Optimal Power Allocation for (N, K)-Limited Access Channels

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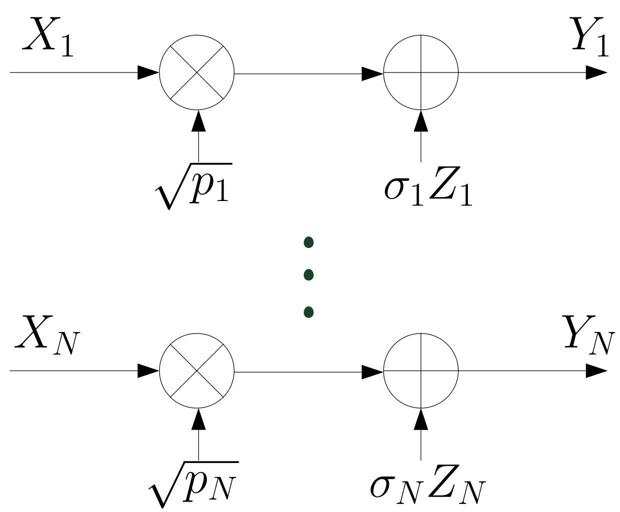
1. Introduction

Water-filling Principle

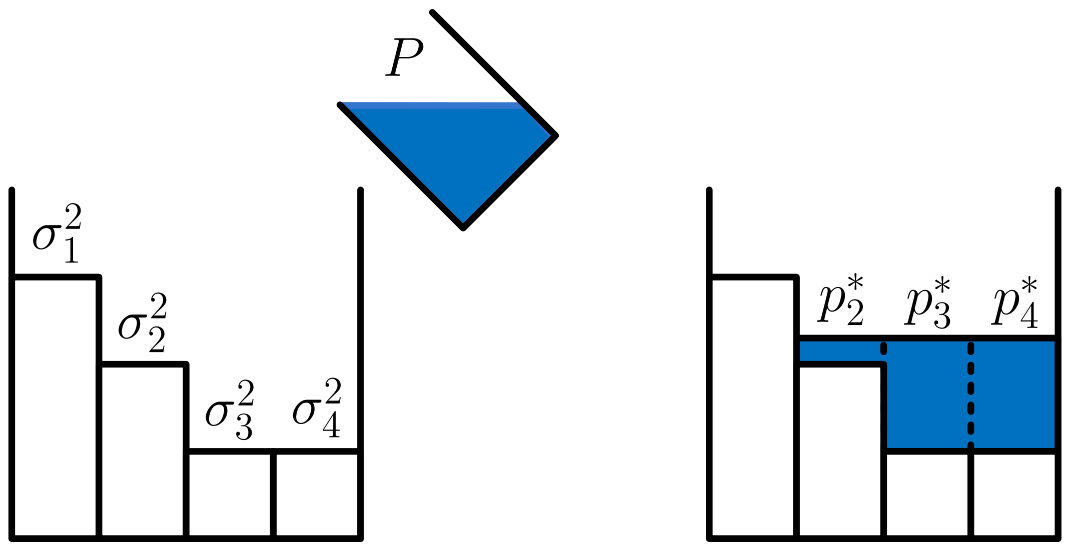
- N parallel additive noise channels:

$$Y_i = \sqrt{p_i} X_i + \sigma_i Z_i \quad \text{for } 1 \le i \le N.$$

  - X_i is channel input with unit power ($E[|X_i|^2] = 1$) for $1 \le i \le N$.
  - Z_i is white Gaussian noise with unit variance for $1 \le i \le N$.
  - Noises are independent from channel to channel.
  - There is a total power constraint $\sum_{i=1}^{N} p_i \le P$.



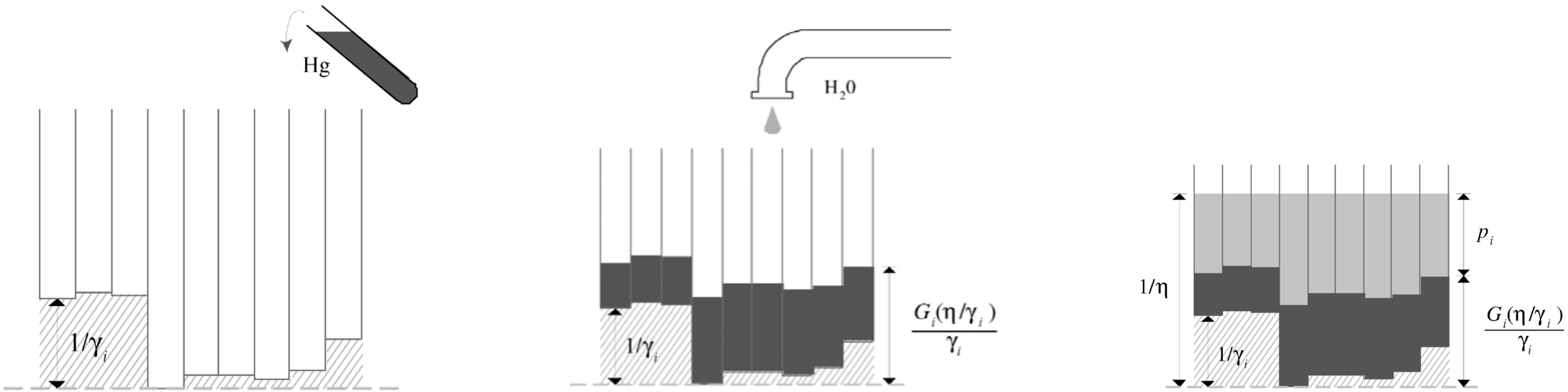
- The power allocation to achieve the capacity under a total power constraint is water-filling scheme.



  - The channel inputs $\{X_i\}_{i=1}^{N}$ should be chosen as i.i.d. Gaussian variables.

Beyond Water-filling

- For parallel Gaussian channels with arbitrary (given) inputs, e.g., BPSK, QPSK, 16-QAM,
  - the optimal power allocation can be interpreted by mercury/water-filling scheme.



$$\frac{1}{\gamma_i} = \sigma_i^2$$

$$G_i(\zeta) = \begin{cases} \frac{1}{\zeta} - \text{MMSE}_i^{-1}(\zeta) & 0 \le \zeta \le 1 \\ 1 & \zeta > 1. \end{cases}$$

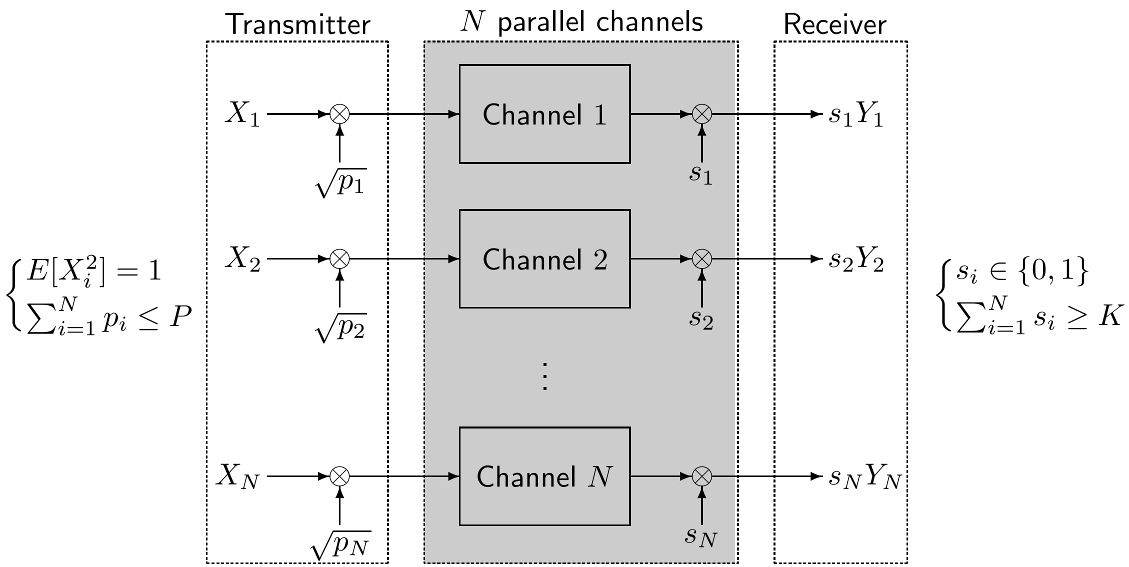
$$\boxed{\frac{\partial \text{Mul\_Info(SNR)}}{\partial \text{SNR}} = \text{MMSE(SNR)}}$$

