachievable} \}
\]

and \(B^*_r(r|X||\tilde{X}) = \infty\) if the above set is empty.

Proposition 2 [14, Theorem 14, eq. (56)]: Fix \(r > 0\). For the general hypothesis testing problem, we have that

\[
B^*_r(r|X||\tilde{X}) = r + \inf \{ R \in \mathbb{R} : \tilde{p}(R) \leq r \}
\]

where

\[
\tilde{p}(R) = \limsup_{n \to \infty} \frac{-1}{n} \log P_{\tilde{X}} \left\{ x^n \in \mathcal{X}^n : \frac{1}{n} \log P_{\tilde{X}}(x^n) \leq R \right\} - R.
\]

Remark 1: Note that Csiszár’s and Nagaoka/Hayashi’s definitions seem different at first glance. In our investigation, we realized that in order to establish our results on the reverse \(\beta\)-cutoff rate for general sources with memory, a formula for the reliability function of the type 1 probability of correct decoding (or type 1 correct exponent) \(D^*_r(E|X||\tilde{X})\) is needed. However, in [14], Nagaoka and Hayashi provided a formula for the reliability function of the type 2 probability of correct decoding \(B^*_r(r|X||\tilde{X})\). This turned out to be an obstacle, since we were not able to derive the reverse \(\beta\)-cutoff rate formula by directly using the formula for \(B^*_r(r|X||\tilde{X})\). To overcome this obstacle, we observed that if we interchange the roles of the null and alternative hypotheses distributions (i.e., \(X \leftrightarrow \tilde{X}\)), and also \(r \leftrightarrow \tilde{E}\) in Nagaoka/Hayashi’s definition (Definition 6), then a formula for \(D^*_r(E|X||\tilde{X})\) can be readily obtained from Nagaoka and Hayashi’s result. More specifically, we have the following.

Definition 7: Fix \(E > 0\). A rate \(r\) is called \(E\)-unachievable if there exists a sequence of acceptance regions \(\mathcal{A}_n = \mathcal{A}_n^c\) (complement of \(\mathcal{A}_n\)) such that

\[
\liminf_{n \to \infty} \frac{-1}{n} \log \lambda_n \geq E \quad \text{and} \quad \limsup_{n \to \infty} \frac{-1}{n} \log (1 - \mu_n) \leq r
\]

where

\[
\lambda_n = \Pr\{\tilde{X}^n \notin \mathcal{A}_n\} = \Pr\{\tilde{X}^n \in \mathcal{A}_n^c\} \quad \text{and} \quad \mu_n = \Pr\{X^n \notin \mathcal{A}_n\} = \Pr\{X^n \in \mathcal{A}_n^c\}.
\]
The infimum of all $E$-unachievable rates is given by

$$B_r^*(E|\tilde{X}|X) = \inf \{ r > 0 : r \text{ is } E\text{-unachievable} \}$$

and $B_r^*(E|\tilde{X}|X) = \infty$ if the above set is empty.

With Definition 7, Proposition 2 becomes as follows.

**Proposition 3:** For any $E > 0$

$$B_r^*(E|\tilde{X}|X) = E + \inf \{ R \in \mathbb{R} : \rho(R) \leq E \}$$

where

$$\rho(R) \triangleq \limsup_{n \to \infty} -\frac{1}{n} \log P_{X^n} \left\{ x^n \in \mathcal{X}^n : \frac{1}{n} \log P_{\tilde{X}^n}(x^n) \leq R \right\} - R.$$ 

**Remark 2:** We can now clearly observe that Definitions 7 and 4 are identical. This indicates that Nagaoka and Hayashi's $B_r^*(E|\tilde{X}|X)$ is, in fact, Csiszár's $D_r^*(E|X||\tilde{X})$. Hence, using Definition 4, Proposition 3 should be as follows.

**Proposition 4:** For any $E > 0$

$$D_r^*(E|X||\tilde{X}) = E + \inf \{ R \in \mathbb{R} : \rho(R) \leq E \}$$

where

$$\rho(R) \triangleq \limsup_{n \to \infty} -\frac{1}{n} \log P_{X^n} \left\{ x^n \in \mathcal{X}^n : \frac{1}{n} \log P_{\tilde{X}^n}(x^n) \leq R \right\} - R.$$ 

The above proposition is a key ingredient for our main results in Section VI.

VI. REVERSE $\beta$-CUTOFF RATE FOR ARBITRARY SOURCES

In the degenerate case where $D_r^*(E|X||\tilde{X}) = 0$, we have that $R_0^{(r)}(\beta|X||\tilde{X}) = \infty$. Similarly, if $D_r^*(E|X||\tilde{X}) = \infty$, then $R_0^{(r)}(\beta|X||\tilde{X}) = 0$. A graphical illustration of $R_0^{(r)}(\beta|X||\tilde{X})$ is given in Fig. 6. We first show the following lemma, which will provide us the key mechanism to establish our reverse $\beta$-cutoff rate result.

**Lemma 2:** Fix $t < 0$. The following two conditions are equivalent:

$$\forall R \in \mathbb{R}, \quad \rho(R) \geq -R(1-t) + tR_0$$

and

$$\forall E > 0 \quad D_r^*(E|X||\tilde{X}) \geq \frac{t}{t-1} (E - R_0).$$

**Proof:**

a) Equation (16) $\Rightarrow$ (17).

$$D_r^*(E|X||\tilde{X}) = E + \inf \{ R \in \mathbb{R} : \rho(R) \leq E \} \geq E + \inf \{ R \in \mathbb{R} : -R(1-t) + tR_0 \leq E \} = \frac{t}{t-1} (E - R_0)$$

where the inequality follows from (16).

b) Equation (17) $\Rightarrow$ (16).

$$\inf \{ R \in \mathbb{R} : \rho(R) \leq E \} \geq \frac{t}{t-1} (E - R_0) - E = \frac{t}{t-1} E - \frac{t}{t-1} R_0.$$ 

Thus,

$$E \leq \rho \left( \frac{1}{t-1} E - \frac{t}{t-1} R_0 \right)$$

since by [14, Remark 15], $\rho(\cdot)$ is strictly decreasing. Letting

$$R = \frac{1}{t-1} E - \frac{t}{t-1} R_0$$

or

$$E = -R(1-t) + tR_0$$

the above inequality can be rewritten as

$$\rho(R) \geq -R(1-t) + tR_0$$

where $R \in \mathbb{R}$.

