Throughput-Oriented Power Allocation Policies for Parallel Gaussian Channels Under Finite-Length and Fixed-Rate Coding Constraints

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- Finding the best strategy for allocating power over parallel independent additive white Gaussian noise (AWGN) channels is a classical problem in information theory.
- The well-known **water-filling** scheme maximize the system's **capacity** when the input of each parallel channel is Gaussian distributed.
- In 2006, Lozano, Tulio and Verdú re-visited this problem by constraining the input to be drawn from **discrete modulation constellations** used in practice. They concluded their study as **mercury water-filling**.
  - Both the mutual information and the minimum mean-square error are functions of the signal-to-noise ratios (SNRs).
  - The derivative of the former measure with respect to the SNR is equal to the latter one.

- There is another more challenging power allocation problem over parallel AWGN channels where only the **total noise variance** is known.
- The worst case where for a given power allocation, noise variance for each channel is chosen such that the capacity is minimized (denoted by **worst-case capacity**) always considered. Power allocation is chosen such that the worst-case capacity is maximized.
- The resulting power allocation policy is to allot **equal signal power** to each channel.

- The **capacity-optimizing** power allocation, although theoretically interesting, is **not realistic** in several aspects.
  - Channel capacity is a function of the total system power, and the **optimal coding scheme** that achieves capacity may be **different** for different capacity values.
  - The optimal rate obtained from a capacity-optimizing power allocation is often a concretely **unrealizable real number**.
  - Capacity is an asymptotic quantity that requires the **coding blocklength** or frame size to **grow without bound**.

- We herein investigate power allocation policies that maximizes over the **effective throughput** and **worst-case effective throughput** for **convolutionally coded** parallel memoryless Gaussian channels with **finite-length** and **fixed-rate** coding constraints.
- Our study however shows that it is possible to obtain a good **approximating expression** for the **error rate** of each coded channel.
- The resulting throughput-oriented power allocation policies are a **variation** of the traditional **water-filling** principle, where the base width and height of each individual vessel (corresponding to each parallel channel) now become functions of the **code characteristics**.

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Capacity-Optimizing: Known Noise Variance in Each Channel

- The capacity of the system is

$$\sum_{i=1}^{K} \frac{1}{2} \log \left(1 + \frac{P_i}{\sigma_i^2} \right), \tag{1}$$

where P_i is chosen such that $\sum_{i=1}^{K} P_i = P_{\text{total}}$ is satisfied.

- Water-filling solution: $P_i = \left(\nu - \sigma_i^2 \right)^+$

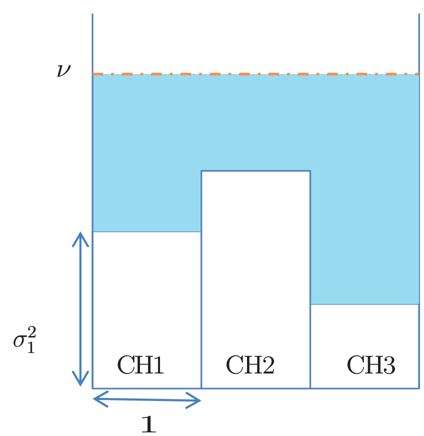

Figure 1: Example of water-filling power allocation for $K = 3$.

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System Model

- The effective throughput is the successfully transmitted information bits per channel use.
- Power constraint: $\sum_{i=1}^{K} P_i = P_{\text{total}}$