- According to experimental measurements, the ambient noise may be non-Gaussian distributed in
  - underwater systems
  - power line channels
  - digital subscriber lines
  - ...
- So, the question is
  - what is the optimal power allocation over general channels (possibly, with fading, non-Gaussian, etc.).
- It is also interested to think
  - what is the optimal power allocation over general channels with "unknown channel states"?

2. (N, K)-Limited Access Channel and Its Capacity Formula

(N, K)-Limited Access Channel

- A system consists of N parallel general channels, and no less than K channel outputs are guaranteed to be received.
  - There is no *a priori* information about which outputs will be blocked, and no knowledge of the statistics of these blockage is provided.
  - The accessible channels remain unchanged within a codeword transmission, but may vary in different codeword blocks.



Capacity formula of (N, K)-Limited Access Channel

- The (N, K)-limited access channel is a kind of compound channel.
  - Example: $(N, K) = (3, 2)$, $[s_1, s_2, s_3]$ is one of

$$\Big\{[1,1,0],\ [1,0,1],\ [0,1,1],\ [1,1,1]\Big\}.$$

  - The capacity for given $\boldsymbol{X} \triangleq [X_1, X_2, \ldots, X_N]$ is

$$\max_{\{\boldsymbol{p}\in\Re_+^N:\sum_{i=1}^N p_i\le P\}} \ \min_{\{\boldsymbol{s}\in\{0,1\}^N:\sum_{i=1}^N s_i\ge K\}} I(\sqrt{\boldsymbol{p}}\circ\boldsymbol{X};\boldsymbol{s}\circ\boldsymbol{Y})$$

    where $I(\cdot;\cdot)$ is the mutual information function, $\sqrt{\boldsymbol{p}} \triangleq [\sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_N}]$, $\boldsymbol{Y} \triangleq [Y_1, Y_2, \ldots, Y_N]$, $\boldsymbol{s} \triangleq [s_1, s_2, \ldots, s_N]$, and "$\circ$" represents component-wise product between two vectors.

- A well-known fact is that for discrete memoryless compound channels, the capacity is not increased if the receiver knows the channel states.

Assumption 1 (Independent Channels)

$$\Pr(\boldsymbol{Y} | \sqrt{\boldsymbol{p}} \circ \boldsymbol{X}) = \prod_{i=1}^{N} \Pr(Y_i | \sqrt{p_i} X_i)$$

- The mutual information can then be simplified to

$$I(\sqrt{\boldsymbol{p}} \circ \boldsymbol{X}; \boldsymbol{s} \circ \boldsymbol{Y}) = \sum_{i=1}^{N} I(\sqrt{p_i} X_i; s_i Y_i) = \sum_{i=1}^{N} s_i \cdot I(\sqrt{p_i} X_i; Y_i)$$

if channel inputs $\{X_i\}_{i=1}^{N}$ are independent.

- For convenience, the mutual information $I(\sqrt{p_i} X_i; Y_i)$ is denoted by $f_i(p_i)$.
- The capacity formula can then be rewritten as

$$\max_{\{\boldsymbol{p} \in \Re_+^N : \sum_{i=1}^{N} p_i \le P\}} \min_{\{\boldsymbol{s} \in \{0,1\}^N : \sum_{i=1}^{N} s_i \ge K\}} \sum_{i=1}^{N} s_i \cdot f_i(p_i).$$

Assumption 2 (Monotonic Mutual Information)
For $1 \leq i \leq N$, $f_i(p)$ is continuous and strictly increasing for $p \geq 0$, and its first derivative (i.e., $f_i'(p)$) exists and is continuous and strictly decreasing in $p \geq 0$.

- There are quite a few practical channels satisfying this assumption, e.g.,
  - scalar AWGN channels with arbitrary inputs;
  - Gaussian fading channels with given inputs;
  - binary-input AWGN channel with hard decision at receiver side;
  - quaternary-input Laplace noise channels.
- Based on the assumption, the capacity formula can then be rewritten as

$$\max_{\{\boldsymbol{p}\in\Re_+^N : \sum_{i=1}^N p_i = P\}} \min_{\{\boldsymbol{s}\in\{0,1\}^N : \sum_{i=1}^N s_i = K\}} \sum_{i=1}^N s_i \cdot f_i(p_i).$$

- Our goal is to find the optimal power allocation $\boldsymbol{p}^* \triangleq [p_1^*, p_2^*, \ldots, p_N^*]$ to achieve the capacity.
  - We shall show $\boldsymbol{p}^*$ can be obtained algorithmically with less than K trials.

3. The Algorithmic Solution of Optimal Power Allocation

The Analysis of the Optimal Power Allocation

Lemma 1 (Necessary Condition for the Optimal Power Allocation)
The optimal power allocation $\boldsymbol{p}^*$ *for an* (N, K)*-limited access channel satifies*

$$f_{a_1}(p^*_{a_1}) \leq f_{a_2}(p^*_{a_2}) \leq \cdots \leq f_{a_K}(p^*_{a_K}) = f_{a_{K+1}}(p^*_{a_{K+1}}) = \cdots = f_{a_N}(p^*_{a_N}).$$

for some permutation $a_1, a_2, \ldots, a_N$ *of sequence* $1, 2, \ldots, N$.

Idea of Proof:

- There exists a permutation $a_1, a_2, \ldots, a_N$ of sequence $1, 2, \ldots, N$ such that

$$f_{a_1}(p^*_{a_1}) \le f_{a_2}(p^*_{a_2}) \le \cdots \le f_{a_K}(p^*_{a_K}) \le f_{a_{K+1}}(p^*_{a_{K+1}}) \le \cdots \le f_{a_N}(p^*_{a_N}).$$

- $\sum_{i=1}^{K} f_{a_i}(p^*_{a_i})$ is the maximal mutual information of the (N, K)-limited access channel because
  - $\boldsymbol{p}^* = [p^*_1, p^*_2, \ldots, p^*_N]$ is the optimal power allocation.
  - $\sum_{i=1}^{K} f_{a_i}(p^*_{a_i})$ is the smallest value among any K selections of $\{f_{a_i}(p^*_{a_i})\}_{i=1}^{N}$.
- If $f_{a_K}(p^*_{a_K}) < f_{a_j}(p^*_{a_j})$ for some $a_j \in \{a_{K+1}, a_{K+2}, \ldots, a_N\}$, we can find another power allocation $\hat{\boldsymbol{p}} = [\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_N]$ such that

$$\min_{\{\boldsymbol{s}\in\{0,1\}^N : \sum_{i=1}^{N} s_i = K\}} \sum_{i=1}^{N} s_i \cdot f_i(\hat{p}_i) > \sum_{i=1}^{K} f_{a_i}(p^*_{a_i}).$$

- According to Lemma 1, i.e.,

$$f_{a_1}(p^*_{a_1}) \le f_{a_2}(p^*_{a_2}) \le \cdots \le f_{a_K}(p^*_{a_K}) = f_{a_{K+1}}(p^*_{a_{K+1}}) = \cdots = f_{a_N}(p^*_{a_N})$$

we can obtain

$$\max_{1\le i\le \ell-1} f_{a_i}(p^*_{a_i}) < f_{a_\ell}(p^*_{a_\ell}) = f_{a_{\ell+1}}(p^*_{a_{\ell+1}}) = \cdots = f_{a_N}(p^*_{a_N}) \qquad (1)$$

is valid for exactly one value of ℓ in $\{1, 2, \ldots, K\}$.