We next employ Lemma 2 to show our main result regarding the reverse $\beta$-cutoff rate.

**Theorem 2 (Reverse $\beta$-Cutoff Rate Formula):** The following hold for the general hypothesis testing problem.

1. $R_0^{(r)}(\beta|X||\tilde{X}) \leq \liminf_{n \to \infty} \frac{1}{n} D_{1/(1-\beta)}(X^n||\tilde{X}^n)$

   for $0 < \beta < 1$.

2. If $\rho(R)$ admits a limit

   $R_0^{(r)}(\beta|X||\tilde{X}) = \liminf_{n \to \infty} \frac{1}{n} D_{1/(1-\beta)}(X^n||\tilde{X}^n)$,

   for $0 < \beta < \beta_{max}$

where

$$\beta_{max} \triangleq \sup \left\{ \gamma \in (0,1) : \liminf_{n \to \infty} \frac{1}{n} D_{1/(1-\gamma)}(X^n||\tilde{X}^n) < \infty \right\}.$$ 

Note that from the above two inequalities, we directly obtain that for $0 < \beta < \beta_{max}$

$$R_0^{(r)}(\beta|X||\tilde{X}) = \liminf_{n \to \infty} \frac{1}{n} D_{1/(1-\beta)}(X^n||\tilde{X}^n)$$

if $\rho(R)$ admits a limit.

**Proof:**

1. **Forward part:**

   1) For $0 < \beta < 1$,

   $$R_0^{(r)}(\beta|X||\tilde{X}) \leq \liminf_{n \to \infty} \frac{1}{n} D_{1/(1-\beta)}(X^n||\tilde{X}^n).$$

   4For the proof of the continuous alphabet case, the same remark given in Observation 2 of Section III applies.
Then (20) is equivalent to

\[
\lambda \triangleq \sup \{ R \in \mathbb{R} : \zeta((t)(R)) > 0 \}
\]

where

\[
\zeta((t)(R)) \triangleq \lim_{n \to \infty} -\frac{1}{n} \log P_X^n \left\{ x^n \in \mathcal{X}^n : \frac{1}{n} \log P_X^n(x^n) \leq R \right\}
\]

is the twisted large deviation spectrum of the normalized log-likelihood ratio with parameter \( t \). It can be shown that \( \lambda \) satisfies \(-\infty < \lambda \leq 0\) (cf. Lemmas 6 and 7 in Appendix B).

By \( \zeta((t)(\lambda + \delta/2)) > \varepsilon \) for some \( \varepsilon > 0 \), we know that

\[
P_X^n \left\{ x^n \in \mathcal{X}^n : \frac{1}{n} \log P_X^n(x^n) \leq \lambda + \delta/2 \right\} \leq e^{-n\varepsilon}
\]

for infinitely many \( n \). Furthermore, \( \zeta((t)(\lambda + \delta/2)) = 0 \) implies that for any \( 0 < \varepsilon < \varepsilon' \)

\[
P_X^n \left\{ x^n \in \mathcal{X}^n : \frac{1}{n} \log P_X^n(x^n) \leq \lambda + \delta/2 \right\} \geq e^{-n\varepsilon'}
\]

for all sufficiently large \( n \). Therefore,

\[
P_X^n \left\{ x^n \in \mathcal{X}^n : \lambda - \delta/2 < \frac{1}{n} \log P_X^n(x^n) \leq \lambda + \delta/2 \right\} \geq e^{-n\varepsilon} - e^{-n\varepsilon'}
\]

for infinitely many \( n \), or equivalently

\[
-\frac{1}{n} \log P_X^n \left\{ x^n \in \mathcal{X}^n : \lambda - \delta/2 < \frac{1}{n} \log P_X^n(x^n) \leq \lambda + \delta/2 \right\} \leq -\frac{1}{n} \log \left[ e^{-n\varepsilon} - e^{-n\varepsilon'} \right]
\]

for infinitely many \( n \), which (by the fact that \( a_n \leq b_n \) for infinitely many \( n \)) implies that \( \lim_{n \to \infty} a_n \leq \limsup_{n \to \infty} b_n \) for any two sequences \( \{a_n\} \) and \( \{b_n\} \).

\[
\liminf_{n \to \infty} -\frac{1}{n} \log P_X^n \left\{ x^n \in \mathcal{X}^n : \lambda - \delta/2 < \frac{1}{n} \log P_X^n(x^n) \leq \lambda + \delta/2 \right\} \leq \limsup_{n \to \infty} -\frac{1}{n} \log \left[ e^{-n\varepsilon} - e^{-n\varepsilon'} \right] = \varepsilon.
\]
Observation:

1) Unlike Theorem 1, where the forward cutoff rate formula holds in full generality, the reverse cutoff rate expression of Theorem 2 requires one assumption that \( \rho(R) \) admits a limit. This assumption is not too stringent; for example, it holds for the class of sources with memory that satisfy the Gärtners-ellis theorem [3, p. 15]. However, it is important to point out that if this assumption does not hold, the reverse cutoff rate formula is not valid in general (see Example 7 in the following text).

2) Remark also that if \( \beta_{\text{max}} \leq \beta < 1 \), it is possible that Part 2 of Theorem 2 may no longer hold (cf. Example 8 below). However, it can be directly verified that for sources (with general alphabet) satisfying the Gärtners-ellis theorem, \( \beta_{\text{max}} = 1 \), and hence Part 2 of Theorem 2 is always valid. Evidently, this is also the case for finite-alphabet sources.

3) The above two observations directly imply that for finite-alphabet i.i.d. sources, Theorem 2 reduces to Csiszar’s result [6]; i.e., the reverse \( \beta \)-cutoff rate is given by the Rényi divergence with parameter \( 1/(1-\beta) \) for all \( 0 < \beta < 1 \).

