Large Deviations Approaches to Performance Analysis of Distributed Detection Systems

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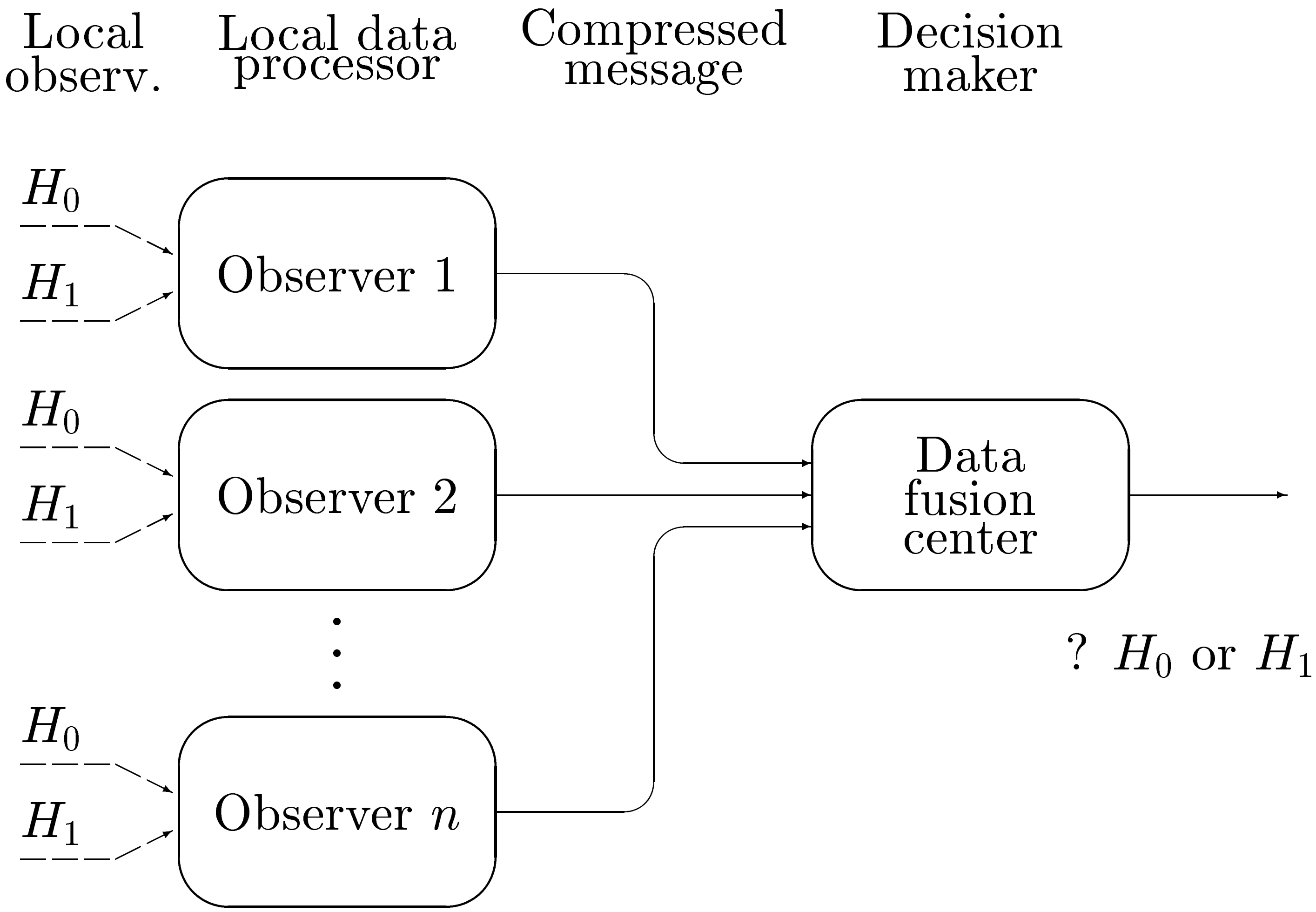
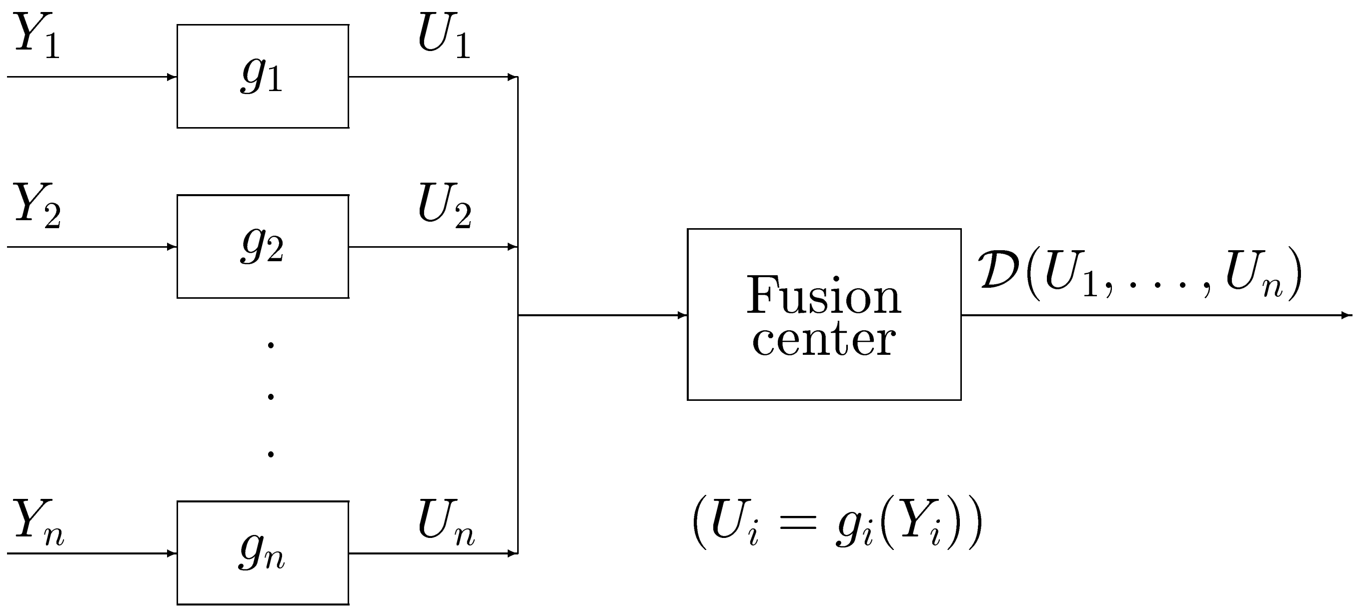
Distributed Detection Systems

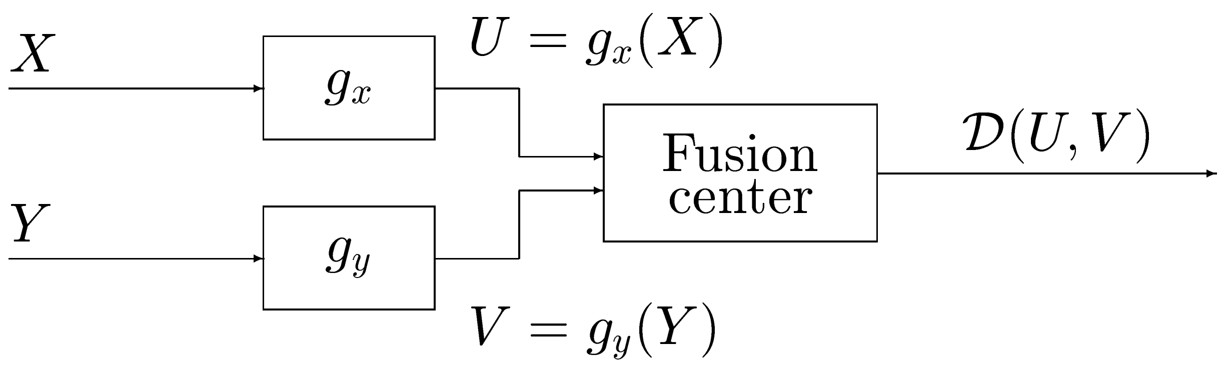


H_0 null hypothesis
H_1 alternative hypothesis

Model 1 $Y_1, \ldots, Y_n$ are i.i.d. under each hypothesis.