Definition 1 (Set of Maximal Mutual Information)

$$\mathbb{A} \triangleq \{a_\ell, a_{\ell+1}, \ldots, a_N\}.$$

- When $\mathbb{A} = \{a_\ell, a_{\ell+1}, \ldots, a_N\}$ is identified in advance, the max-min power allocation problem is simplified to

$$\max_{\boldsymbol{p} \in \mathcal{P}(\mathbb{A})} \left\{ \sum_{i \notin \mathbb{A}} f_i(p_i) + (K + |\mathbb{A}| - N) \max_{1 \le j \le N} f_j(p_j) \right\} \quad (2)$$

where

$$\mathcal{P}(\mathbb{A}) \triangleq \left\{ \boldsymbol{p} \in \Re_+^N : \begin{array}{c} (i)\ \ \sum_{i=1}^N p_i = P \\ (ii)\ \ f_i(p_i) < \max_{1 \le j \le N} f_j(p_j) \text{ for } i \notin \mathbb{A} \\ (iii) f_i(p_i) = \max_{1 \le j \le N} f_j(p_j) \text{ for } i \in \mathbb{A} \end{array} \right\}.$$

- Although the max-min problem is simplified to maximization problem, there are still too many constraints in $\mathcal{P}(\mathbb{A})$.

Definition 2 (Aggregate Mutual Information Function) *The aggregate mutual information function associated with mutual information functions $\{f_i\}_{i\in\mathbb{B}}$ is defined through its inverse function as follows:*

$$F_{\mathbb{B}}^{(\mathtt{inv})}(y) \triangleq \sum_{i\in\mathbb{B}} f_i^{(\mathtt{inv})}(y) \quad \text{for } y \geq 0.$$

where $f^{(\mathtt{inv})}$ denotes the inverse function of f.

- Example: If $f_1(p_1) = f_2(p_2) = f_3(p_3) = y$ and $\mathbb{B} = \{1,2,3\}$, then
  - $F_{\mathbb{B}}(p_1 + p_2 + p_3) = y$;
  - $p_i = f_i^{(\mathtt{inv})}(y) = f_i^{(\mathtt{inv})}(F_{\mathbb{B}}(p_1 + p_2 + p_3))$ for $1 \leq i \leq 3$.
- Example: If $f_i(p) = \log\left(1 + \frac{p}{\sigma_i^2}\right)$ for $1 \leq i \leq 3$ and $\mathbb{B} = \{1,2,3\}$, then $F_{\mathbb{B}}(p) = \log\left(1 + \frac{p}{\sigma_1^2+\sigma_2^2+\sigma_3^2}\right)$.

- Based on the aggregation function with some fundamental analysis effort, the power allocation problem shown in (2) can be further simplified to

$$\max_{\boldsymbol{q}\in\tilde{\mathcal{Q}}(\mathbb{A})} \left\{ \sum_{i\notin\mathbb{A}} f_i(q_i) + (K + |\mathbb{A}| - N) F_{\mathbb{A}}(q_{\mathbb{A}}) \right\} \tag{3}$$

where

$$\tilde{\mathcal{Q}}(\mathbb{A}) \triangleq \left\{ \boldsymbol{q} = (\text{list of } q_i\ \forall i \notin \mathbb{A},\ q_{\mathbb{A}}) \in \Re_+^{N-|\mathbb{A}|+1} : \sum_{i\notin\mathbb{A}} q_i + q_{\mathbb{A}} = P \right\}.$$

- So far, we have realized that if $\mathbb{A}$ is obtained, then the max-min problem can be simplified to a maximization problem with single power-sum constraint.
  - The relation between $\boldsymbol{p}^*$ and $\boldsymbol{q}^*$, the maximizer of (3), is as follows:

$$p_i^* = \begin{cases} q_i^* & \text{for } i \notin \mathbb{A} \\ f_i^{(\mathtt{inv})}(F_{\mathbb{A}}(q_{\mathbb{A}}^*)) & \text{for } i \in \mathbb{A}. \end{cases}$$

- Can we know $\mathbb{A}$ in advance without knowing $\boldsymbol{p}^*$? No (in general)!
- There are $\sum_{\ell=1}^{K}\binom{N}{\ell-1}$ possible $\mathbb{A}=\{a_\ell, a_{\ell+1}, \ldots, a_K\}$ since $\ell \in \{1, 2, \ldots, K\}$.
  - We can solve the maximization problems in the form of (3) for each possible choice of $\mathbb{A}$ and select the best power allocation. (Too complicated!)
- We found that the number of required maximization problems in the form of (3) can be reduced to only $(N-|\mathbb{A}|+1)$.

(N, K)	$\sum_{\ell=1}^{K}\binom{N}{\ell-1}$	$(N-\lvert\mathbb{A}\rvert+1)=\ell$
$(8, 5)$	163	$1 \sim 5$
$(64, 30)$	$\approx 4.9097 \times 10^{18}$	$1 \sim 30$

Algorithm to Find the Optimal Power Allocation

Step 1. Initialize $m = 1$ and $\mathbb{B}_1 = \{1, 2, \ldots, N\}$.

Step 2. Obtain the maximizer $\tilde{\boldsymbol{q}}^{(m)} = \left[\tilde{q}_i^{(m)} \text{ for } i \notin \mathbb{B}_m, \quad \tilde{q}_{\mathbb{B}_m}^{(m)}\right]$:

$$\max_{\boldsymbol{q} \in \tilde{\mathcal{Q}}(\mathbb{B}_m)} \left\{ \sum\nolimits_{i \notin \mathbb{B}_m} f_i(q_i) + (K - m + 1) F_{\mathbb{B}_m}(q_{\mathbb{B}_m}) \right\}$$

where

$$\tilde{\mathcal{Q}}(\mathbb{B}_m) \triangleq \left\{ \boldsymbol{q} \in \Re_+^{N - |\mathbb{B}_m| + 1} : \sum\nolimits_{i \notin \mathbb{B}_m} q_i + q_{\mathbb{B}_m} = P \right\}.$$

Step 3. Calculate $\tilde{\boldsymbol{p}}^{(m)} = \left[\tilde{p}_1^{(m)}, \ldots, \tilde{p}_N^{(m)}\right]$ through

$$\tilde{p}_i^{(m)} = \begin{cases} \tilde{q}_i^{(m)} & \text{for } i \notin \mathbb{B}_m; \\ f_i^{(\mathtt{inv})}\left(F_{\mathbb{B}_m}\left(\tilde{q}_{\mathbb{B}_m}^{(m)}\right)\right) & \text{for } i \in \mathbb{B}_m. \end{cases}$$

Step 4. Assign $\mathbb{B}_{m+1} = \mathbb{B}_m \setminus \{j_m\}$, where

$$j_m = \arg\min_{i \in \mathbb{B}_m} f_i'\left(\tilde{p}_i^{(m)}\right).$$

Step 5. If

Stop Criterion: $$(K-m)F'_{\mathbb{B}_{m+1}}\left(\tilde{q}_{\mathbb{B}_m}^{(m)} - \tilde{p}_{j_m}^{(m)}\right) \leq f'_{j_m}\left(\tilde{p}_{j_m}^{(m)}\right), \quad (4)$$

then set $\mathbb{A} = \mathbb{B}_m$ and $\boldsymbol{p}^* = \tilde{\boldsymbol{p}}^{(m)}$ and stop the algorithm;

else set $m = m + 1$ and go to Step 2.