4) Finally, it directly follows from the definition of \( R_0^{(\beta)}(\mathbf{X}||\mathbf{X}) \) and Part 1 of Theorem 2 (which holds without any assumption), that a convex lower bound for the reliability function \( D_*(E[\mathbf{X}|\mathbf{X}]) \) can be established in terms of the liminf Rényi divergence rate, as noted in Observation 4 of Section III (with the exception that the supremum is over \( 0 < \beta < 1 \)). Hence, for any sources with memory whose \( \liminf \) Rényi divergence rate is explicitly known (e.g., for general time-invariant finite-alphabet Markov sources [17], [18]), a computable lower bound to \( D_*(E[\mathbf{X}|\mathbf{X}]) \) can be determined.

VII. EXAMPLES FOR THE REVERSE \( \beta \)-CUTOFF RATE

In this section, we provide four examples to illustrate and facilitate the understanding of Theorem 2. In Example 5, we present a case (involving discrete alphabet sources) where \( \rho(R) \) admits a limit and \( R_0^{(\beta)}(\mathbf{X}||\mathbf{X}) \) is equal to the Rényi divergence rate. In Example 6, we illustrate the situation (involving continuous alphabet sources) where the limit of \( \rho(R) \) does not exist within

\[
R_0^{(\beta)}(\mathbf{X}||\mathbf{X}) = \liminf_{n \to \infty} (1/n) D_1(1-\beta) \left( \mathbf{X}^n || \mathbf{X}^n \right).
\]

In Example 7, we construct an example where \( \rho(R) \) does not admit a limit and

\[
R_0^{(\beta)}(\mathbf{X}||\mathbf{X}) \neq \liminf_{n \to \infty} (1/n) D_1(1-\beta) \left( \mathbf{X}^n || \mathbf{X}^n \right).
\]

This example shows that without the assumption on the existence of the limit of \( \rho(R) \), Theorem 2 may not hold in general (particularly Part 2

\[
\liminf_{n \to \infty} -\frac{1}{n} \log P_{\mathbf{X}^n}(x^n) \left\{ x^n \in \mathcal{X}^n : \lambda - \delta/2 < \frac{1}{n} \log P_{\mathbf{X}^n}(x^n) \leq \lambda + \delta/2 \right\} = 0. \tag{22}
\]
We have the following three cases.

1) $\alpha + 2 - 2\alpha > 0$. Note that $e^{-cn}$ and $e^{-2n}$ approach 0 as $n \to \infty$, and that $e^{-cn} e^{-2n(1-\alpha)} = e^{-n(c+2-2\alpha)}$, which also approaches 0 as $n \to \infty$. Hence, the $\alpha$-divergence rate is equal to 0 since the argument of the logarithm $\to 1$ as $n \to \infty$.

2) $\alpha + 2 - 2\alpha < 0$. In this case, since $e^{-n(c+2-2\alpha)} \to \infty$ as $n \to \infty$, the argument of the logarithm, for large $n$, is dominated by $e^{-cn}$. Hence,

$$\lim_{n \to \infty} \frac{1}{n} D_\alpha(X^n \| \tilde{X}^n) = \lim_{n \to \infty} \frac{n(\alpha + 2 - 2\alpha)}{n(\alpha - 1)} = \frac{\alpha + 2 - 2\alpha}{1 - \alpha}.$$

3) $\alpha + 2 - 2\alpha = 0$. Clearly, the $\alpha$-divergence rate is equal to 0 in this case.

We next determine $D_0^\alpha(E|X||\tilde{X})$ using Proposition 4.

$$D_0^\alpha(E|X||\tilde{X}) = E + \inf \{ R \in \mathbb{R} : \zeta(R) - R \leq E \}$$

$$= \begin{cases} E, & \text{for } 0 < E \leq c \\ c, & \text{for } c < E \leq 2 \end{cases}$$

Now, the reverse $\beta$-cutoff rate is the $E$-axis intercept of the line of slope $\beta$ passing by the point $(2, c)$ as illustrated in Fig. 7. A straightforward calculation yields

$$R_0^{(\beta)}(\beta|X||\tilde{X}) = -\frac{c}{\beta} + 2.$$

For $\alpha = 1/(1 - \beta)$, we get that

$$R_0^{(\beta)}(\beta|X||\tilde{X}) = \frac{\alpha + 2 - 2\alpha}{1 - \alpha}.$$

Since by definition, $R_0^{(\beta)}(\beta|X||\tilde{X}) \geq 0$, we directly obtain that

$$R_0^{(\beta)}(\beta|X||\tilde{X}) = \lim_{n \to \infty} \frac{1}{n} D_0^{1/(1-\beta)}(X^n \| \tilde{X}^n), \quad \text{for } 0 < \beta < 1.$$

Note that for this example, $\beta_{\max} = 1$ since the $\alpha$-divergence rate is always finite.

**Example 6:** Consider the hypothesis testing problem between two i.i.d. (memoryless) Gaussian sources

$$X = \{ X_n \}_{n=1}^{\infty} \quad \text{and} \quad \tilde{X} = \{ X_n \}_{n=1}^{\infty}$$

such that for $n$ odd, $X$ and $\tilde{X}$ have distributions $N(-1, 1)$ and $N(1, 1)$, respectively; for $n$ even, $X$ and $\tilde{X}$ have distributions $N(-2, 1)$ and $N(2, 1)$, respectively. Then

$$z(x) = \log \frac{dP_X(x)}{dP_{\tilde{X}}(x)}$$

$$= \begin{cases} \log \frac{1}{\sqrt{2\pi e}} - \frac{(x+1)^2}{2}, & \text{if } n \text{ is odd} \\ \log \frac{1}{\sqrt{2\pi e}} - \frac{(x-2)^2}{2}, & \text{if } n \text{ is even} \end{cases} \quad = 2x, \quad \text{if } n \text{ is odd}$$