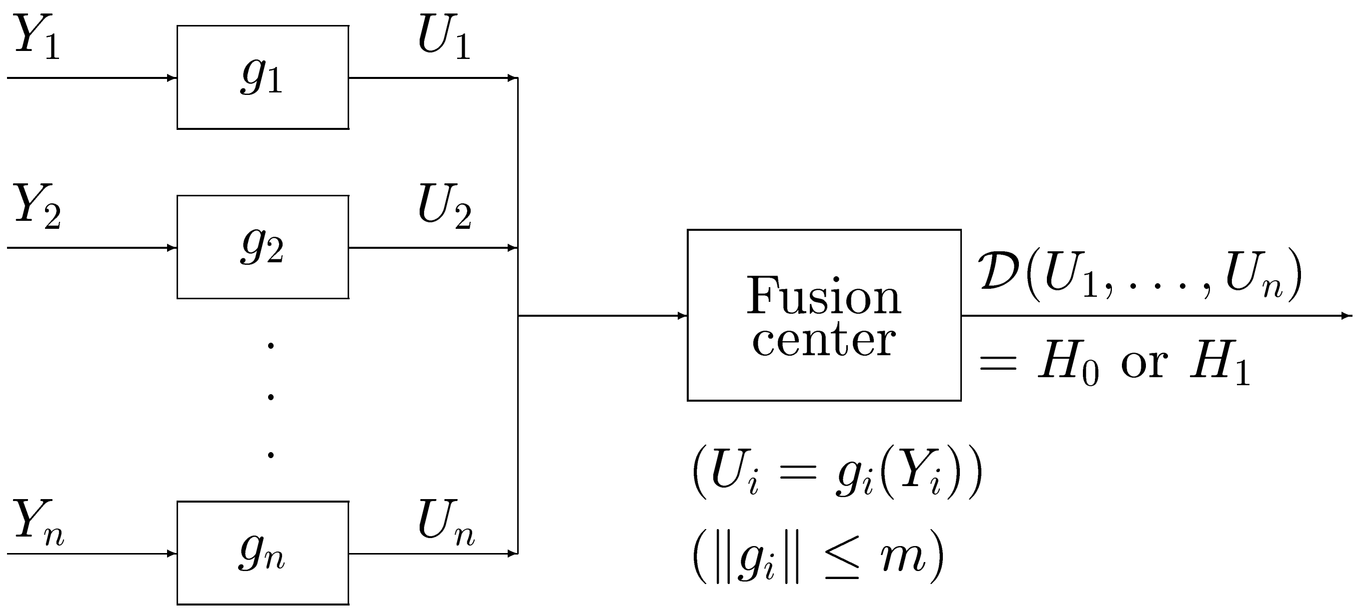


Model 2 X and Y are correlated Gaussian with different mean under each hypothesis.



Problem Description (Model 1)

$\{Y_i\}_{i=1}^n$ i.i.d. under each hypothesis.



Choose $\left[\begin{array}{l} m\text{-ary quantizers } g_1, \ldots, g_n \\ \text{global decision rule } \mathcal{D} \end{array}\right.$

so as to minimize

- Neyman-Pearson type II error $\beta_n^*(\alpha)$
  subject to a constant bound α on type I error.
- Bayes error probability $\gamma_n^*(\pi)$
  under null prior π.

Known Analytical Results

- $\exists$ optimal system employing *likelihood ratio quantizers*. (Tenney & Sandell 1981)

- Optimal system *need not* employ identical quantizers:

  If Bayes error is given by

  $\gamma_n^*(\pi)$ (absolutely) optimal system
  $\gamma_n^\diamond(\pi)$ best symmetric (identical-quantizer) system

  then $\gamma_n^\diamond(\pi) > \gamma_n^*(\pi)$ in general.
  (Tsitsiklis 1986)

- Similarly, $\beta_n^\diamond(\alpha) > \beta_n^*(\alpha)$ in general, where $\beta_n^\diamond(\alpha)$ and $\beta_n^*(\alpha)$ are the counterparts of $\gamma_n^\diamond(\pi)$ and $\gamma_n^*(\pi)$ in Neyman-Pearson testing.

- The two systems are *exponentially* equivalent (equality of error exponents) under some moment condition, i.e.,

$$\lim_{n\to\infty} -\frac{1}{n}\log\gamma_n^\diamond(\pi) = \lim_{n\to\infty} -\frac{1}{n}\log\gamma_n^*(\pi)$$
$$\lim_{n\to\infty} -\frac{1}{n}\log\beta_n^\diamond(\alpha) = \lim_{n\to\infty} -\frac{1}{n}\log\beta_n^*(\alpha)$$

  (Tsitsiklis 1988)

Empirical Results

The ratio $\gamma_n^*(\pi)/\gamma_n^\diamond(\pi)$ is (apparently) lower-bounded by a constant which is often reasonably close to unity.

(It is, of course, upper bounded by unity.)

- (Tsitsiklis 1988) Under some moment condition on the statistics of the local observations,

$$\lim_{n\to\infty} -\frac{1}{n}\log\gamma_n^\diamond(\pi) = \lim_{n\to\infty} -\frac{1}{n}\log\gamma_n^*(\pi).$$

Main Result 1 (Bayes testing)

Let $\begin{cases} Y_i \sim P & \text{under } H_0 \\ Y_i \sim Q & \text{under } H_1 \end{cases}.$

For all $\pi \in (0,1)$,

$$0 < \liminf_{n\to\infty} \frac{\gamma_n^*(\pi)}{\gamma_n^\diamond(\pi)} \le \limsup_{n\to\infty} \frac{\gamma_n^*(\pi)}{\gamma_n^\diamond(\pi)} \le 1$$

is true as long as $P \equiv Q$.

(Moment condition, as conjectured by Tsitsiklis, is superfluous.)

Related Counterexample

A fairly simple choice of P and Q gives

$$\liminf_{n\to\infty} \frac{\gamma_n^*(\pi)}{\gamma_n^\diamond(\pi)} < 1.$$

• It is easy to show

$$\lim_{n\to\infty} -\frac{1}{n}\log\beta_n^{\diamond}(\alpha) = \lim_{n\to\infty} -\frac{1}{n}\log\beta_n^{*}(\alpha)$$

under some moment condition.