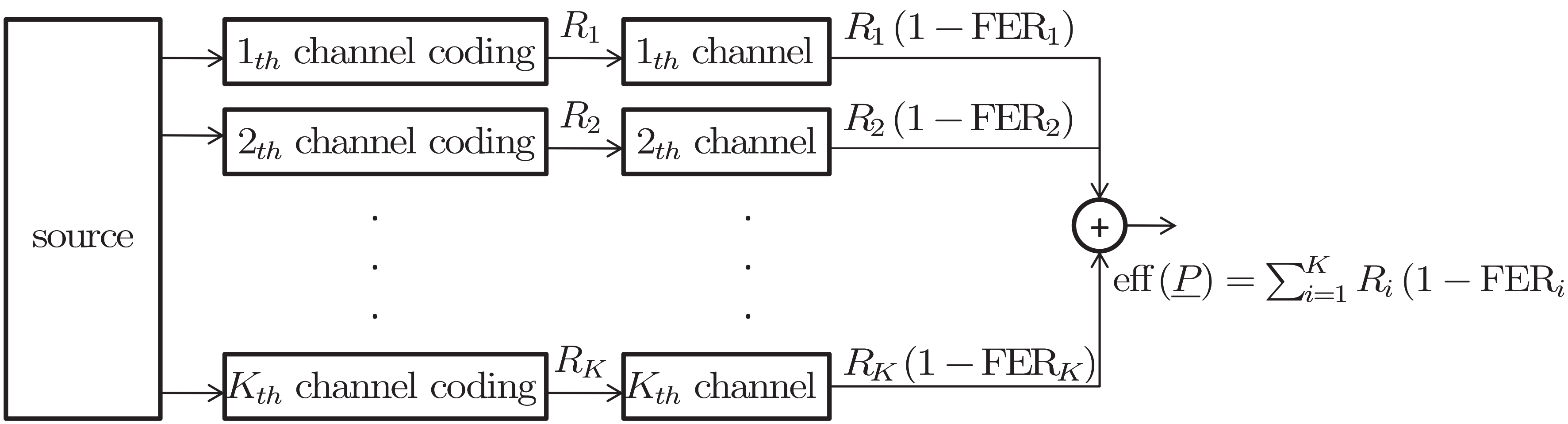

Figure 2: System Model

FER Approximation

- In AWGN channels, the FER at **high SNRs** can be well approximated by the **event error rate** as

$$\mathrm{FER}_i \approx A_{d_{\mathrm{free}}} e^{-d_{\mathrm{free}} \frac{P_i}{2\sigma_i^2}}, \qquad (2)$$

where d_{free} is the free distance of the convolutional code, $A_{d_{\mathrm{free}}}$ is the number of codewords with Hamming weight equal to d_{free}.

- However, the approximated FER in (2) is far from accurate for **moderate SNRs** and finite frame sizes. We choose to fix this inaccuracy by replacing $A_{d_{\mathrm{free}}}$ and d_{free} with the refined parameters A_i and d_i respectively such that the adjusted curve defined below,

$$\mathrm{FER}_i \approx \min \left\{ 1, e^{-d_i \frac{P_i}{2\sigma_i^2} + \log A_i} \right\}, \qquad (3)$$

is close to the true FER in the **least squares sense** over the range of operating SNRs.

FER Approximation

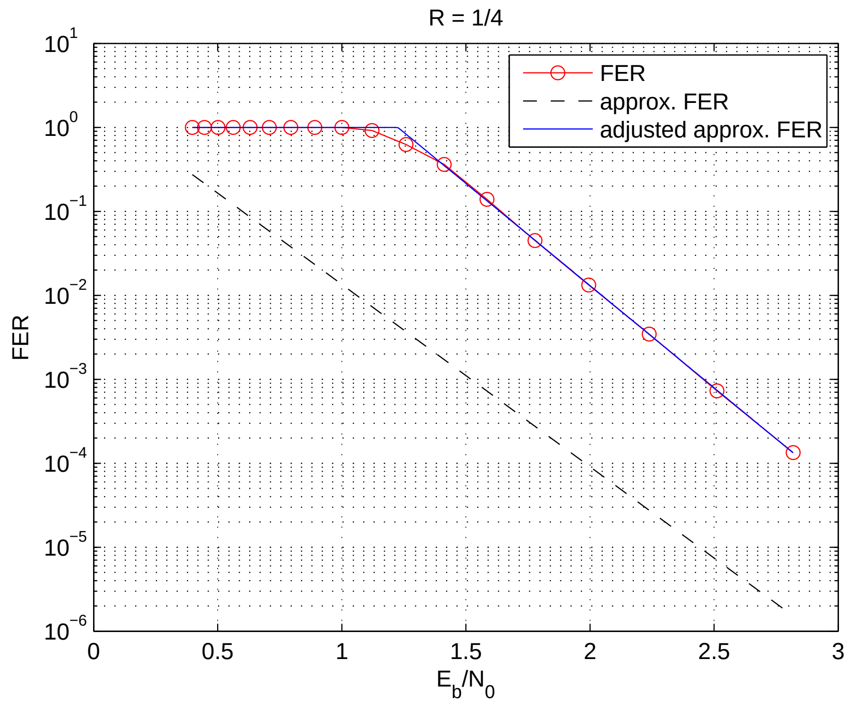

Figure 3: FER and its approximations for a $(4, 1, 6)$ convolutional code with generator polynomial(octal form) being [177 127 155 171], $d_{\text{free}} = 20$ and $A_{\text{dfree}} = 2$. The adjusted parameters are $d = 22.42$ and $A = 962.51$. The frame size is $N = 4(500 + 6)$. E_b/N_0 is plotted in linear scale.