$$= 4x, \quad \text{if } n \text{ is even}.$$
Hence, under distribution $P_X$, the log-likelihood ratios $Z_1, Z_2, \ldots$ are i.i.d. with $N(-2, 4)$ for $n$ odd, and with $N(-8, 16)$ for $n$ even. So, it follows using the Gärtnér–Elliass theorem that
\[
- \frac{1}{n} \log P_X^n \left\{ z^n \in \mathcal{A}^n : \frac{1}{n} \log \frac{P_X^n(x^n)}{P_X^n(x^n)} \leq R \right\} = \frac{1}{n} \log \operatorname{Pr}\left\{\frac{Z_1 + Z_2 + \ldots + Z_n}{n} \leq R\right\} = \frac{(R + 2)^2}{8} 1\{R < -2\}, \quad \text{if } n \text{ is odd}
\]
\[
= \frac{(R + 2)^2}{8} 1\{R < -8\}, \quad \text{if } n \text{ is even}
\]
where $1\{\cdot\}$ denotes the indicator function. We then obtain that
\[
\rho(R) = \limsup_{n \to \infty} - \frac{1}{n} \log P_X^n \left\{ x^n \in \mathcal{A}^n : \frac{1}{n} \log \frac{P_X^n(x^n)}{P_X^n(x^n)} \leq R \right\} = R
\]
\[
= \frac{(R + 2)^2}{8} 1\{R < -2\} - R.
\]
Given the above $\rho(R)$ (note that the limit of $\rho(R)$ does not exist), we can determine the correct exponent $D_1^*(E|X||X)$ using Proposition 4 as follows:
\[
D_1^*(E|X||X) = E + \inf\{R \in \mathbb{R} : \rho(R) \leq E\} = (\sqrt{E} - \sqrt{2}E)^2 1\{E \geq 2\}.
\]
Solving for the $E$-axis intercept of the support line to $D_1^*(E|X||X)$ of slope $\beta$ yields the reverse $\beta$-cutoff rate
\[
P_{\text{th}}(\beta) = \frac{2}{1 - \beta}.
\]
Finally, a simple calculation gives that
\[
\liminf_{n \to \infty} \frac{1}{n} \log P_X^n \left( x^n \in \mathcal{A}^n : \frac{1}{n} \log \frac{P_X^n(x^n)}{P_X^n(x^n)} \leq R \right) = \frac{2}{1 - \beta}
\]
which is finite for $0 < \beta < \beta_{\text{max}} = 1$ (note that $\limsup_{n \to \infty}$ Rényi divergence rate equals $\beta(1 - \beta)$). Hence, we obtain that $P_{\text{th}}^0(\beta) = \frac{2}{1 - \beta}$ is equal to the $\liminf_{n \to \infty}$ Rényi divergence rate for all $0 < \beta < 1$.

**Example 7:** Consider the hypothesis testing problem between sources $X = \{X_i\}_{i=1}^\infty$ and $\tilde{X} = \{\tilde{X}_i\}_{i=1}^\infty$, where for odd $n$, $X$ and $\tilde{X}$ are, respectively, unit-variance memoryless Gaussian with means $-\nu$ and $\nu$, and for even $n$, $X$ and $\tilde{X}$ have respective distributions $P_X^n(x_0^n) = e^{-\nu^2}, P_X^n(x_1^n) = 1 - e^{-\nu^2}$, and $P_{\tilde{X}}(x_0^n) = e^{-\nu^2}, P_{\tilde{X}}(x_1^n) = 1 - e^{-\nu^2}$, where $x_0^n \neq x_1^n$. Then for odd $n$
\[
Z_n(x^n) = \frac{1}{n} \log \frac{P_{\tilde{X}}^n(x^n)}{P_X^n(x^n)} = \frac{2}{n} \sum_{i=1}^n x_i,
\]
and for even $n$
\[
Z_n(x^n) = \frac{1}{n} \log \frac{P_{\tilde{X}}^n(x^n)}{P_X^n(x^n)} = \frac{2}{n} \sum_{i=1}^n x_i.
\]
Hence, under distribution $P_X$, the normalized log-likelihood ratio $Z_n(X^n)$ is normal distributed with mean $-2\nu^2$ and variance $4\nu^2/n$ for odd $n$; for even $n$.
\[
\Pr\left[ Z_n(x^n) \leq -\theta \right] = e^{-\theta n/2} \text{ and for } n \text{ odd}
\]
\[
\Pr\left[ Z_n(x^n) \leq -\theta \right] = e^{-\theta n} \text{ and for } n \text{ even}
\]
In addition, under distribution $P_{\tilde{X}}$, the normalized log-likelihood ratio $Z_n(\tilde{X}^n)$ is normal distributed with mean $2\nu^2$ and variance $4\nu^2/n$ for odd $n$; for even $n$
\[
\Pr\left[ Z_n(\tilde{X}^n) \leq -\theta \right] = e^{-2\theta n} \text{ and for } n \text{ odd}
\]
\[
\Pr\left[ Z_n(\tilde{X}^n) \leq -\theta \right] = 1 - e^{-2\theta n} \text{ and for } n \text{ even}
\]
**Direct Derivation of the Correct Exponent:** It is known from the Neyman–Pearson lemma [16] that the log-likelihood test is optimal. Hence, the type 1 and type 2 error probabilities ($\mu_n$ and $\lambda_n$) can be written as
\[
\mu_n = \Pr[Z_n(X^n) > R_n] + (1 - \omega_n) \Pr[Z_n(X^n) = R_n]
\]
and
\[
\lambda_n = \Pr[Z_n(\tilde{X}^n) < R_n] + \omega_n \Pr[Z_n(\tilde{X}^n) = R_n]
\]
where $\omega_n \in [0, 1]$ is some randomization factor, and $R_n$ is the log-likelihood ratio threshold. Now the requirement in Definition 4 for
\[
\liminf_{n \to \infty} - (1/n) \log \mu_n = E > 0
\]
can be met for
\[
R_n = 2\nu^2 - 2\nu \sqrt{2E} \quad \text{and } \quad \omega_n = 0
\]
if $n$ is odd, and for
\[
R_n = \frac{1}{2} \log (1 + e^{-\theta n}) \quad \text{and } \quad \omega_n = e^{-\theta n}, \quad \text{if } 0 < E < 2\theta
\]
\[
R_n = \frac{1}{2} \log (1 + e^{-\theta n}) \quad \text{and } \quad \omega_n = e^{-\theta n}, \quad \text{if } E = 2\theta
\]
\[
R_n = -\theta \quad \text{and } \quad \omega_n = e^{-\theta n}, \quad \text{if } E \geq 2\theta
\]
if $n$ is even. Thus, if we take $\theta = 18\nu^2$ and apply the Gärtnér–Elliass theorem, we get the following (type 1) correct exponent
\[
D_1^*(E|X||X) = \limsup_{n \to \infty} - \frac{1}{n} \log (1 - \mu_n)
\]
\[
= \liminf_{n \to \infty} - \frac{1}{n} \log \Pr[Z_n(X^n) < R_n] + \omega_n \Pr[Z_n(X^n) = R_n]
\]
\[
= \max \left\{ \left( \sqrt{E} - \sqrt{2}E \right)^2 \cdot 1\{E > 2\nu^2\}, E \cdot 1\{0 < E \leq \theta\}, \right. \theta \cdot 1\{\theta < E \leq 2\theta\} + \left( E - \theta \right) \cdot 1\{E > 2\theta\} \right\}
\]
\[
= \left\{ \begin{array}{ll}
E, & \text{if } 0 < E \leq 18\nu^2 \\
18\nu^2, & \text{if } 18\nu^2 < E \leq 32\nu^2 \\
\left( \sqrt{E} - \sqrt{2}E \right)^2, & \text{if } 32\nu^2 < E \leq 50\nu^2 \\
E - 18\nu^2, & \text{for } E \geq 50\nu^2.
\end{array} \right.
\]
**The Correct Exponent using Nagaoka and Hayashi’s Formula:** We next verify our above direct derivation of the correct exponent $D_1^*(E|X||X)$ by applying Nagaoka and Hayashi’s formula in Proposition 4
\[
\zeta(R) = \limsup_{n \to \infty} - \frac{1}{n} \log P_X^n \left\{ x^n \in \mathcal{A}^n : \frac{1}{n} \log \frac{P_X^n(x^n)}{P_X^n(x^n)} \leq R \right\}
\]
\[
= \limsup_{n \to \infty} - \frac{1}{n} \log \Pr[Z_n(X^n) \leq R]
\]
\[
= \max \left\{ \frac{(R + 2\nu^2)^2}{8\nu^2} \cdot 1\{R < -2\nu^2\}, \theta \cdot 1\{-\theta \leq R \leq 0\} + \infty \cdot 1\{R < -\theta\} \right\}
\]
\[
= \left\{ \begin{array}{ll}
0, & \text{for } R > 0 \\
\theta \left( \frac{R + 2\nu^2}{\nu^2} \right)^2, & \text{for } -2\nu^2 < R \leq 0 \\
\infty, & \text{for } R < -2\nu^2 - 2\nu \sqrt{2\theta}.
\end{array} \right.
\]
Setting \( \theta = 18 \nu^2 \), the following expression for \( \rho(R) \) is obtained:

\[
\rho(R) = \begin{cases} 
-\frac{R}{18}, & R > 0 \\
18 \nu^2 - R, & -18 \nu^2 \leq R \leq 0 \\
(1 - \frac{2}{3} \nu^2), & -18 \nu^2 \leq R < -14 \nu^2 \\
\infty, & R < -18 \nu^2.
\end{cases}
\]

Hence, by Proposition 4

\[
D_p^*(E|X|\hat{X}) = E + \inf \{ R \in \mathbb{R} : \zeta(R) - R \leq E \}.
\]

\[
E = \begin{cases} 
E, & \text{for } -18 \nu^2 \leq E \leq 32 \nu^2 \\
E - 18 \nu^2, & \text{for } E > 32 \nu^2
\end{cases}
\]

and indeed our direct derivation coincides with Nagaoka and Hayashi’s expression.

**Examination of the Reverse Cutoff Rate:** By Lemma 2, the \( \beta \)-reverse cutoff rate \( R_{\beta}^*(\beta|X|\hat{X}) \) is the smallest \( R_0 \) satisfying \( \zeta(R) = \rho(R) + R \geq t(R + R_0) \), where \( t = \beta/(\beta - 1) \) for \( 0 < \beta < 1 \). We thus get that

\[
R_{\beta}^*(\beta|X|\hat{X}) = \begin{cases} 
0, & \text{for } -\frac{2}{3} \nu^2 \leq t < 0 \\
\frac{2}{3} \nu^2, & \text{for } -\frac{3}{2} \leq t < -\frac{2}{3} \\
(1 - t) \cdot 18 \nu^2, & \text{for } -4 \leq t < -3 \\
\frac{2}{3} \nu^2, & \text{for } E > 32 \nu^2
\end{cases}
\]

We next compute the \( \liminf (1 - t) \)-divergence rate (where \( \beta = t / (t - 1) \)) to compare it with \( R_{\beta}^*(\beta|X|\hat{X}) \)

\[
1/n D_{1-t}^*(X^n|\hat{X}^n) = -1/n \log \left( \int_{x \in X^n} e^{t \log |dP_{X|\hat{X}}(x^n) / dP_{X|\hat{X}}(x^n)|} dP_{X|\hat{X}}(x^n) \right) = -1/n \log E \left( e^{ntZ_n(X^n)} \right)
\]

\[
= \begin{cases} 
2(1 - t) \nu^2, & \text{for } n \text{ odd} \\
\frac{2}{3} \nu^2, & \text{for } n \text{ even}
\end{cases}
\]