Main result 2 (Neyman-Pearson testing)

1. Under an additional regularity condition on the statistics of the local observations,

$$0 < \liminf_{n\to\infty} \frac{\beta_n^{*}(\alpha)}{\beta_n^{\diamond}(\alpha)} \le \limsup_{n\to\infty} \frac{\beta_n^{*}(\alpha)}{\beta_n^{\diamond}(\alpha)} \le 1.$$

2. In the absence of the regularity condition, it is possible that the ratio $\beta_n^{*}(\alpha)/\beta_n^{\diamond}(\alpha)$ goes to zero.

Related Results

Under some moment condition on the statistics of the local observations, the type II error exponent

$$\lim_{n\to\infty} -\frac{1}{n}\log\beta_n^{*}(\alpha)$$

(if it exists) is a function of the type I error bound α.

Input Layer

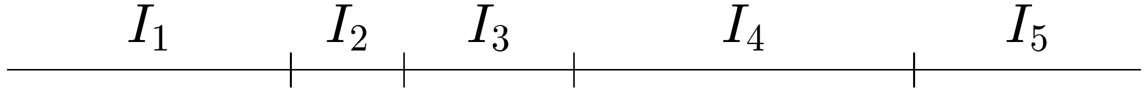
- Local observation $Y_i \in (\mathcal{Y}, \mathcal{B})$.
- Log-likelihood ratio observed by ith sensor:

$$Z(Y_i) \stackrel{\triangle}{=} \log \frac{dP}{dQ}(Y_i).$$

  ($Z(Y_i)$ is well-defined and a.s. finite when $P \equiv Q$.)
- P and Q are nontrivial w.r.t. m-ary quantization.
  Trivial case: if $|\mathcal{Y}| \leq m$, then choose m-ary quantizer as $U_i = g_i(Y_i) = Y_i$. (no quantization)

Quantization

- Set of all possible m-ary likelihood ratio quantizers (LRQ's) : $\mathcal{F}_m$
- m-ary LRQ f partitions $\Re$ into consecutive intervals $I_1, \ldots, I_m$, e.g., $m = 5$



- LRQ output (message) is given by

$$U = g(Y) = u \quad \text{iff} \quad Z(Y) \in I_u.$$

  Induced distributions on $\mathcal{U}_m = \{1, \ldots, m\}$

$$\begin{array}{l} U \sim P_f \;\text{ under } H_0 \\ U \sim Q_f \;\text{ under } H_1 \end{array}.$$

- Post-quantization log-likelihood ratio of ith sensor:

$$Z_{f_i}(u) \stackrel{\triangle}{=} \log \frac{P_{f_i}(u)}{Q_{f_i}(u)}.$$

(ith sensor employs LRQ f_i.)

Neyman-Pearson testing

Boundedness Assumption $\exists\, 1 \geq \delta \geq 0$ for which

$$\sup_{f \in \mathcal{F}_m} E_P[|Z_f|^{2+\delta}] < \infty.$$

Remark. Define $\lambda_t \stackrel{\triangle}{=} ((-\infty, t], (t, \infty))$.

Boundedness Assumption is equivalent to

$$\limsup_{t \to \infty} E_P[Z_{\lambda_t}^{2+\delta}] < \infty.$$

Main Results Distinguish between Four Cases

Case A. $$\limsup_{t\to\infty} E_P[Z_{\lambda_t}] = \infty.$$

Case B. $$0 < \limsup_{t\to\infty} E_P[Z_{\lambda_t}] < \infty, \ \limsup_{t\to\infty} E_P[Z_{\lambda_t}^2] = \infty.$$

Case C. $$\limsup_{t\to\infty} E_P[Z_{\lambda_t}] = 0, \ \limsup_{t\to\infty} E_P[Z_{\lambda_t}^2] = \infty.$$

Case D. $$\limsup_{t\to\infty} E_P[Z_{\lambda_t}^{2+\delta}] < \infty.$$

Example $Y_i \in (\mathcal{Y}, \mathcal{B}) = ((0,1], \mathrm{Borel}(0,1])$

$$P\{Y \le y\} = y$$

$$Q\{Y \le y\} = \exp\left\{\frac{a+1}{a}\left(1 - \frac{1}{y^a}\right)\right\}$$

a. $a > 1$: $\limsup_{t\to\infty} E_P[Z_{\lambda_t}] = \infty$. (Case A)

b. $a = 1$: $\limsup_{t\to\infty} E_P[Z_{\lambda_t}] = 2$,

$\limsup_{t\to\infty} E_P[Z^2_{\lambda_t}] = \infty$. (Case B)

c. $1/2 < a < 1$: $\limsup_{t\to\infty} E_P[Z_{\lambda_t}] = 0$,

$\limsup_{t\to\infty} E_P[Z^2_{\lambda_t}] = \infty$. (Case C)

d. $0 < a \le 1/2$: $\limsup_{t\to\infty} E_P[Z^{2+\delta}_{\lambda_t}] < \infty$. (Case D)

Theorem 1 (continued) *Case B.*

If $0 < \limsup_{t\to\infty} E_P[Z_{\lambda_t}] < \infty$ and $\limsup_{t\to\infty} E_P[Z^2_{\lambda_t}] = \infty$,

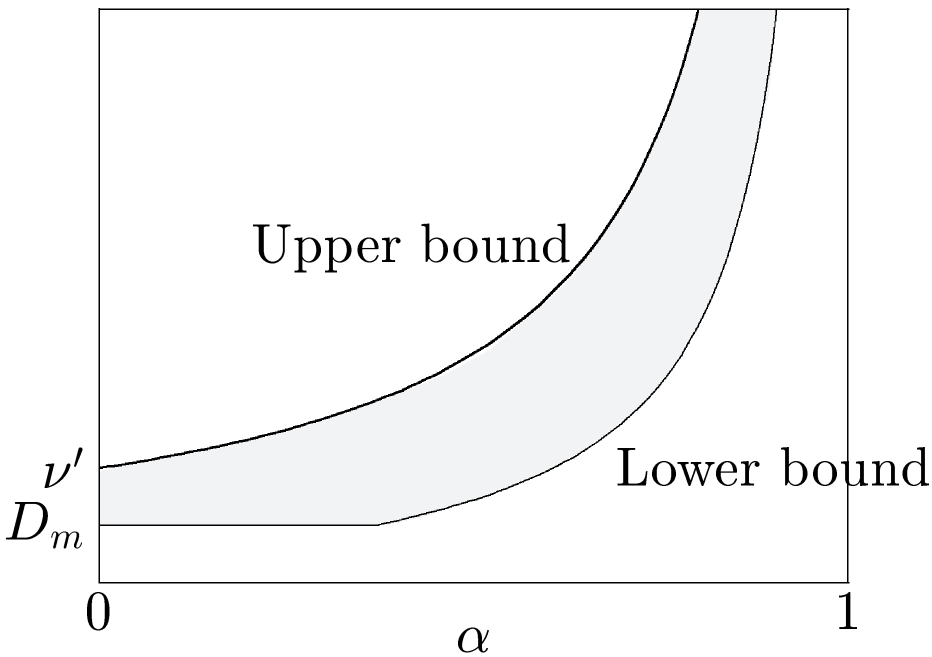
there exists an increasing sequence of integers $\{n_k, k \in \mathbf{N}\}$ such that

$$\liminf_{k\to\infty} -\frac{1}{n_k}\log\beta^{\diamond}_{n_k}(\alpha) \;\geq\; D_m \vee \frac{\nu}{\log(1/\alpha)}, \quad (\textit{Lower bound})$$

and

$$\limsup_{n\to\infty} -\frac{1}{n}\log\beta^{*}_{n}(\alpha) \;\leq\; \frac{\nu'}{1-\alpha}, \qquad (\textit{Upper bound})$$

where
$$D_m \stackrel{\triangle}{=} \sup_{f\in\mathcal{F}_m} E_P[Z_f] = \sup_{f\in\mathcal{F}_m} D(P_f\|Q_f)$$
$$\nu \stackrel{\triangle}{=} \limsup_{t\to\infty} E_P[Z_{\lambda_t}]$$
$$\nu' \stackrel{\triangle}{=} \sup_{f\in\mathcal{F}_m} E_P[|Z_f|]$$