Key of the Proof:

- $\mathbb{A} = \mathbb{B}_{N-|\mathbb{A}|+1} \subset \mathbb{B}_{N-|\mathbb{A}|} \subset \cdots \subset \mathbb{B}_1$
- For $1 \leq m < N - |\mathbb{A}| + 1$,

$$(K-m)F'_{\mathbb{B}_{m+1}}\left(\tilde{q}_{\mathbb{B}_m}^{(m)} - \tilde{p}_{j_m}^{(m)}\right) > f'_{j_m}\left(\tilde{p}_{j_m}^{(m)}\right),$$

and when $m = N - |\mathbb{A}| + 1$,

$$(K-m)F'_{\mathbb{B}_{m+1}}\left(\tilde{q}_{\mathbb{B}_m}^{(m)} - \tilde{p}_{j_m}^{(m)}\right) \leq f'_{j_m}\left(\tilde{p}_{j_m}^{(m)}\right).$$

□

Note

- The algorithm will definitely stop when m reaches K.

4. Optimal Power Allocation Over Additive Noise Channels

Additive noises of the same family

$$Y_i = \sqrt{p_i} X_i + \sigma_i Z_i \quad \text{for } 1 \le i \le N$$

where $\{X_i\}_{i=1}^N$ and $\{Z_i\}_{i=1}^N$ are both i.i.d. complex random variables with unit second moments, and they are independent of each other.

- Without loss of generality, $\sigma_1^2 \ge \sigma_2^2 \ge \ldots \ge \sigma_N^2$ is assumed.
- If both X_i and Z_i are discrete, then Assumption 2 may not hold.
  - For example, consider $X_i, Z_i \in \{\pm 1\} \Rightarrow f_i(p_i) = H(X_i)$ for every $p_i \ne \sigma_i$.
- Regarding the validity of Assumption 2, we only consider Z_i is a continuous random variable.

- In such channels,

$$f_i(p_i) = g\left(\frac{p_i}{\sigma_i^2}\right) \quad \text{for } 1 \le i \le N$$

  where

$$g(\rho) \triangleq I(\sqrt{\rho}X_i; \sqrt{\rho_i}X_i + Z_i)$$

- Since the mutual information function is only governed by a single function g (rather than N functions $\{f_i\}_{i=1}^N$), finding $\mathbb{A}$ (by our algorithm) can be further simplified to a water-filling procedure.
- The optimal power allocation thus can be obtained by solving (3).
- Together with the procedure for the identification of $\mathbb{A}$, the optimal power allocation can then be interpreted by a "two-phase water-filling" scheme.

Step 4. Assign $\mathbb{B}_{m+1} = \mathbb{B}_m \setminus \{j_m\}$, where

$$j_m = \arg\min_{i \in \mathbb{B}_m} f_i'\left(\tilde{p}_i^{(m)}\right) = \boxed{\arg\max_{i \in \mathbb{B}_m} \sigma_i^2}.$$

Step 5. If

$$(K-m)F'_{\mathbb{B}_{m+1}}\left(\tilde{q}_{\mathbb{B}_m}^{(m)} - \tilde{p}_{j_m}^{(m)}\right) \le f'_{j_m}\left(\tilde{p}_{j_m}^{(m)}\right)$$

$$\Rightarrow \boxed{(K-m)\sigma_m^2 \le \sum_{i=m+1}^{N} \sigma_i^2}$$

then set $\mathbb{A} = \mathbb{B}_m$ and $\boldsymbol{p}^* = \tilde{\boldsymbol{p}}^{(m)}$ and stop the algorithm;

else set $m = m + 1$ and go to Step 2.

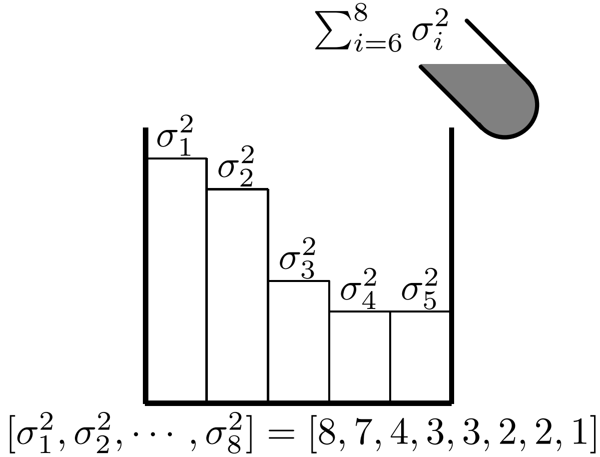
- For additive noise of the same family (recall that $\sigma_1^2 \geq \sigma_2^2 \geq \ldots \geq \sigma_N^2$), we can obtain that

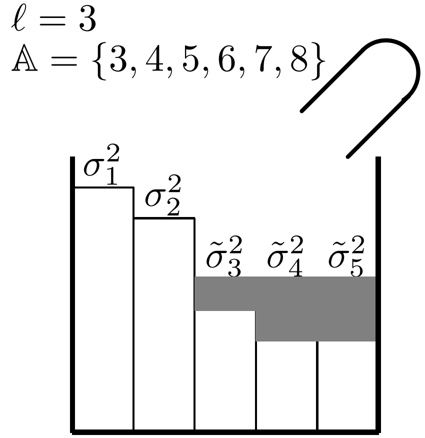
$$\mathbb{A} = \{\ell, \ell+1, \cdots, N\}$$

where

$$\ell \triangleq \min\left\{ i \in \{1, 2, \cdots, K\} \middle| \sigma_i^2 \leq \tilde{\sigma}_K^2 \text{ for every } 1 \leq i \leq K \right\}$$

and $\tilde{\sigma}_i^2 \triangleq \sigma_i^2 + [\lambda - \sigma_i^2]^+$ with λ chosen to satisfy $\sum_{i=1}^{K}[\lambda - \sigma_i^2]^+ = \sum_{i=K+1}^{N} \sigma_i^2$, and $[y]^+ \triangleq \max\{0, y\}$.