Since

\[
\lim_{n \to \infty} \frac{1}{n} \log \left( e^{(1 - t)\theta n} + (1 + e^{-\theta n})^t (1 - e^{-\theta n}) \right) = \theta \frac{t + 1}{t} \cdot 1 \{ t < 1 \}
\]

we obtain (with \( \theta = 18 \nu^2 \)) that

\[
\liminf_{n \to \infty} \frac{1}{n} D_{1-t}^*(X^n|\hat{X}^n) \leq \limsup_{n \to \infty} \frac{1}{n} \log P_{X|\hat{X}} \left( x^n \in X^n : \frac{1}{n} \log \frac{P_{X|\hat{X}}(x^n)}{P_{Y|\hat{Y}}(x^n)} \leq R \right)
\]

\[
= \liminf_{n \to \infty} -1/n \log P_{X^n} \left( x^n \in X^n : 1/n \log \frac{P_{X^n}(x^n)}{P_{Y^n}(x^n)} \leq R \right)
\]

\[
= \liminf_{n \to \infty} -1/n \log \Pr \left[ Z_n(X^n) \leq R \right]
\]

\[
= \begin{cases} 
0, & \text{for } R > 0 \\
\theta, & \text{for } -\theta < R \leq 0 \\
\limsup_{n \to \infty} -1/n \log \int_{0}^{\infty} \log |dP_{X}(x^n)| / (\theta n) - R / \theta n \, dP_{X}(x^n), & \text{for } R \leq -\theta
\end{cases}
\]

\[
= \theta \cdot 1 \{ -\theta < R \leq 0 \} - R \cdot 1 \{ R \leq -\theta \}
\]

Note from this example, where \( \rho(R) \) does not admit a limit, that

\[
R_{\beta}^*(\beta|X|\hat{X}) \neq \liminf_{n \to \infty} 1/n D_{1-t}^*(X^n|\hat{X}^n)
\]

for \( t < 0 \) (i.e., Part 1 of Theorem 2 is valid), while Part 2 of Theorem 2 does not hold (e.g., for \( t = -8 \), \( R_{\beta}^*(\beta|X|\hat{X}) = 14 \nu^2 \) while \( \liminf_{n \to \infty} 1/n D_{1-t}^*(X^n|\hat{X}^n) = 15.75 \nu^2 \)). Hence, if the limit of \( \rho(R) \) does not exist

\[
R_{\beta}^*(\beta|X|\hat{X}) \neq \liminf_{n \to \infty} 1/n D_{1-t}^*(X^n|\hat{X}^n)
\]

in general.

**Example 8:** In this example, we illustrate the situation where \( \beta_{\max} = 1/2 < 1 \) and show that if \( \beta_{\max} < \beta < 1 \), Part 2 of Theorem 2 may not hold.

Consider the hypothesis testing problem between two sources \( X = \{ X_1 \}_{n=1}^\infty \) and \( \hat{X} = \{ \hat{X}_1 \}_{n=1}^\infty \). Both \( X \) and \( \hat{X} \) have a point mass on \( \hat{x}^n \), and have a density over the hyper-line \( \cdot \hat{x}^n = \{ (a \cdot \hat{x}, a \cdot \hat{x}, \ldots, a \cdot \hat{x}) \} \) for \( a \geq 0 \), where \( \hat{x}^n \not\in \mathbb{R}^n : x^n = a \cdot \hat{x}^n \) for some \( a \geq 0 \) and \( \hat{x}^n \neq 0 \) is fixed. The densities for \( X \) and \( \hat{X} \) over \( \{ x^n \in \mathbb{R}^n : x^n = a \cdot \hat{x}^n \} \) for some \( a \geq 0 \) are described by

\[
p_{X \hat{X}}(a \cdot \hat{x}^n) = \theta n e^{-\theta a n} \quad \text{and}
\]

\[
p_{X \hat{X}}(a \cdot \hat{x}^n) = 2 \theta n e^{-2\theta a n}
\]

respectively. Furthermore, \( P_{X \hat{X}}(\hat{x}^n) = 1 - e^{-\theta n} \), and \( P_{X \hat{X}}(\hat{x}^n) = 1 - e^{-2\theta n} \). Then under distribution \( P_X \)

\[
Z_n(a \cdot \hat{x}^n) = \frac{1}{n} \log P_{X \hat{X}}(a \cdot \hat{x}^n) = \frac{1}{n} \log \frac{2 \theta n e^{-2\theta a n}}{\theta n e^{-\theta a n}} = 2 \log(2) - (a + 1) \theta
\]

with density \( \theta n e^{-\theta a n} \cdot 1 \{ a \geq 0 \}

and

\[
Z_n(\hat{x}^n) = \frac{1}{n} \log P_{X \hat{X}}(\hat{x}^n) = \frac{1}{n} \log \frac{1 - e^{-2\theta n}}{1 - e^{-\theta n}} = \frac{1}{n} \log(1 + e^{-\theta n}) \quad \text{with prob.} 1 - e^{-\theta n}.
\]

Then by the Gáirnér–Ellis theorem, we get the expression at the bottom of the page. Note that the limit of \( \rho(R) \) (or \( \zeta(R) \)) does exit. A direct application of Lemma 2 yields that \( R_{\beta}^*(\beta|X|\hat{X}) = 0 \) for \( -1 < t < 0 \), and that \( R_{\beta}^*(\beta|X|\hat{X}) = \infty \) for \( t < -1 \) (where \( t = \beta/(\beta - 1) \)). We next compute the \( \liminf (1 - t) \)-divergence rate, where \( \beta = t /(t - 1) \)

\[
\frac{1}{n} D_{1-t}^*(X^n|\hat{X}^n) = -\frac{1}{nt} \log \left( \int_{x \in X^n} e^{t \log |dP_{X^n}(x^n) / dP_{Y^n}(x^n)|} dP_{X^n}(x^n) \right) = -\frac{1}{nt} \log E \left( e^{ntZ_n(X^n)} \right)
\]
\[
\begin{align*}
&= -\frac{1}{m_t} \log \left[ \int_0^\infty e^{-(1/n) \log(2)-(a+1)\theta} \\
&\quad \cdot \theta ne^{-(a+1)\theta} \, da + (1 + e^{-a \theta})^t (1 - e^{-\theta}) \right] \\
&= -\frac{1}{m_t} \log \left[ \theta \int_0^\infty e^{-(a+1)\theta} \int_0^\infty e^{-a \theta} \, da \\
&\quad + (1 + e^{-\theta})^t (1 - e^{-\theta}) \right] \\
&= -1 \cdot 1 \{ -1 < t < 0 \} + \infty \cdot 1 \{ t \leq -1 \}. 
\end{align*}
\]