Preliminaries on Case D: $\sup_{f \in \mathcal{F}_m} E_P[|Z_f|^{2+\delta}] < \infty$

- (Recall) $\mathcal{F}_m \in [0,1]^{2m}$: set of all possible m-ary LRQ's.

$$f = (P\{U{=}1\}, \ldots, P\{U{=}m\}, Q\{U{=}1\}, \ldots, Q\{U{=}m\}) .$$

- Euclidean distance:

$$\|f - f'\| \stackrel{\triangle}{=} \left[\sum_{i=1}^{2m} (f_i - f'_i)^2 \right]^{1/2}, \qquad f \text{ and } f' \in \mathcal{F}_m.$$

- Define

$$\begin{aligned} \mu(f) &\stackrel{\triangle}{=} E_P[Z_f] \\ \sigma^2(f) &\stackrel{\triangle}{=} E_P[|Z_f - \mu(f)|^2] \\ \tau_\delta(f) &\stackrel{\triangle}{=} E_P[|Z_f - \mu(f)|^{2+\delta}] \end{aligned}$$

Observations

- $D_m \stackrel{\triangle}{=} \sup_{f \in \mathcal{F}_m} E_P[Z_f] = \sup_{f \in \mathcal{F}_m} D(P_f \| Q_f) = \sup_{f \in \mathcal{F}_m} \mu(f).$
- $\mu(f)$ is convex over $\mathcal{F}_m$.
- $\mu(f)$ and $\sigma^2(f)$ are continuous over $\mathcal{F}_m$.
- $\tau_\delta(f)$ is continuous at any point $f \in \mathcal{F}_m$ with strictly positive coordinates.
- $\mathcal{O}_m \stackrel{\triangle}{=} \{f \in \mathcal{F}_m : \mu(f) = D_m\}$ is *non-empty closed* subset of $\mathcal{F}_m$. Every element in $\mathcal{O}_m$ has strictly positive entries.

Observations (continued)

- (Neighborhood of $\mathcal{O}_m$) Let

$$\mathcal{N}_a(\mathcal{O}_m) \triangleq \left\{ f \in \mathcal{F}_m \; : \; \min_{f_{\mathcal{O}} \in \mathcal{O}_m} \|f - f_{\mathcal{O}}\| \leq a \right\}.$$

  Then

  1. $\sigma^2(f)$ and $\tau_\delta(f)$ are bounded away from zero on $\mathcal{N}_a(\mathcal{O}_m)$.
  2. $\exists \; A_1 > 0$ such that $(\forall \; f_{\mathcal{N}} \in \mathcal{N}_a(\mathcal{O}_m)$ and $f \in \mathcal{F}_m)$

$$\left| \sigma^2(f_{\mathcal{N}}) - \sigma^2(f) \right| \leq A_1 \cdot \|f_{\mathcal{N}} - f\|.$$

Regularity Assumption There exists $A_2 > 0$ such that for all $f \in \mathcal{F}_m$,

$$D_m - \mu(f) \geq A_2 \cdot \left(\min_{f_{\mathcal{O}} \in \mathcal{O}_m} \|f - f_{\mathcal{O}}\| \right)^2 .$$

Remarks.

- When $|\mathcal{Y}| < \infty$, Regularity Assumption is satisfied.
- For $m = 2$ and $m = 3$, Regularity Assumption is not violated in any of parametric tests investigated with the aid of MATHEMATICA including testing for a fixed signal in Gaussian noise.
- It is possible to construct examples that violate Regularity Assumption.

Let ith sensor in an n-sensor distributed system employs LRQ f_{ni}. Define

$$
\begin{aligned}
m_n &= \sum_{i=1}^{n} \mu(f_{ni}) \\
s_n^2 &= \sum_{i=1}^{n} \sigma^2(f_{ni}) \\
r_{n,\delta} &= \sum_{i=1}^{n} \tau_\delta(f_{ni})
\end{aligned}
$$

Basic Lemma for Neyman-Pearson testing

If $\lim_{n\to\infty} m_n/n = D_m$, $\exists\, C_n = C_n(\delta, \alpha)$ such that

$$
\frac{1}{C_n\sqrt{n}} \exp\left\{-s_n\Phi^{-1}(\alpha) - m_n\right\} \le \beta_n(\alpha) \le \frac{C_n}{\sqrt{n}} \exp\left\{-s_n\Phi^{-1}(\alpha) - m_n\right\},
$$

where $\log C_n = O(r_{n,\delta}/s_n^{1+\delta})$ and $\Phi(\cdot)$ is the cdf of $\mathcal{N}(0,1)$.

Pf. Large deviations technique and Berry-Esseen Theorem.

Remark. In both absolutely optimal and best identical-quantizer systems,

$$
\lim_{n\to\infty} m_n/n = D_m \quad \text{and} \quad r_{n,\delta}/s_n^{1+\delta} = O(n^{(1-\delta)/2}).
$$

Remark. (On centralized Neyman-Pearson testing)

If $Z_1, \ldots, Z_n$ i.i.d. and $E_P[Z^3] < \infty$,

$$\beta_n(\alpha) \asymp \frac{1}{\sqrt{n}} \exp\left\{-\sigma\Phi^{-1}(\alpha)\sqrt{n} - D\,n\right\},$$

where $\sigma^2 \triangleq \mathrm{Var}_P[Z]$ and $D \triangleq E_P[Z]$.

Theorem 1 (continued) *Case D.*

Under Boundedness Assumption, i.e.,

$$\sup_{f\in\mathcal{F}_m} E_P[|Z_f|^{2+\delta}] < \infty,$$

if $\alpha \le 1/2$, or if $\alpha > 1/2$ and Regularity Assumption holds, then

$$\frac{\beta_n^*(\alpha)}{\beta_n^\diamond(\alpha)} \ge \exp\left\{-C(\delta,\alpha)\cdot n^{(1-\delta)/2}\right\}.$$

In particular, if $\sup_{f\in\mathcal{F}_m} E_P[|Z_f|^3] < \infty$ (i.e., $\delta = 1$), then the ratio $\beta_n^*(\alpha)/\beta_n^\diamond(\alpha)$ is bounded from below.