- After obtaining $\mathbb{A}$, we can obtain the optimal power allocation as follows (by solving (3)):

$$p_i^* = \begin{cases} q_i^* & \text{for } \text{for } i \notin \mathbb{A} \\ f_i^{(\mathtt{inv})}\left(F_{\mathbb{A}}(q_{\mathbb{A}}^*)\right) & \text{for } i \in \mathbb{A} \end{cases}$$

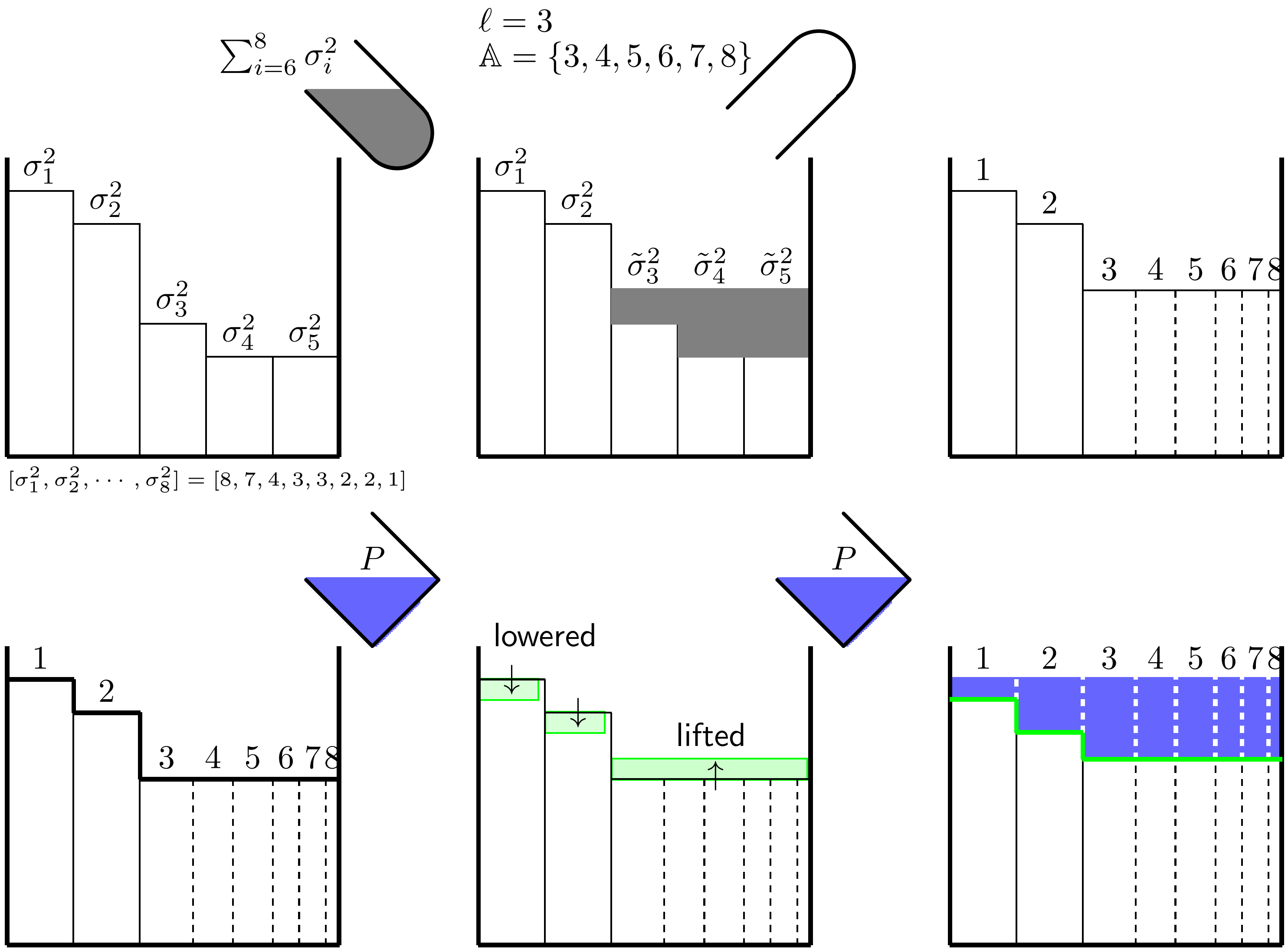
$$= \begin{cases} q_i^* & \text{for } 1 \le i < \ell \\ \sigma_i^2 \cdot g^{(\mathtt{inv})}\left(g\left(\frac{q_{\mathbb{A}}^*}{\sum_{i\in\mathbb{A}} \sigma_i^2}\right)\right) & \text{for } \ell \le i \le N \end{cases}$$

$$= \begin{cases} q_i^* & \text{for } 1 \le i < \ell \\ \frac{\sigma_i^2}{\sum_{i\in\mathbb{A}} \sigma_i^2} \cdot q_{\mathbb{A}}^* & \text{for } \ell \le i \le N. \end{cases}$$

where $\boldsymbol{q}^* \triangleq (q_1^*, q_2^*, \ldots, q_{\ell-1}^*, q_{\mathbb{A}}^*)$ is the maximizer of

$$\max_{\{\boldsymbol{q}\in\Re_+^{\ell}:\sum_{i=1}^{\ell-1} q_i + q_{\mathbb{A}} = P\}} \left\{ \sum_{i=1}^{\ell-1} g\left(\frac{q_i}{\sigma_i^2}\right) + (K-\ell+1) g\left(\frac{q_{\mathbb{A}}}{\sum_{i=\ell}^{N} \sigma_i^2}\right) \right\}.$$

Two-Phase Water-filling Scheme



Theorem 1 (Two-Phase Water Filling Scheme)

- *Noise-Power Re-distribution Phase*
- *Signal-Power Allocation Phase*

Remarks

- The adjustment of the base heights in the *Signal-Power Allocation Phase* follows

$$L_i(\nu) \triangleq \begin{cases} \sigma_i^2 \cdot G(\nu\sigma_i^2) & \text{for } 1 \leq i < \ell; \\ \tilde{\sigma}_K^2 \cdot G(\nu\tilde{\sigma}_K^2) & \text{for } \ell \leq i \leq N, \end{cases}$$

where $1/\nu$ is the water level, and

$$G(\zeta) \triangleq \begin{cases} \frac{1}{\zeta} - g'^{(\mathtt{inv})}(\zeta) & \text{if } 0 < \zeta < g'(0); \\ \frac{1}{g'(0)} & \text{if } \zeta \geq g'(0). \end{cases}$$

Remarks (Continued.)

- If $\{Z_i\}$ Gaussian, then $L_i(\nu) \geq \tilde{\sigma}_i^2$.
  - Hence, one can visualize the lifting of the vessel base as mercury pouring, which is so-named mercury/water-filling for (N, N)-limited access channels with additive Gaussian noises.
- However, the vessel base may be lowered for noises other than Gaussian!

Example (4-ary Input Laplacian Additive Noise Channel) The assumption of X_i uniform in $\{\frac{\pm 1 \pm \imath}{\sqrt{2}}\}$ and $Z_i \sim e^{-2(|\text{Re}(z)|+\text{Im}(z)|)}$ for $z \in \mathbb{C}$ implies

$$g'(\rho) = \sqrt{\frac{2}{\rho}} \cdot e^{-\sqrt{2\rho}} \cdot \text{Gd}\left(\sqrt{2\rho}\right) \quad \text{for } \rho > 0$$

where $\text{Gd}(x) \triangleq 2\tan^{-1}(e^x) - \frac{\pi}{2}$ is the Gudermannian function.

Example (Continued) Case 1: $(N, K) = (4, 4)$.

Assume $P = 1.5$ and $[\sigma_1^2,\ \sigma_2^2,\ \sigma_3^2,\ \sigma_4^2] = [1.2,\ 1.0,\ 0.4,\ 0.1]$.

Since $N = K = 4$, we get $\tilde{\sigma}_i^2 = \sigma_i^2$ for $1 \le i \le 4$.

Hence, $\ell = 4$ and $\mathbb{A} = \{4\}$.

