Determination of the Asymptotic Largest Minimum Distance of Block Codes

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Abstract — **In this paper, we present a general formula for the asymptotic largest minimum distance (in blocklength) of deterministic block codes under *generalized* distance functions (not necessarily additive, symmetric and bounded). An alternative expression for the same formula, which is useful in characterizing the general Varshamov-Gilbert bound, is next addressed.**

I. Introduction

The problem of determining the asymptotic largest minimum distance among block codes can be described as follows. Determine the asymptotic ratio (in n) of n-fold largest minimum distance among M selected codewords divided by the code blocklength n, subject to a fixed rate $R \triangleq \log(M)/n$ over a given code alphabet and a given measurable function on the 'distance' between two code symbols.

Research on this problem has been done for years. Up to the present, only bounds on this ratio are established. Recently, channels without statistical assumptions such as memoryless, information stability, stationarity, causality, and ergodicity, etc., are successfully handled by employing the notions of *liminf in probability* and *limsup in probability* of the information spectrum [2]. Inspired by such probabilistic methodology, together with random-coding scheme with expurgation, a spectrum formula on the largest minimum distance of deterministic block codes for *generalized* distance functions (not necessarily additive, symmetric and bounded) is established [1]. An alternative expression for the same formula in term of large deviations is next addressed.

II. Formulas for the Asymptotic Largest Minimum Distance of Block Codes

Denote the n-tuple code alphabet by $\mathcal{X}^n$. For any two elements $\hat{x}^n$ and x^n in $\mathcal{X}^n$, we use $\mu_n(\hat{x}^n, x^n)$ to denote the n-fold measure on the "distance" of these two elements. A codebook with block length n and size M is represented by

$$\mathcal{C}_{n,M} \triangleq \left\{ \mathbf{c}_0^{(n)}, \mathbf{c}_1^{(n)}, \mathbf{c}_2^{(n)}, \ldots, \mathbf{c}_{M-1}^{(n)} \right\},$$

where $\mathbf{c}_m^{(n)} \triangleq (c_{m1}, c_{m2}, \ldots, c_{mn})$, and each c_{mk} belongs to $\mathcal{X}$. We define the *minimum distance* and the *largest minimum distance* respectively as

$$d_m(\mathcal{C}_{n,M}) \triangleq \min_{\substack{0 \le \hat{m} \le M-1 \\ \hat{m} \ne m}} \mu_n\left(\mathbf{c}_{\hat{m}}^{(n)}, \mathbf{c}_m^{(n)} \right)$$

and

$$d_{n,M} \triangleq \max_{\mathcal{C}_{n,M}} \min_{0 \le m \le M-1} d_m(\mathcal{C}_{n,M}).$$

Note that there is no assumption on the code alphabet $\mathcal{X}$ and the sequence of the functions $\{\mu_n(\cdot,\cdot)\}_{n\ge 1}$. For simplicity, $\hat{X}^n$ and X^n are used specifically to denote two independent random variables having common distribution P_{X^n} throughout. The natural logarithm is employed unless otherwise stated.

Theorem 1 (distance-spectrum formula)

$$\sup_{\mathbf{X}} \bar{\Lambda}_{\mathbf{X}}(R) \ge \limsup_{n\to\infty} \frac{d_{n,M}}{n} \ge \sup_{\mathbf{X}} \bar{\Lambda}_{\mathbf{X}}(R+\delta)$$

and

$$\sup_{\mathbf{X}} \underline{\Lambda}_{\mathbf{X}}(R) \ge \liminf_{n\to\infty} \frac{d_{n,M}}{n} \ge \sup_{\mathbf{X}} \underline{\Lambda}_{\mathbf{X}}(R+\delta)$$

for every $\delta > 0$, where

$$\bar{\Lambda}_{\mathbf{X}}(R) \triangleq \inf\left\{ a \in \Re : \limsup_{n\to\infty} \left(Pr\left\{ \frac{\mu(\hat{X}^n, X^n)}{n} > a \right\} \right)^M = 0 \right\}$$

and

$$\underline{\Lambda}_{\mathbf{X}}(R) \triangleq \inf\left\{ a \in \Re : \liminf_{n\to\infty} \left(Pr\left\{ \frac{\mu(\hat{X}^n, X^n)}{n} > a \right\} \right)^M = 0 \right\}.$$

We next derive an alternative expression for the formulas derived above.

Lemma 1 (large deviation formulas for $\bar{\Lambda}_{\mathbf{X}}(R)$ and $\underline{\Lambda}_{\mathbf{X}}(R)$)

$$\bar{\Lambda}_{\mathbf{X}}(R) = \inf\left\{ a \in \Re : \bar{\ell}_{\mathbf{X}}(a) < R \right\}$$

and

$$\underline{\Lambda}_{\mathbf{X}}(R) = \inf\left\{ a \in \Re : \underline{\ell}_{\mathbf{X}}(a) < R \right\}$$

where $\bar{\ell}_{\mathbf{X}}(a)$ and $\underline{\ell}_{\mathbf{X}}(a)$ are respectively the sup- and the inf-large deviation spectrums of $(1/n)\mu_n(\hat{X}^n, X^n)$, defined as

$$\bar{\ell}_{\mathbf{X}}(a) \triangleq \limsup_{n\to\infty} -\frac{1}{n} \log Pr\left\{ \frac{1}{n}\mu_n(\hat{X}^n, X^n) \le a \right\}$$

and

$$\underline{\ell}_{\mathbf{X}}(a) \triangleq \liminf_{n\to\infty} -\frac{1}{n} \log Pr\left\{ \frac{1}{n}\mu_n(\hat{X}^n, X^n) \le a \right\}.$$

References

[1] P.-N. Chen, T.-Y. Lee and Y.-S. Han, "Distance-spectrum formulas on the largest minimum distance of block codes," to appear, *IEEE Trans. Inform. Theory*, May 2000.

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