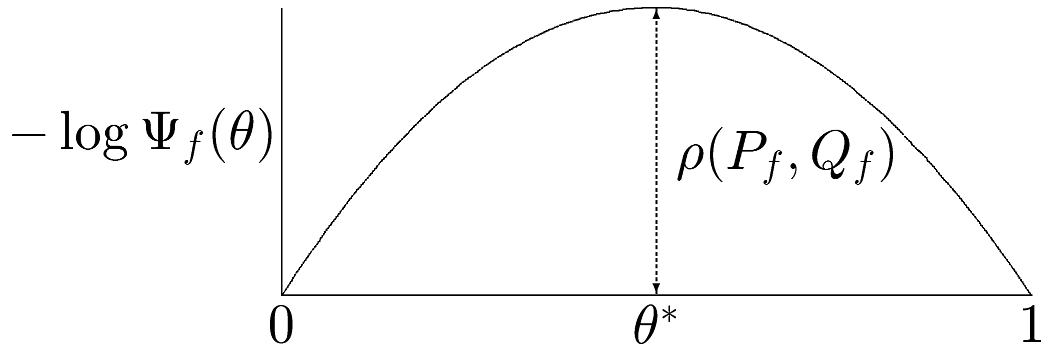
Proof. To show both systems satisfy the hypothesis of the Basic Lemma for Neyman-Pearson Testing, and then apply this lemma. □

Chernoff exponent

Moment generating function of Z_f:

$$\Psi_f(\theta) = E_Q \left[\exp \{ \theta Z_f \} \right].$$

Both $\Psi_f(\theta)$ and $\log \Psi_f(\theta)$ are finite and convex in θ.



Chernoff exponent is defined by

$$\rho(P_f, Q_f) \stackrel{\triangle}{=} -\log \left[\min_{\theta \in (0,1)} \Psi_f(\theta) \right].$$

Significance

- $\rho(P_f, Q_f)$ is the Bayes error exponent in the i.i.d. case, when LRQ f is used by *all* sensors.
- The optimal (both $\diamond$ and $*$) error exponent is

$$\rho_m \stackrel{\triangle}{=} \max_{f \in \mathcal{F}_m} \rho(P_f, Q_f).$$

Error Analysis for Nonidentical Quantizers

- $Z_{f_1}, \ldots, Z_{f_n}$ are idependent random variables.

  Moment generating function of $\sum_{i=1}^{n} Z_{f_i}$ is

$$M_n(\theta) = \prod_{i=1}^{n} \Psi_{f_i}(\theta).$$

  and has a unique minimum at $\theta_n \in (0,1)$.

$$M_n(\theta_n) \geq \prod_{i=1}^{n} \min_{\theta \in (0,1)} \Psi_{f_i}(\theta) \geq \exp\{-n\rho_m\}$$

- Fusion center uses MAP rule:
  decide H_0 iff $Z_{f_1} + \cdots + Z_{f_n} > \eta \stackrel{\triangle}{=} \log[(1-\pi)/\pi]$

  Bayes error probability:

$$\gamma_n(\pi) = \pi P \left\{ \sum_{i=1}^{n} Z_{f_i} \leq \eta \right\} + (1-\pi) Q \left\{ \sum_{i=1}^{n} Z_{f_i} > \eta \right\}$$

Exponential Tilting

Tilted distribution: Recall that P_{f_i} and Q_{f_i} are the post-quantization distribution pairs w.r.t. quantizer f_i. The *tilted* distribution of Z_{f_i} is defined by:

$$Q_i^{(\theta_n)}(z) \triangleq \frac{\exp\{\theta_n z\} Q_{f_i}(z)}{\Psi_{f_i}(\theta_n)}.$$

Under the product measure $Q^{(\theta_n)} = Q_1^{(\theta_n)} \times \cdots \times Q_n^{(\theta_n)}$:

$$E_{Q^{(\theta_n)}}\left[\sum_{i=1}^{n} Z_{f_i}\right] = 0.$$

Equivalently

$$\int_{\Re} z \; dF_n^{(\theta_n)}(z) = 0$$

where $F_n^{(\theta_n)}(z)$ is the cdf of $\sum_{i=1}^{n} Z_{f_i}$ under $Q^{(\theta_n)}$.

Bounds on Bayes Error Probability

Rewrite $\gamma_n(\pi)$ using

$$P\left\{\sum_{i=1}^{n} Z_{f_i} \leq \eta\right\} = M_n(\theta_n) \int_{[z \leq \eta]} \exp\{(1-\theta_n)z\}\, dF_n^{(\theta_n)}(z)$$

$$Q\left\{\sum_{i=1}^{n} Z_{f_i} > \eta\right\} = M_n(\theta_n) \int_{[z > \eta]} \exp\{-\theta_n z\}\, dF_n^{(\theta_n)}(z)$$

Upper bound: For the i.i.d. case ($f_i = f$), CLT-based techniques yield (Bahadur & Rao 1960)

$$c \leq \gamma_n(\pi)\sqrt{n}\exp\{n\rho(P_f, Q_f)\} \leq C$$

with $0 < c \leq C < \infty$. Hence the upper bound

$$\gamma_n^{\diamond}(\pi) \leq \frac{c^{\diamond}}{\sqrt{n}} \exp\{-n\rho_m\}.$$

Lower bound: To prove the desired result, need lower bound the form

$$\gamma_n^{*}(\pi) \geq \frac{c^{*}}{\sqrt{n}} \exp\{-n\rho_m\}.$$

- (Easiest part)

$$\gamma_n^{*}(\pi) \geq [\pi \wedge (1-\pi)]\ M_n(\theta_n) \int_{\Re} \exp\{-|z|\} dF_n^{(\theta_n)}(z).$$

- (More difficult part) Show that integral above is $O(1/\sqrt{n})$.

Basic Lemma for Bayes Testing

If $\exists$ uniform (in n and $i \leq n$) bounds $a > 0$ and $b < \infty$ such that

C1. $|Z_{f_i}| \leq b$ a.s.; and
C2. $\mathrm{Var}_{Q^{(\theta_n)}}[Z_{f_i}] \geq a^2$,

then

$$\liminf_{n\to\infty} \sqrt{n} \int_{\Re} \exp\{-|z|\} \, dF_n^{(\theta_n)}(z) > 0.$$

and thus for some $c^* > 0$,

$$\gamma_n^*(\pi) \geq \frac{c^*}{\sqrt{n}} \exp\{-n\rho_m\}.$$

Proof. Utilize a CLT (due to Esseen) for independent, but not necessarily identically distributed, summands. C1 and C2 are related to Lindeberg's condition. □

Major obstacle: There is no guarantee that in an absolutely optimal system, the f_i's will satisfy conditions C1 and C2.

Solution: Divide sensors into

- good sensors (obeying C1 and C2)
- bad sensors (not necessarily obeying C1 and C2)

and treat separately.

The Split

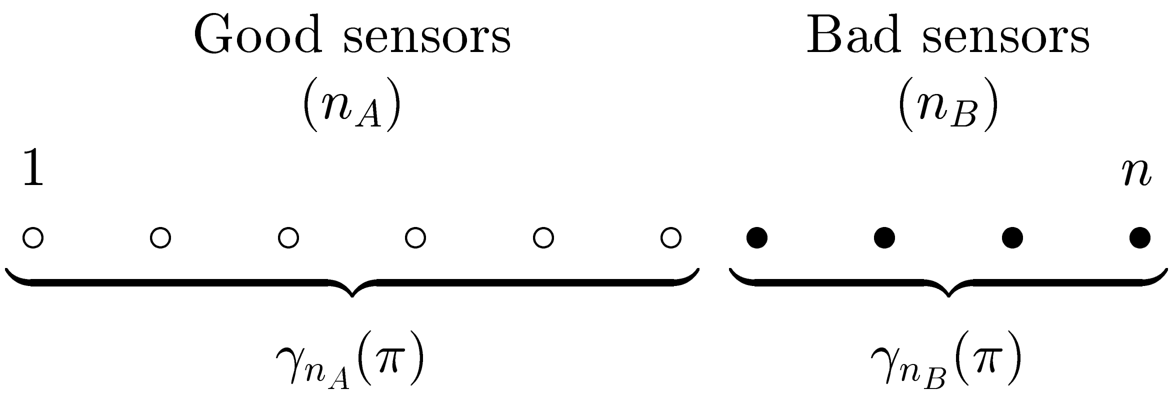
Consider the optimal quantizers, and fix $\delta > 0$.