Therefore,
\[
R_0^{(r)}(\beta|X||\bar{X}) \leq \liminf \frac{1}{n} D_{1-\epsilon t}(X^n||\bar{X}^n)
\]

for all \( t < 0 \) (i.e., \( 0 < \beta < 1 \)) and Part 1 of Theorem 2 is satisfied. However,
\[
R_0^{(r)} \geq \liminf \frac{1}{n} D_{1-\epsilon t}(X^n||\bar{X}^n)
\]

only for \( -1 < t < 0 \); in other words, Part 2 of Theorem 2 is only valid for \( 0 < \beta < \beta_{\text{max}} = 1/2 \). Notably, for \( t = -1 \)
\[
0 = R_0^{(r)}(\beta|X||\bar{X}) < \liminf \frac{1}{n} D_{1-\epsilon t}(X^n||\bar{X}^n) = \infty.
\]

VIII. Conclusion

We examined the forward and reverse \( \beta \)-cutoff rates for the hypothesis testing problem between arbitrary sources with memory (not necessarily Markovian, ergodic, stationary, etc.) and arbitrary alphabet (concatenable or continuous). We showed that the forward \( \beta \)-cutoff rate is given by the lim inf Rényi \( \alpha \) divergence rate, where \( \alpha = \frac{1}{1-\beta} \) and \( \beta < 0 \). We also demonstrated that if the large-deviation spectrum \( \rho(R) \) admits a limit, then the reverse \( \beta \)-cutoff rate is given by the lim sup \( \alpha \) divergence rate, where \( \alpha = \frac{1}{\beta} \) and \( 0 < \beta < \beta_{\text{max}} \). For \( \beta_{\text{max}} \leq \beta < 1 \), we provided an upper bound on the reverse \( \beta \)-cutoff rate. As expected, for finite-alphabets i.i.d. sources, our theorems reduce to Csiszár’s result [6]. Our forward and reverse \( \beta \)-cutoff rates results and the methods used to derive them are illustrated via several examples. An interesting consequence of this work is that while the formula of the type 1 error (respectively, correct) exponent is not explicitly known for general sources with memory, our forward (respectively, reverse) \( \beta \)-cutoff rate result, expressed in terms of the lim inf Rényi divergence rate, can be used to provide a useful convex lower bound (which is computable in principle) to the error (respectively, correct) exponent.

Future work may include the study of Csiszár’s channel coding \( \beta \)-cutoff rates [6] for arbitrary discrete channels with memory using our information spectrum techniques.

APPENDIX A

Properties of \( \tau \) and \( \sigma(R) \)

Lemma 3: For \( 0 < \tau < 1 \)
\[
\tau \triangleq \sup \{ R : \eta(t)(R) > 0 \} \leq \liminf \frac{1}{n} D_{1-\epsilon t}(X^n||\bar{X}^n).
\]

Proof: For any \( \nu > 0 \)
\[
P_X^{(t)} \left\{ x^n \in \mathcal{X}^n : \frac{1}{n} \log \frac{P_{X^n}(x^n)}{P_{\bar{X}^n}(x^n)} > \frac{1}{n} D_{1-\epsilon t}(X^n||\bar{X}^n) + 2\nu \right\}
\]

\[
\leq P_X^{(t)} \left\{ x^n \in \mathcal{X}^n : \frac{1}{n} \log \frac{P_{X^n}(x^n)}{P_{\bar{X}^n}(x^n)} > \frac{1}{n} D_{1-\epsilon t}(X^n||\bar{X}^n) + \nu \right\}
\]

for \( n \in \mathcal{N} \) sufficiently large

where \( \mathcal{N} \) is defined in (9). But
\[
P_X^{(t)} \left\{ x^n \in \mathcal{X}^n : \frac{1}{n} \log \frac{P_X(x^n)}{P_{\bar{X}^n}(x^n)} > \frac{1}{n} D_{1-\epsilon t}(X^n||\bar{X}^n) + \nu \right\}
\]

\[
= P_X^{(t)} \left\{ x^n \in \mathcal{X}^n : \frac{1}{n} \log \frac{P_X(x^n)}{P_{\bar{X}^n}(x^n)} + D_{1-\epsilon t}(X^n||\bar{X}^n) > \nu \right\}
\]

\[
= P_X^{(t)} \left\{ x^n \in \mathcal{X}^n : \frac{1}{n} \log \frac{P_X(x^n)}{P_{\bar{X}^n}(x^n)} + D_{1-\epsilon t}(X^n||\bar{X}^n) < -\nu t \right\}
\]

\[
\leq e^{-\nu t} P_X \left\{ x^n \in \mathcal{X}^n : P_X(x^n) < e^{-\nu t} P_X(x^n) \right\}
\]

\[
\leq e^{-\nu t} P_X \left\{ x^n \in \mathcal{X}^n : P_X(x^n) < e^{-\nu t} P_X(x^n) \right\}
\]

(23)

where (23) follows from (8). Thus, for \( n \in \mathcal{N} \) sufficiently large

\[
P_X^{(t)} \left\{ x^n \in \mathcal{X}^n : \frac{1}{n} \log \frac{P_X(x^n)}{P_{\bar{X}^n}(x^n)} \leq \liminf \frac{1}{n} D_{1-\epsilon t}(X^n||\bar{X}^n) + 2\nu \right\}
\]

\[
\geq 1 - e^{-\nu t}
\]

which implies
\[
\eta(t) \left( \liminf \frac{1}{n} D_{1-\epsilon t}(X^n||\bar{X}^n) + 2\nu \right)
\]

\[
= \liminf \frac{1}{n} \log P_X^{(t)} \left\{ x^n \in \mathcal{X}^n : \frac{1}{n} \log \frac{P_X(x^n)}{P_{\bar{X}^n}(x^n)} \right\}
\]

\[
\leq \liminf \frac{1}{n} D_{1-\epsilon t}(X^n||\bar{X}^n) + 2\nu \}
\]

\[
\leq \limsup \frac{1}{n} \log (1 - e^{-\nu t}) = 0.
\]

Consequently
\[
\sup \left\{ R : \eta(t)(R) > 0 \right\} \leq \liminf \frac{1}{n} D_{1-\epsilon t}(X^n||\bar{X}^n) + 2\nu.
\]

The proof is completed by noting that \( \nu \) can be made arbitrarily small.