As a result,

$$L_i(\nu) = \begin{cases} 1.0081 \text{ for } i = 1 & (< \tilde{\sigma}_1^2 = 1.2;\ \text{i.e., lowered}) \\ 0.9411 \text{ for } i = 2 & (< \tilde{\sigma}_2^2 = 1.0;\ \text{i.e., lowered}) \\ 0.8071 \text{ for } i = 3 & (> \tilde{\sigma}_3^2 = 0.4;\ \text{i.e., lifted}) \\ 0.9521 \text{ for } i = 4 & (> \tilde{\sigma}_4^2 = 0.1;\ \text{i.e., lifted}) \end{cases}$$

and

$$\boldsymbol{p}^* = [p_1^*,\ p_2^*,\ p_3^*,\ p_4^*] = [\underbrace{0.294}_{\#4},\ \underbrace{0.361}_{\#2},\ \underbrace{0.495}_{\#1},\ \underbrace{0.350}_{\#3}].$$

Example (Continued) Case 2: $(N, K) = (4, 3)$.

Assume $P = 1$ and $[\sigma_1^2,\ \sigma_2^2,\ \sigma_3^2,\ \sigma_4^2] = [1.2,\ 1.0,\ 0.4,\ 0.1]$.

We get $[\tilde{\sigma}_1^2,\ \tilde{\sigma}_2^2,\ \tilde{\sigma}_3^2] = [1.2,\ 1.0,\ 0.5]$.

Hence, $\ell = 3$ and $\mathbb{A} = \{3, 4\}$.

As a result,

$$L_i(\nu) = \begin{cases} 0.9643 \text{ for } i = 1 & (< \tilde{\sigma}_1^2 = 1.2; \text{i.e., lowered}) \\ 0.8969 \text{ for } i = 2 & (< \tilde{\sigma}_2^2 = 1.0; \text{i.e., lowered}) \\ 0.7586 \text{ for } i = 3 & (> \tilde{\sigma}_3^2 = 0.5; \text{i.e., lifted}) \\ 0.7586 \text{ for } i = 4 & (> \tilde{\sigma}_3^2 = 0.5; \text{i.e., lifted}) \end{cases}$$

and

$$\boldsymbol{p}^* = [p_1^*,\ p_2^*,\ p_3^*,\ p_4^*] = [\underbrace{0.2423}_{\#3},\ \underbrace{0.3097}_{\#2},\ \underbrace{0.3584}_{\#1},\ \underbrace{0.0896}_{\#4}].$$

Remarks (Continued.)

- when $\ell = 1$ (equivalently, $\mathbb{A} = \{1, 2, \ldots, N\}$, $\boldsymbol{p}^*$ can be determined without any maximization labor, and the optimal power allocation follows equal SNR principle as

$$\frac{p_i^*}{\sigma_i^2} = \frac{P}{\sum_{j=1}^{N} \sigma_j^2} \quad \text{for } 1 \leq i \leq N.$$

5. Implication from the Optimal Power Allocation

Implication from the Optimal Power Allocation

- Our algorithm implies that the sequence $j_1, j_2, j_3, \ldots$ is a sorting sequence of the channels in decending degree of "noisiness."

> Step 4. Assign $\mathbb{B}_{m+1} = \mathbb{B}_m \setminus \{j_m\}$, where
>
> $$j_m = \arg\min_{i \in \mathbb{B}_m} f_i'\left(\tilde{p}_i^{(m)}\right)$$

This coincides with the implication from additive noise channels:

$$\sigma_{j_1}^2 \geq \sigma_{j_2}^2 \geq \sigma_{j_3}^2 \cdots$$

For general channels, how to compare their degree of "noisiness" by their mutual information functions?

Theorem 2 *For a general (N, K)-limited access channel, if for $y \geq 0$,*

$$f'_{k_1} f_{k_1}^{(\texttt{inv})}(y) \leq f'_{k_2} f_{k_2}^{(\texttt{inv})}(y) \leq \cdots \leq f'_{k_N} f_{k_N}^{(\texttt{inv})}(y),$$

then $j_M = k_M$ for $M = 1, 2, 3, \ldots$.

Example. For additive additive noise of the same family,

$$f'_i f_i^{(\texttt{inv})}(y) = \frac{1}{\sigma_i^2} \cdot g' g^{(\texttt{inv})}(y).$$

- $j_1, j_2, \ldots, j_N$ is in general a function of P except for some special cases, e.g.,

$$f'_{k_1} f_{k_1}^{(\mathtt{inv})}(y) \leq \cdots \leq f'_{k_N} f_{k_N}^{(\mathtt{inv})}(y) \text{ for all } y \geq 0.$$

- Instead, we may concern whether they can be determined and fixed in the $\underbrace{\text{low-power}}_{\text{i.e., } P<\delta}$ and $\underbrace{\text{high-power}}_{\text{i.e., } P>\Delta}$ regimes.

Lemma 2 (General Degree of Noisiness Condition)

- *If for every* $1 \le k_i < k_j \le N$,

$$\limsup_{y \to 0} \operatorname{sgn}\left(f'_{k_i} f^{(\mathtt{inv})}_{k_i}(y) - f'_{k_j} f^{(\mathtt{inv})}_{k_j}(y)\right) \le 0,$$

then $j_i = k_i$ *in the low-power regime.*

- *If for every* $1 \le k_i < k_j \le N$,

$$\limsup_{y \to \min\{\omega_{k_i}, \omega_{k_j}\}} \operatorname{sgn}\left(f'_{k_i} f^{(\mathtt{inv})}_{k_i}(y) - f'_{k_j} f^{(\mathtt{inv})}_{k_j}(y)\right) \le 0.$$

then $j_i = k_i$ *in the high-power regime,*

provided $\lim_{p \to \infty} f'_i(p) = 0$ *for* $1 \le i \le N$.

1. $\operatorname{sgn}(\rho)$ equals either 1, 0 or -1 depending on whether $\rho > 0$, $= 0$ or < 0.
2. $\omega_i \triangleq \lim_{p \to \infty} f_i(p)$

Corollary 1 (Simplified Degree of Noisiness Condition)

- *If for every* $1 \le k_i < k_j \le N$,

$$\begin{cases} \text{either} & f'_{k_i}(0) < f'_{k_j}(0) \\ \text{or} & f'_{k_i}(0) = f'_{k_j}(0) \text{ and } f''_{k_i}(0) < f''_{k_j}(0) \\ \text{or} & (\exists\, \delta > 0)\ f'_{k_i}(p) \le f'_{k_j}(p) \text{ for } 0 < p < \delta \end{cases}$$

then $j_i = k_i$ *in the low-power regime.*

- *If for every* $1 \le k_i < k_j \le N$,

$$\omega_{k_i} = \lim_{p\to\infty} f_{k_i}(p) < \omega_{k_j} = \lim_{p\to\infty} f_{k_j}(p),$$

then $j_i = k_i$ *in the high-power regime,*

provided $\lim_{p\to\infty} f'_i(p) = 0$ *for* $1 \le i \le N$.

Example 1 (Fading Channels)

$$Y_i = (\beta_i H_i)(\sqrt{p_i} X_i) + \sigma_i Z_i \quad \text{for } 1 \le i \le N = 6$$

where $\{H_i\}_{i=1}^{6}$ and $\{Z_i\}_{i=1}^{6}$ unit-second-moment i.i.d. with the latter being Gaussian, and $\{X_i\}_{i=1}^{6}$ are respectively BPSK, QPSK, 16-QAM, Gaussian, Gaussian, Gaussian.