Good sensors are those employing a f such that

$$(\forall\ u \in \mathcal{U}_m)\ \ \min\{P_f(u), Q_f(u)\} \geq \delta$$

Bad sensors are those employing a f such that

$$(\exists\ u \in \mathcal{U}_m)\ \ \min\{P_f(u), Q_f(u)\} < \delta$$



Fact:

$$\gamma_n^*(\pi) \geq 2\gamma_{n_A}(\pi)\gamma_{n_B}(1/2).$$

- Good sensors satisfy conditions C1 and C2 in the hypothesis of the Basic Lemma, so

$$\gamma_{n_A}(\pi) \geq \frac{c_A}{\sqrt{n_A}} \exp\left\{-n_A\, \rho_m\right\}.$$

- Need similar result for $\gamma_{n_B}(1/2)$.

Conditioning Argument

- For each bad sensor, define the event

$$\Delta_i \stackrel{\triangle}{=} [\min\{P_{f_i}(U_i), Q_{f_i}(U_i)\} \geq \delta]$$

and let

$$\Delta \stackrel{\triangle}{=} \bigcap_{\text{bad } i} \Delta_i.$$

- Easy to show that

$$\gamma_{n_B}(1/2) \geq \gamma_c \cdot \min\{P(\Delta), Q(\Delta)\}$$

where

$$\gamma_c \stackrel{\triangle}{=} \frac{1}{2} P\left\{ \sum_{\text{bad } i} Z_{f_i} \leq \eta \middle| \Delta \right\} + \frac{1}{2} Q\left\{ \sum_{\text{bad } i} Z_{f_i} > \eta \middle| \Delta \right\}.$$

- By virtue of $P \equiv Q$,

$$\min\{P(\Delta), Q(\Delta)\} \geq \exp\{-n_B \cdot \varepsilon_1(\delta)\}$$

where $\varepsilon_1(\delta) \to 0$ as $\delta \to 0$.

- It remains to build a (good) lower bound for γ_c. This is done by "restoration" of bad sensors.

Restoration (To bound γ_c)

Construct a new system of n_B sensors such that:

- For each i, the observation space is restricted to Δ_i, i.e.,

$$\mathcal{Y}_i^{\Delta} \triangleq \{y \in \mathcal{Y} : f_i(y) \in \Delta_i\}.$$

- Each new observation Y_i^{Δ} has distribution

$$P_i^{\Delta}(y) = \frac{P(y)}{P(\Delta_i)}$$

  for $y \in \mathcal{Y}_i^{\Delta}$ under H_0; and similarly for Q under H_1;

- Each new quantizer is the restriction of its predecessor to Δ_i. In particular:

  1. it is an ℓ-ary quantizer, where

$$\ell \leq m - 1.$$

     Recall that for a bad sensor, $\exists\, u'$ such that $\min\{P_{f_i}(u'), Q_{f_i}(u')\} < \delta$, and hence $u' \notin \Delta_i$.

  2. it is a likelihood ratio quantizer.

By a simple argument, if $\gamma_{n_B}^{\Delta}(1/2)$ is the Bayes error of the new system, then

$$\gamma_c \geq \gamma_{n_B}^{\Delta}(1/2)$$

and therefore

$$\gamma_{n_B}(1/2) \geq \exp\{-n_B \cdot \varepsilon_1(\delta)\}\, \gamma_{n_B}^{\Delta}(1/2).$$

Error Exponent of $\gamma_{n_B}^{\Delta}(1/2)$

- Basic Lemma applies to the new system.
- The moment generating function of each new post-quantization log-likelihood ratio is vanishingly (in δ) different from its predecessor:

$$\sup_{\theta \in (0,1)} \left| \Psi_{f_i}^{\Delta}(\theta) - \Psi_{f_i}(\theta) \right| \leq \varepsilon_2(\delta).$$

  This gives

$$\left| \rho\left(P_{f_i}^{\Delta}, Q_{f_i}^{\Delta}\right) - \rho\left(P_{f_i}, Q_{f_i}\right) \right| \leq \varepsilon_3(\delta). \tag{1}$$

- The new LRQ is an ℓ-ary quantizer, $\ell \leq m-1$, implies that

$$\rho\left(P_{f_i}, Q_{f_i}\right) < \rho_{m-1} + \varepsilon_4(\delta) \tag{2}$$

  where $\rho_{m-1} \triangleq \sup_{f \in \mathcal{F}_{m-1}} \rho(P_f, Q_f)$.

Conclude: (from Basic Lemma, (1) and (2))

$$\gamma_{n_B}^{\Delta}(1/2) \geq \frac{c_5}{\sqrt{n_B}} \exp\left\{-n_B(\rho_{m-1} + \varepsilon_5(\delta))\right\}$$

and (hence)

$$\gamma_{n_B}(1/2) \geq \frac{c_6}{\sqrt{n_B}} \exp\left\{-n_B(\rho_{m-1} + \varepsilon_6(\delta))\right\}.$$

$$\begin{aligned}\gamma_n^*(\pi) &\geq 2\ \gamma_{n_A}(\pi)\ \gamma_{n_B}(1/2) \\ &\geq 2\frac{c_A}{\sqrt{n_A}}\exp\{-n_A\,\rho_m\}\frac{c_6}{\sqrt{n_B}}\exp\{-n_B(\rho_{m-1}+\varepsilon_6(\delta))\} \\ &= \frac{2c_Ac_6}{\sqrt{n_An_B}}\exp\{-[n_A\rho_m+n_B(\rho_{m-1}+\varepsilon_6(\delta))]\} \\ \gamma_n^\diamond(\pi) &\leq \frac{c^\diamond}{\sqrt{n}}\exp\{-n\rho_m\}\end{aligned}$$

$$\Rightarrow\ \frac{\gamma_n^*(\pi)}{\gamma_n^\diamond(\pi)} \geq \frac{c}{\sqrt{n_B}}\exp\{n_B(\rho_m-\rho_{m-1}-\varepsilon_6(\delta))\}\,.$$

From the facts that $\rho_m > \rho_{m-1}$ and $\varepsilon_6(\delta) \to 0$ as $\delta \to 0$, we obtain

Theorem 2

$$1 \geq \limsup_{n\to\infty}\frac{\gamma_n^*(\pi)}{\gamma_n^\diamond(\pi)} \geq \liminf_{n\to\infty}\frac{\gamma_n^*(\pi)}{\gamma_n^\diamond(\pi)} > 0$$

Side Result

In an absolutely optimal system, for any fixed $\varepsilon > 0$, only boundedly many (in n) quantizers can have Chernoff exponent smaller than $\rho_m - \varepsilon$.