\textbf{Lemma 4:} For \( 0 < \tau < 1 \), if \( \liminf \frac{1}{n} D_{1-\epsilon t}(X^n||\bar{X}^n) < K \), then
\[
\tau \triangleq \sup \{ R : \eta(t)(R) > 0 \} > -\infty.
\]

\textbf{Proof:} By (8), we get that
\[
P_X^{(t)}(x^n) \leq e^{D_{1-\epsilon t}(X^n||\bar{X}^n)} e^{(1-t)\log P_X(x^n)}
\]

Hence,
\[
P_X^{(t)} \left\{ x^n \in \mathcal{X}^n : \frac{1}{n} \log \frac{P_X(x^n)}{P_{\bar{X}^n}(x^n)} \leq R \right\}
\]

\[
\leq e^{D_{1-\epsilon t}(X^n||\bar{X}^n)} e^{(1-t)\log P_X(x^n)}
\]

\[
\times P_X \left\{ x^n \in \mathcal{X}^n : \frac{1}{n} \log \frac{P_X(x^n)}{P_{\bar{X}^n}(x^n)} \leq R \right\}
\]

\[
\leq e^{D_{1-\epsilon t}(X^n||\bar{X}^n)} e^{(1-t)\log R}
\]

which implies that
\[
\eta(t)(R) \geq -\tau \limsup \frac{1}{n} D_{1-\epsilon t}(X^n||\bar{X}^n) - (1-t)R.
\]

Therefore,
\[
\tau \geq -\frac{1}{1-t} \limsup \frac{1}{n} D_{1-\epsilon t}(X^n||\bar{X}^n).
\]
This shows that \( \tau = -\infty \) implies that
\[
\limsup_{n \to \infty} \frac{1}{n} D_{1^{-t}}(X^n \| \tilde{X}^n) = \lim_{n \to \infty} \frac{1}{n} D_{1^{-t}}(X^n \| \tilde{X}^n) = \infty
\]
contradicting the assumption that
\[
\liminf_{n \to \infty} \frac{1}{n} D_{1^{-t}}(X^n \| \tilde{X}^n) < K.
\]

**Lemma 5:** We have the following:
\[
\sup \{ R \in \mathbb{R} : \sigma(R) > 0 \} \geq 0.
\]

**Proof:** For any \( \nu > 0 \)
\[
P_{X^n}^{(t)} \left\{ x^n \in \mathcal{X}^n : \frac{1}{n} \log \frac{P_{X^n}(x^n)}{P_{\tilde{X}^n}(x^n)} > R \right\}
\]
\[
= P_{X^n} \left\{ x^n \in \mathcal{X}^n : P_{X^n}(x^n) \leq e^{-nR} P_{\tilde{X}^n}(x^n) \right\}
\]
\[
\leq e^{-nR} P_{\tilde{X}^n} \left\{ x^n \in \mathcal{X}^n : P_{X^n}(x^n) \leq e^{-nR} P_{\tilde{X}^n}(x^n) \right\}
\]
which implies \( \sigma(\nu) \geq \nu \). Hence, the lemma holds.

**APPENDIX B**

**PROPERTIES OF \( \lambda \)**

**Lemma 6:** For \( t < 0, \lambda \leq 0 \).

**Proof:** Observe that for \( R > 0 \)
\[
P_{X^n}^{(t)} \left\{ x^n \in \mathcal{X}^n : \frac{1}{n} \log \frac{P_{X^n}(x^n)}{P_{\tilde{X}^n}(x^n)} > R \right\}
\]
\[
\leq e^{-nR(1-t)} \times P_{X^n} \left\{ x^n \in \mathcal{X}^n : \frac{1}{n} \log \frac{P_{X^n}(x^n)}{P_{\tilde{X}^n}(x^n)} > R \right\}
\]
which immediately implies that \( \lambda \leq 0 \).

**Lemma 7:** For \( 0 > t > t_{\min} = \frac{\beta_{\max}}{(\beta_{\max} - 1)} \), \( \lambda > -\infty \).

**Proof:** If \( \lambda = -\infty \), then \( t(\lambda) = 0 \) for every \( R \in \mathbb{R} \). Hence, by choosing any \( v = (t - t_{\min})/2 \), we have
\[
P_{X^n}^{(t)} \left\{ x^n \in \mathcal{X}^n : \frac{1}{n} \log \frac{P_{X^n}(x^n)}{P_{\tilde{X}^n}(x^n)} \leq R \right\}
\]
\[
\leq e^{-nR(1-t)} \times P_{X^n} \left\{ x^n \in \mathcal{X}^n : \frac{1}{n} \log \frac{P_{X^n}(x^n)}{P_{\tilde{X}^n}(x^n)} \leq R \right\}
\]
which implies that \( \lambda = t(\lambda) = 0 \).

A contradiction to the definition of \( t_{\min} \) is obtained, since by

\[\exp \left\{ -\alpha D_{1^{-\alpha}}(X^n \| \tilde{X}^n) \right\} \leq \left( \frac{P_{X^n}(x^n)}{P_{\tilde{X}^n}(x^n)} \right)^{1-\gamma} \left( \frac{P_{\tilde{X}^n}(x^n)}{P_{X^n}(x^n)} \right)^{1/(1-\gamma)} \]

or equivalently
\[|\alpha| D_{1^{-\alpha}}(X^n \| \tilde{X}^n) \leq |\gamma| D_{1^{-\gamma}}(X^n \| \tilde{X}^n) \]
and hence by definition of \( t_{\min} \)
\[
\liminf_{n \to \infty} \frac{1}{n} D_{1^{-t}}(X^n \| \tilde{X}^n) < \infty.
\]

**ACKNOWLEDGMENT**

The authors would like to thank the anonymous reviewers for their helpful comments; in particular, they are grateful to one of the reviewers for informing them about [14] whose results were instrumental in significantly improving Theorem 2.

**REFERENCES**