- Assume $\frac{\sigma_1^2}{\beta_1^2} = \frac{\sigma_2^2}{\beta_2^2} > \frac{\sigma_3^2}{\beta_3^2} = \frac{\sigma_4^2}{\beta_4^2} > \frac{\sigma_5^2}{\beta_5^2} \ge \frac{\sigma_6^2}{\beta_6^2}$.
- As a result, $[j_1, j_2, j_3, j_4, j_5, j_6] = [1, 2, 3, 4, 5, 6]$ in both low- and high-power regimes.
- Implication: For example, under equal effective noise power, QPSK modulations should be favored than BPSK modulations when the power budget is either extremely tight or extremely rich.

Lemma 3 (Conditions on Stop Criterion: Low-Power Regime) *In the low-power regime, for the already pre-determined $j_1, j_2, j_3, \ldots$, we have the following logical statements to help determining $\mathbb{A}$.*

- *If*

$$\limsup_{y \to 0} \operatorname{sgn}\left(K - M - \sum_{i \in \mathbb{B}_{M+1}} \frac{f'_{j_M} f_{j_M}^{(\mathtt{inv})}(y)}{f'_i f_i^{(\mathtt{inv})}(y)} \right) \le 0 \qquad (5)$$

  then stop criterion in Step 5 holds (i.e., (4) holds).

- *If*

$$\liminf_{y \to 0} \operatorname{sgn}\left(K - M - \sum_{i \in \mathbb{B}_{M+1}} \frac{f'_{j_M} f_{j_M}^{(\mathtt{inv})}(y)}{f'_i f_i^{(\mathtt{inv})}(y)} \right) > 0 \qquad (6)$$

  then stop criterion in Step 5 fails (i.e., (4) does not hold).

Lemma 4 (Conditions on Stop Criterion: High-Power Regime) *Assume* $\lim_{p\to\infty} f_i'(p) = 0$ *for* $1 \le i \le N$*. Then, in the high-power regime, for the already pre-determined* $j_1, j_2, \ldots, j_N$*, we have the following logical statements to help further determining* $\mathbb{A}$*.*

- *If*

$$\limsup_{y\to\Omega(\mathbb{B}_M)} \operatorname{sgn}\left(K - M - \sum_{i\in\mathbb{B}_{M+1}} \frac{f'_{j_M} f_{j_M}^{(\mathtt{inv})}(y)}{f'_i f_i^{(\mathtt{inv})}(y)}\right) \le 0 \tag{7}$$

  then stop criterion in Step 5 holds (i.e., (4) *holds).*

- *If*

$$\liminf_{y\to\Omega(\mathbb{B}_M)} \operatorname{sgn}\left(K - M - \sum_{i\in\mathbb{B}_{M+1}} \frac{f'_{j_M} f_{j_M}^{(\mathtt{inv})}(y)}{f'_i f_i^{(\mathtt{inv})}(y)}\right) > 0 \tag{8}$$

  then stop criterion in Step 5 fails (i.e., (4) *does not hold).*

1. $\Omega(\mathbb{B}_M) := \min_{i\in\mathbb{B}_M} \omega_i = \min_{i\in\mathbb{B}_M} \lim_{p\to\infty} f_i(p)$.

Corollary 2 (Simplified Stop Criterion Condition)

- (5) *is valid if* $K - M < \sum_{i \in \mathbb{B}_{M+1}} \frac{f'_{j_M}(0)}{f'_i(0)}$.
- (6) *is valid if* $K - M > \sum_{i \in \mathbb{B}_{M+1}} \frac{f'_{j_M}(0)}{f'_i(0)}$.
- (7) *is valid if* $K - M < \sum_{i \in \mathbb{B}_{M+1}} \liminf_{y \to \Omega(\mathbb{B}_M)} \frac{f'_{j_M} f_{j_M}^{(\mathtt{inv})}(y)}{f'_i f_i^{(\mathtt{inv})}(y)}$.
- (8) *is valid if* $K - M > \sum_{i \in \mathbb{B}_{M+1}} \limsup_{y \to \Omega(\mathbb{B}_M)} \frac{f'_{j_M} f_{j_M}^{(\mathtt{inv})}(y)}{f'_i f_i^{(\mathtt{inv})}(y)}$.

We assume $\lim_{p\to\infty} f'_i(p) = 0$ *for* $1 \le i \le N$ *for the last two.*

Example 2 (Continue...)

$$Y_i = (\beta H_i)(\sqrt{p_i} X_i) + \sigma_i Z_i \quad \textit{for } 1 \le i \le N = 6,$$

where $\{H_i\}_{i=1}^6$ *and* $\{Z_i\}_{i=1}^6$ *unit-second-moment i.i.d. with the latter being Gaussian, and* $\{X_i\}_{i=1}^6$ *are respectively BPSK, QPSK, 16-QAM, Gaussian, Gaussian, Gaussian.*

- Assume $\frac{\sigma_1^2}{\beta_1^2} = \frac{\sigma_2^2}{\beta_2^2} > \frac{\sigma_3^2}{\beta_3^2} = \frac{\sigma_4^2}{\beta_4^2} > \frac{\sigma_5^2}{\beta_5^2} \ge \frac{\sigma_6^2}{\beta_6^2}$.
- As a result, $[j_1, j_2, j_3, j_4, j_5, j_6] = [1, 2, 3, 4, 5, 6]$ in both low- and high-power regimes.
- When $K = 5$, $\mathbb{A} = \begin{cases} \{3,4,5,6\} & \text{in low-power regime} \\ \{4,5,6\} & \text{in high-power regime} \end{cases}$

Lemma 5 (Optimal Power Ratio in the Low-Power Regime)

After the determination of $\mathbb{A}$, the optimal power allocation q_i^* for the ith channel outside $\mathbb{A}$ asymptotically satisfies

$$\lim_{P \to 0} \frac{q_i^*}{q_{\mathbb{A}}^*} = \begin{cases} \dfrac{(K + |\mathbb{A}| - N) F''_{\mathbb{A}}(0)}{f''_i(0)} & \text{if } f'_i(0) = (K + |\mathbb{A}| - N) F'_{\mathbb{A}}(0) \\ 0 & \text{otherwise,} \end{cases}$$

provided the second derivative $f''_i(p)$ exists and

$$f''_i(0) \triangleq \lim_{p \to 0} f''_i(p).$$

In addition, for two channels with indices i, j in $\mathbb{A}$, the re-distribution of $q_{\mathbb{A}}^*$ yields that

$$\lim_{P \to 0} \frac{p_i^*}{p_j^*} = \frac{f'_j(0)}{f'_i(0)}.$$

Observation 1 (Optimal Power Ratio in the High-Power Regime)

1. *For channels outside $\mathbb{A}$, the following statements hold.*

   - *If $F'_{\mathbb{A}}(q)$ and each f'_i outside $\mathbb{A}$ vanish polynomially fast, i.e.,*

   $$\lim_{q\to\infty} q^{m_i} f'_i(q) = c_i \text{ for } i \notin \mathbb{A} \text{ and } \lim_{q\to\infty} q^{m_{\mathbb{A}}} F'_{\mathbb{A}}(q) = \frac{c_{\mathbb{A}}}{K - N + |\mathbb{A}|},$$

   where m_i, $m_{\mathbb{A}}$, c_i and $c_{\mathbb{A}}$ are all positive,

   or if $F'_{\mathbb{A}}(q)$ and each f'_i outside $\mathbb{A}$ vanish exponentially fast, i.e.,

   $$\lim_{q\to\infty} q^{m_i} \ln\left(f'_i(q)\right) = c_i \text{ for } i \notin \mathbb{A} \text{ and } \lim_{q\to\infty} q^{m_{\mathbb{A}}} \ln\left(F'_{\mathbb{A}}(q)\right) = c_{\mathbb{A}}$$

   where m_i, $m_{\mathbb{A}}$, c_i and $c_{\mathbb{A}}$ are all negative, then for $i, j \notin \mathbb{A}$,

   $$\lim_{P\to\infty} \frac{q_i^*}{q_j^*} = \begin{cases} 0 & \text{if } |m_i| > |m_j| \\ \left(\frac{c_i}{c_j}\right)^{1/m_i} & \text{if } |m_i| = |m_j| \\ \infty & \text{if } |m_i| < |m_j| \end{cases}$$