In particular, if the observation space $\mathcal{Y}$ is finite, the existence of an absolutely optimal system using the *same* quantizer on *all but a bounded* number of sensors.

Counterexamples

We have also constructed P and Q on finite $\mathcal{Y}$ such that

- $\liminf_{n\to\infty} [\gamma_n^*(\pi)/\gamma_n^\diamond(\pi)] < 1$.
- infinitely often (in n), the absolutely optimal system employs *at least two* distinct quantizers.

——— *End of Model 1* ———

Known Results

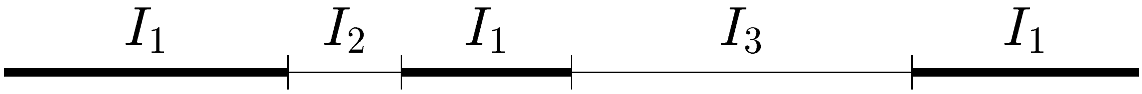
- The optimal g_x and g_y are not necessarily functions of the marginal (log-)likelihood ratios.
  (Tenney & Sandell 1981)
- For the additive Gaussian noise, the marginal LR is a strictly monotone function of the local observation. Therefore, a mapping based on marginal LR can be written as a function of the local observation, i.e.,
  $$g_x : \Re \mapsto \mathcal{U} \text{ and } g_y : \Re \mapsto \mathcal{V}.$$

Motivation

If *contiguous* marginal LRQ's (whose partition regions are contiguous) are optimal, one only need to find $|\mathcal{U}| - 1$ and $|\mathcal{V}| - 1$ break points (for quantizers g_x and g_y respectively) to build the optimal system, which considerably simplifies the optimal design.

Note

g_x and g_y being marginal LRQ's does not imply that they are also *contiguous* marginal LRQ's, e.g., $\mathcal{U} = \{1, 2, 3\}$



$$g_x(x) = u \quad \text{iff} \quad x \in I_u.$$

Statistical Model

$$H_0 : P_{xy} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{pmatrix} \right)$$

$$H_1 : Q_{xy} \sim \mathcal{N}\left(\begin{pmatrix} \mu \\ \eta \end{pmatrix}, \begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{pmatrix} \right)$$

Lemma (Main result)

If

$$\sigma_{xy}(\eta\sigma_x^2 - \mu\sigma_{xy})(\mu\sigma_y^2 - \eta\sigma_{xy}) \geq 0, \qquad \text{(Condition C)}$$

contiguous marginal LRQ's are optimal.

- (*Basic observation*) Suppose $\gamma_{\text{AND}}(\pi)$ is the Bayes error under binary quantization and AND fusion rule, i.e.,

$$\mathcal{D}(g_x(x), g_y(y)) = H_1 \text{ only when } g_x(x) = g_y(y) = 1.$$

Let $\left[\begin{array}{lcl} \mathcal{A} & \stackrel{\triangle}{=} & \{x \in \Re : g_x(x) = 1\} \\ \mathcal{B} & \stackrel{\triangle}{=} & \{y \in \Re : g_y(y) = 1\} \end{array} \right.$. Then

$$\gamma_{\text{AND}}(\pi) = (1-\pi) + \int_{\mathcal{A}} \Bigg[\pi \underbrace{\int_{\mathcal{B}} P_{xy}(x,y)dy}_{H_0 : P'_x} - (1-\pi) \underbrace{\int_{\mathcal{B}} Q_{xy}(x,y)dy}_{H_1 : Q'_x} \Bigg] dx.$$

$$\mathcal{A} : \left(\int_{\mathcal{B}} P_{xy}(x,y)dy \right) \Big/ \left(\int_{\mathcal{B}} Q_{xy}(x,y)dy \right) \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi}{1-\pi}$$

(*Key idea*) For any set $\mathcal{B} \in \Re$, define

$$f_{\mathcal{B}}(x) \stackrel{\triangle}{=} \log \frac{\displaystyle\int_{\mathcal{B}} P_{xy}(x,y)dy}{\displaystyle\int_{\mathcal{B}} Q_{xy}(x,y)dy}.$$

If $f_{\mathcal{B}}(x)$ is a monotone function of x, then optimal $\mathcal{A}$ must be a contiguous marginal LRP.

(*Verification*) Take the derivative of $f_{\mathcal{B}}(x)$. Using a large deviations tool, obtain:

$$\frac{d\, f_{\mathcal{B}}(x)}{dx} = \frac{\sigma_{xy}}{A}\left(E_{\widehat{P}}[Y] - E_{\widehat{P}^{(\theta)}}[Y]\right) - \frac{\mu\sigma_y^2 - \eta\sigma_{xy}}{A},$$

where

$$A = \sigma_x^2\sigma_y^2 - \sigma_{xy}^2$$

$$\widehat{P}(y) = \frac{\exp\left\{-\dfrac{\sigma_y^2}{2A}y^2 + \dfrac{\sigma_{xy}}{A}xy\right\}}{\displaystyle\int_{\mathcal{B}} \exp\left\{-\dfrac{\sigma_y^2}{2A}y^2 + \dfrac{\sigma_{xy}}{A}xy\right\} dy}$$

$$\widehat{P}^{(\theta)}(y) = \frac{\exp\{\theta y\}\,\widehat{P}(y)}{\displaystyle\int_{\mathcal{B}} \exp\{\theta y\}\,\widehat{P}(y)dy}$$

$$\theta = \frac{\eta\sigma_x^2 - \mu\sigma_{xy}}{A}$$

Now:

$$\theta > 0 \implies E_{\widehat{P}}[Y] \leq E_{\widehat{P}(\theta)}[Y]$$
$$\theta < 0 \implies E_{\widehat{P}}[Y] \geq E_{\widehat{P}(\theta)}[Y]$$

If the hypothesis

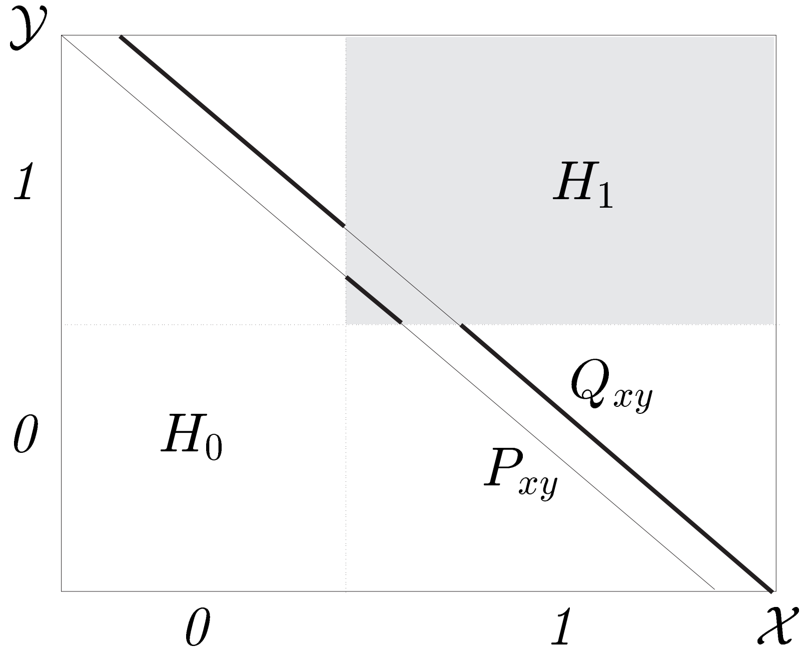
$$\sigma_{xy}\theta(\mu\sigma_y^2 - \eta\sigma_{xy}) = \frac{\sigma_{xy}(\eta\sigma_x^2 - \mu\sigma_{xy})(\mu\sigma_y^2 - \eta\sigma_{xy})}{A} \geq 0, \quad \text{(C)}$$

