General Formulas for Csiszár's Source Coding Cutoff Rates

Po-Ning Chen
Dept. of Communication Eng.
National Chiao Tung Univ.
Hsin Chu, Taiwan 30050, R.O.C.
email: poning@cc.nctu.edu.tw

Fady Alajaji
Dept. of Mathematics and Statistics
Queen's Univ., Kingston
Ontario K7L 3N6, Canada
email: fady@polya.mast.queensu.ca

Abstract — **In this work, Csiszár's fixed-length source coding β-cutoff rates are investigated for the class of arbitrary discrete sources with memory. It is demonstrated that the limsup and liminf Rényi entropy rates provide the formulas for the forward and reverse β-cutoff rates, respectively. Consequently, new fixed-length source coding operational characterizations for the Rényi entropy rates are established.**

I. Introduction

In [2], Csiszár establishes the concept of generalized fixed-length source coding cutoff rates (forward and reverse) for discrete memoryless sources. More specifically, given $\beta > 0$, he defines the forward β-cutoff rate for a source $\{X_i\}_{i=1}^{\infty}$ as the number R_0 that provides the best possible lower bound in the form $\beta(R - R_0)$ to the source reliability function. This definition implies that the source error probability is guaranteed to exponentially decay with a linear exponent of specified slope β for $R > R_0$. He also provides a similar definition for the reverse β-cutoff rate (where $\beta > 0$) with respect to the source unreliability function (the exponent of the vanishing probability of correct decoding). He then demonstrates that the forward and reverse β-cutoff rates are respectively given by $H_{1/(1+\beta)}(X_1)$ and $H_{1/(1-\beta)}(X_1)$, where $H_\alpha(X_1)$ denotes the Rényi entropy of order α.

In this work, we extend Csiszár's results [2] by investigating the β-cutoff rate for arbitrary (not necessarily, stationary, ergodic, etc.) discrete-time finite-alphabet sources $\mathbf{X} \triangleq \{X^n = (X_1^{(n)}, \ldots, X_n^{(n)})\}_{n=1}^{\infty}$ [3]. We demonstrate that the limsup and liminf Rényi entropy rates provide the expressions for the forward and reverse β-cutoff rates, respectively. These results also provide simple, and in certain cases, computable lower bounds to the source reliability and unreliability functions.

II. Main results

Definition 1 An (n, M) fixed-length source code for X^n is a collection of M n-tuples $\mathcal{C}_n = \{c_1^n, \ldots, c_M^n\}$. The error probability of the code is $P_e(\mathcal{C}_n) \triangleq P_{X^n}[X^n \notin \mathcal{C}_n]$.

Definition 2 Fix $e > 0$. $R > 0$ is *e-achievable* for a source $\mathbf{X}$, if there exists a sequence of (n, M_n) fixed-length source code $\mathcal{C}_n$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \log M_n \le R \quad \text{and} \quad \liminf_{n\to\infty} -\frac{1}{n} \log P_e(\mathcal{C}_n) \ge e.$$

Fix $\beta > 0$. The *forward β-cutoff rate* for $\mathbf{X}$, denoted by $R_0^{(f)}(\beta|\mathbf{X})$, is defined as the *smallest* $R_0 \ge 0$ such that every $R > 0$ is *$\beta(R - R_0)$-achievable.*

This work was supported in part by Queen's University, NSERC of Canada and NSC of Taiwan, R.O.C.

Theorem 1 (Forward β-cutoff rate [1]) Fix $\beta > 0$. For an arbitrary source $\mathbf{X}$,

$$R_0^{(f)}(\beta|\mathbf{X}) = \limsup_{n\to\infty} \frac{1}{n} H_{1/(1+\beta)}(X^n),$$

where

$$H_\alpha(X^n) \triangleq \frac{1}{1-\alpha} \log \sum_{x^n \in \mathcal{X}^n} P_{X^n}^{\alpha}(x^n)$$

is the (n-dimensional) Rényi entropy of order α.

Definition 3 Fix $e > 0$. $R > 0$ is *reverse e-achievable* for a source $\mathbf{X}$, if there exists a sequence of (n, M_n) fixed-length source code $\mathcal{C}_n$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \log M_n \le R \quad \text{and} \quad \liminf_{n\to\infty} -\frac{1}{n} \log(1 - P_e(\mathcal{C}_n)) \le e.$$

Fix $\beta > 0$. The *reverse β-cutoff rate* for $\mathbf{X}$, denoted by $R_0^{(r)}(\beta|\mathbf{X})$, is defined as the *largest* R_0 such that every $R > 0$ is *reverse $\beta(R - R_0)$-achievable.*

Theorem 2 (Reverse β-cutoff rate [1]) Fix $0 < \beta < 1$. For any source $\mathbf{X}$,

$$R_0^{(r)}(\beta|\mathbf{X}) = \liminf_{n\to\infty} \frac{1}{n} H_{1/(1-\beta)}(X^n).$$

III. Conclusions

In closing, we would like to make the following observations.

- It is important to point out that if the source $\mathbf{X}$ is a time-invariant Markov source of arbitrary order, then its Rényi entropy rate exists and can be computed [4]. Thus in this case, the β-cutoff rates for this source can be obtained.

- A convex lower bound can be obtained on the source reliability function. It consists of the supremum of all the support lines with slope β which pass through the point $(R_0^{(f)}(\beta|\mathbf{X}), 0)$, given by $\sup_{\beta>0}[\beta(R - R_0^{(f)}(\beta|\mathbf{X}))]$ for every $R > 0$. We can thus conclude that for the class of sources $\mathbf{X}$ for which the Rényi entropy rate can be calculated (e.g., the class of Markov sources), a computable lower bound to the source reliability function can also be obtained. A similar remark applies for the source unreliability function.

References

[1] P.-N. Chen and F. Alajaji, "Csiszár's Cutoff Rates for Arbitrary Discrete Sources," submitted, 1999.

[2] I. Csiszár, "Generalized cutoff rates and Rényi's information measures," *IEEE Trans. Inform. Theory*, pp. 26–34, Jan. 1995.

[3] T. S. Han, *Information-Spectrum Methods in Information Theory*, (in Japanese), Baifukan Press, Tokyo, 1998.

[4] Z. Rached, F. Alajaji and L. L. Campbell, "Rényi's divergence and entropy rates for finite alphabet Markov sources," submitted, 2000.

0-7803-5857-0/00/$10.00 ©2000 IEEE.