Observation 1 (Continue...)

and

$$\lim_{P\to\infty}\frac{q_i^*}{q_{\mathbb{A}}^*}=\begin{cases}0 & \text{if } |m_i|>|m_{\mathbb{A}}|\\ \left(\frac{c_i}{c_{\mathbb{A}}}\right)^{1/m_i} & \text{if } |m_i|=|m_{\mathbb{A}}|\\ \infty & \text{if } |m_i|<|m_{\mathbb{A}}|\end{cases}$$

- *If for* $i,j\notin\mathbb{A}$, f_i' *vanishes exponentially fast while* f_j' *and* $F_{\mathbb{A}}'$ *decay to zero at a polynomial speed, then*

$$\lim_{P\to\infty}\frac{q_i^*}{q_j^*}=\lim_{P\to\infty}\frac{q_i^*}{q_{\mathbb{A}}^*}=0.$$

2. *For channels* i,j *inside* $\mathbb{A}$,

$$\begin{cases}\lim_{P\to\infty}p_i^*=f_i^{(\mathtt{inv})}(\Omega(\mathbb{A}))<\infty & \text{if } \omega_i>\Omega(\mathbb{A})=\min_{k\in\mathbb{A}}\omega_k\\ \lim_{P\to\infty}\frac{p_i^*}{p_j^*}=\lim_{y\to\Omega(\mathbb{A})}\frac{f_j'f_j^{(\mathtt{inv})}(y)}{f_i'f_i^{(\mathtt{inv})}(y)} & \text{if } \omega_i=\omega_j=\Omega(\mathbb{A})\end{cases}$$

6. Concluding Remarks

- The optimal power allocation for (N, K)-limited access channels can be obtained algorithmically.
- For additive noises of the same family, the power allocation can be further simplified to two-phase water-filling scheme.
- General behaviors of the optimal power allocation in low- and high-power regimes are also established.
- The result in this work can be applied to the optimal power allocation associated with "profit" functions $\{f_i\}$ (which are not necessarily mutual information functions) as long as they are of the form

$$\max_{\left\{\boldsymbol{p}\in\Re_+^N:\sum_{i=1}^N p_i\le P\right\}} \min_{\left\{\boldsymbol{s}\in\{0,1\}^N:\sum_{i=1}^N s_i\ge K\right\}} \sum_{i=1}^N s_i \cdot f_i(p_i).$$

  I.e., maximize the sum of the worst K profits.

- One possible future work is to relax the independence assumption among channels to a dependence one.
  - A good start would be to investigate the additive color noise channel modeled by

$$Y_i = \sqrt{p_i} X_i + \sigma_i Z_i \quad \text{for } 1 \le i \le N$$

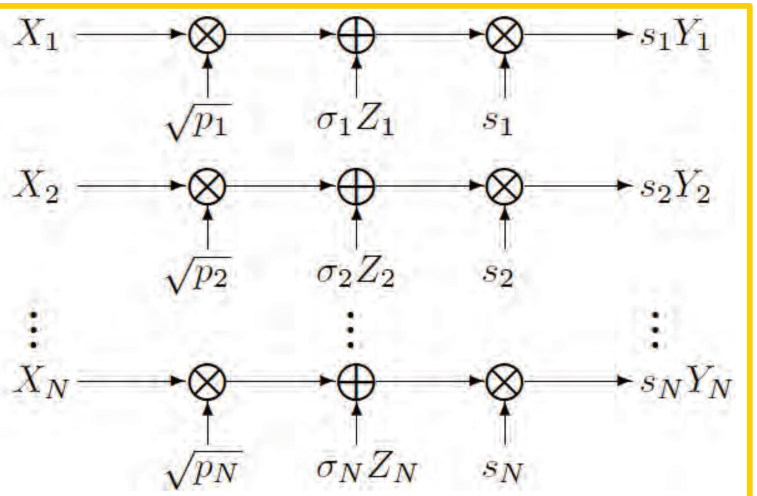
    where $\{Z_i\}_{i=1}^N$ are dependent random variables and $\sigma_1^2 \ge \sigma_2^2 \ge \ldots \ge \sigma_N^2$.

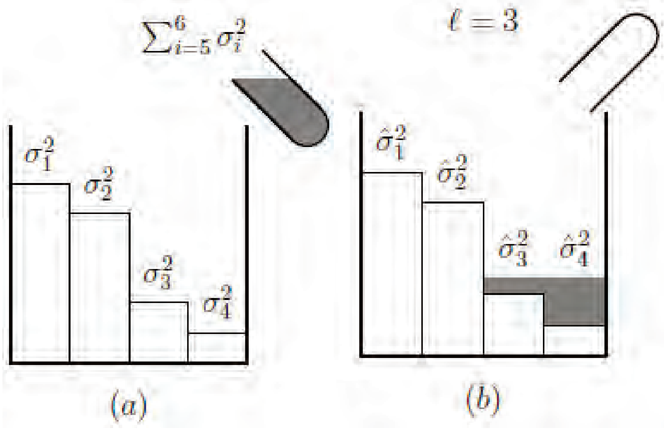
- According to our preliminary study, we found that if the system mutual information remains unchanged when simultaneously switching (p_i, σ_i^2) and (p_j, σ_j^2), also s_i and s_j, for every $1 \le i < j \le N$,
  - then the max-min problem can be transformed to an equivalent problem for K parallel channels without limited access constraint via a water-filling noise-power-redistribution process.

- Further investigation along this direction might be worthwhile.

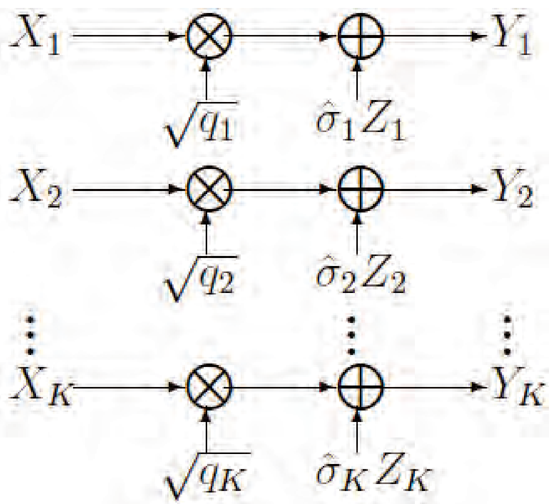
$$\begin{cases} E[X_i^2] = 1 \\ \sum_{i=1}^{N} p_i \le P \end{cases}$$

$$\begin{cases} s_i \in \{0,1\} \\ \sum_{i=1}^{N} s_i \ge K \end{cases}$$





$$\begin{cases} E[X_i^2] = 1 \\ \sum_{i=1}^{K} q_i \le P \end{cases}$$



$$p_i^* = \begin{cases} q_i^* & \text{for } 1 \le i < \ell \\ \frac{\sigma_i^2/\hat{\sigma}_K^2}{(K-\ell+1)} \sum_{j=\ell}^{K} q_j^* & \text{for } \ell \le i \le N. \end{cases}$$