holds, then

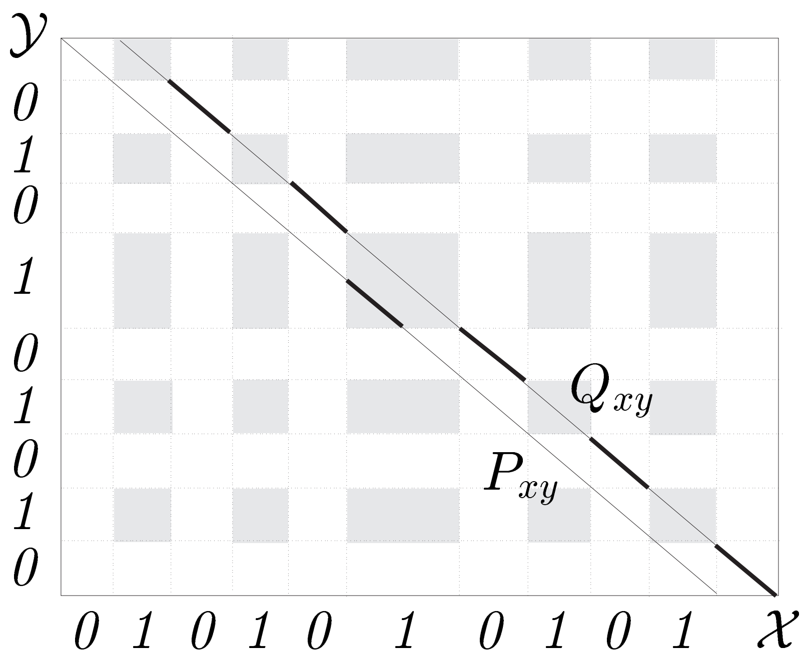
$$\frac{d\, f_{\mathcal{B}}(x)}{dx} = \frac{\sigma_{xy}}{A}\left(E_{\widehat{P}}[Y] - E_{\widehat{P}(\theta)}[Y]\right) - \frac{\mu\sigma_y^2 - \eta\sigma_{xy}}{A}$$

is either *always nonnegative* or *always nonpositive*, which implies $f_{\mathcal{B}}(x)$ is a monotone function of x for any $\mathcal{B}$. $\square$

- The best contiguous marginal LRP.



- Based on the best contiguous LRP, a *better* non-contiguous marginal LRP is constructed.
  (Note that some of the bold segments of the support line Q_{xy} have been removed.)



On Symmetric Distributed System

If P_{xy} is such that $P_x = P_y$ and the same is true for Q_{xy}, then for the optimal design

$$g_x = g_y.$$

(Further simplification of the optimal design.)

Proof: Find the optimal design by taking the derivative of Bayes error w.r.t. the break points, say s and t. Due to the symmetry of the statistics, (s, t) satisfies

$$s = f(s, t) \quad (3)$$

$$t = f(t, s) \quad (4)$$

for some function $f(\cdot, \cdot)$.

By virtue of the inequality

$$(\forall s \neq t) \quad \frac{f(s,t) - f(t,s)}{s - t} < 1,$$

we comfirm that the solutions of the simutaneous equations (3) and (4) must be located on the line $s = t$, i.e., $g_x = g_y$. □

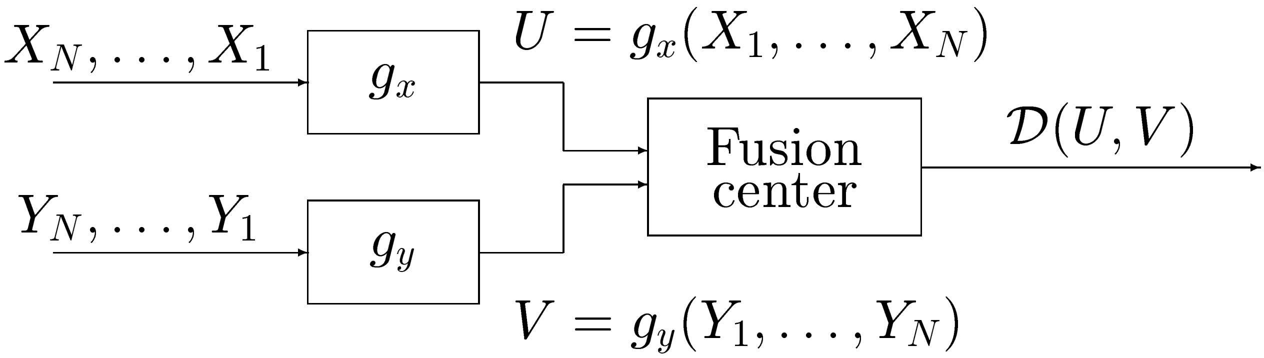
Conclude

Suppose the marginal statistics under null hypothesis (also alternative hypothesis) are the same, and $\sigma_{xy} \geq 0$. Then there is no loss of optimality by employing identical marginal LRQ's whose partition regions are contiguous.

Some Other Related Results

1. An example is also demonstrated where the marginal LRQ's (whether *contiguous* or not) are suboptimal for the case of two observations per sensor.

2. We also consider the temporal asymptotics for a distributed system with temporally i.i.d. statistics.



We obtain that:

- If $\|g_x\| \wedge \|g_y\| \geq O(N^{1/2+b})$ for some $b > 0$, the Neyman-Pearson type II error of the distributed system employing contiguous marginal LRQ's is at most a fixed multiple of that achieved by the centralized system in which the local observations $X_1, \ldots, X_N$ and $Y_1, \ldots, Y_N$ are directly available to the fusion center.

- If $\|g_x\| \wedge \|g_y\| \geq O(N)$, same conclusion can be drawn for the Bayes error.

Conclusions

1. **Parallel distributed detection** (Model 1)

- The dependence of the Neyman-Pearson type II error exponent on the type I error bound in the absence of the boundedness moment condition is surprising.

- The boundedness of the performance ratio between the absolutely optimal and best identical-quantizer systems implies that the degree of asymptotic equivalence of these two systems is far greater than what is implied by the equality of error exponents.

- The boundedness moment condition, as conjectured by Tsitsiklis, is superfluous for Bayes testing; however, for Neyman-Pearson testing, an additional Regularity Assumption is required in order for the type II error ratio between the two systems to be bounded.

- An extension of our results to L-ary hypothesis testing ($L > 2$) can be obtained by applying multidimensional Berry-Esseen Theorem. In such case, the *best identical-quantizer* system is understood as system employing at most $L(L-1)/2$ distinct fixed quantizers.

Conclusions (continued)

2. Two-sensor distributed detection in additive Gaussian noise (Model 2)

- A sufficient condition under which contiguous marginal LRQ's are optimal is derived.
- A counterexample for which contiguous marginal LRQ's are suboptimal is demonstrated in absence of the sufficient condition.
- A symmetric distributed system is shown to have symmetric optimal solutions under such condition.