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Throughput-Oriented Water-Filling I

- $\mathcal{O}$: the optimal set of active channels in use (assumed given)
- Based on the adjusted approximation formula in (3), $\text{eff}(\underline{P})$ becomes

$$\text{eff}(\underline{P}) = \sum_{i=1}^{K} R_i \left(1 - \min\left\{1, A_i e^{-\frac{d_i P_i}{2\sigma_i^2}}\right\}\right) \quad (4)$$

$$= \sum_{i \in \mathcal{O}} R_i \left(1 - A_i e^{-\frac{d_i P_i}{2\sigma_i^2}}\right) \quad (5)$$

- $P_i > P_{\text{thres},i} \triangleq \frac{2\sigma_i^2}{d_i} \log(A_i), \quad \forall\, i \in \mathcal{O}$
- By applying technique of Kuhn-Tucker conditions, the optimal choice of power allocation that maximizes (5) is

$$P_i^* = \frac{2\sigma_i^2}{d_i} \left(\nu - \log \frac{\sigma_i^2}{d_i A_i R_i}\right)^+ \quad (6)$$

where $(x)^+ \triangleq \max\{0, x\}$, and ν is chosen such that $\sum_{i \in \mathcal{O}} P_i^* = P_{\text{total}}$.

Throughput-Oriented Water-Filling I

- $$\nu \geq \nu_{\min} \triangleq \max_{i \in \mathcal{O}} \log \frac{\sigma_i^2}{d_i R_i}, \quad \forall i \in \mathcal{O}$$
- graphical interpretation:

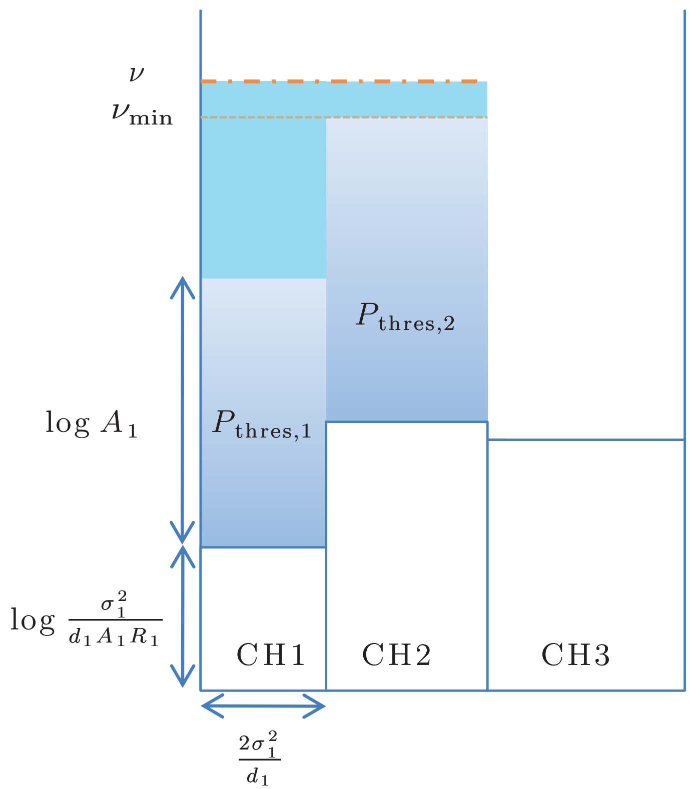

Figure 4: Example of throughput-optimizing water-filling with $\mathcal{O} = \{1, 2\}$.

Throughput-Oriented Water-Filling I

- When total power go without bound, the optimal power allocation should make the SNR in channel i be inversely proportional to d_i.

$$\lim_{P_{\text{total}} \to \infty} \frac{\frac{P_i^*}{\sigma_i^2}}{\frac{P_j^*}{\sigma_j^2}} = \frac{\frac{1}{d_i}}{\frac{1}{d_j}} \tag{7}$$

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Code Information

Table 1: The information of the used convolutional codes in the simulation.

code	d_{free}	$A_{d_{\text{free}}}$	adjusted d	adjusted A	codeword length N	generator polynomial (octal)
$(2, 1, 6)$	10	11	10.63 11.02	1478.07 4750.45	2(500+6) 2(1000+6)	[133 171]
$(3, 1, 6)$	14	1	15.79 16.12	593.83 1449.97	3(500+6) 3(1000+6)	[133 171 145]
$(4, 1, 6)$	20	2	22.42 22.13	962.51 1401.29	4(500+6) 4(1000+6)	[117 127 155 171]

- The information of the used convolutional codes in the simulation in listed in Table 2.

Case I-Settings

- $K = 3$
- $\sigma_1^2 = 1$, $\sigma_2^2 = 3.5$ and $\sigma_3^2 = 6$
- convolutional codes used in three channels
  - Ch1: convolutional (2,1,6) code
  - Ch2: convolutional (3,1,6) code
  - Ch3: convolutional (4,1,6) code
- $R_1 = 1/2$, $R_2 = 1/3$ and $R_3 = 1/4$
- $N_1 = 2(1000 + 6)$, $N_2 = 3(1000 + 6)$ and $N_3 = 4(1000 + 6)$

Case I-Simulation Results

- In Figure. 5, we depict the effective throughputs for the seven possible choices of the active channel set $\mathcal{O}$.

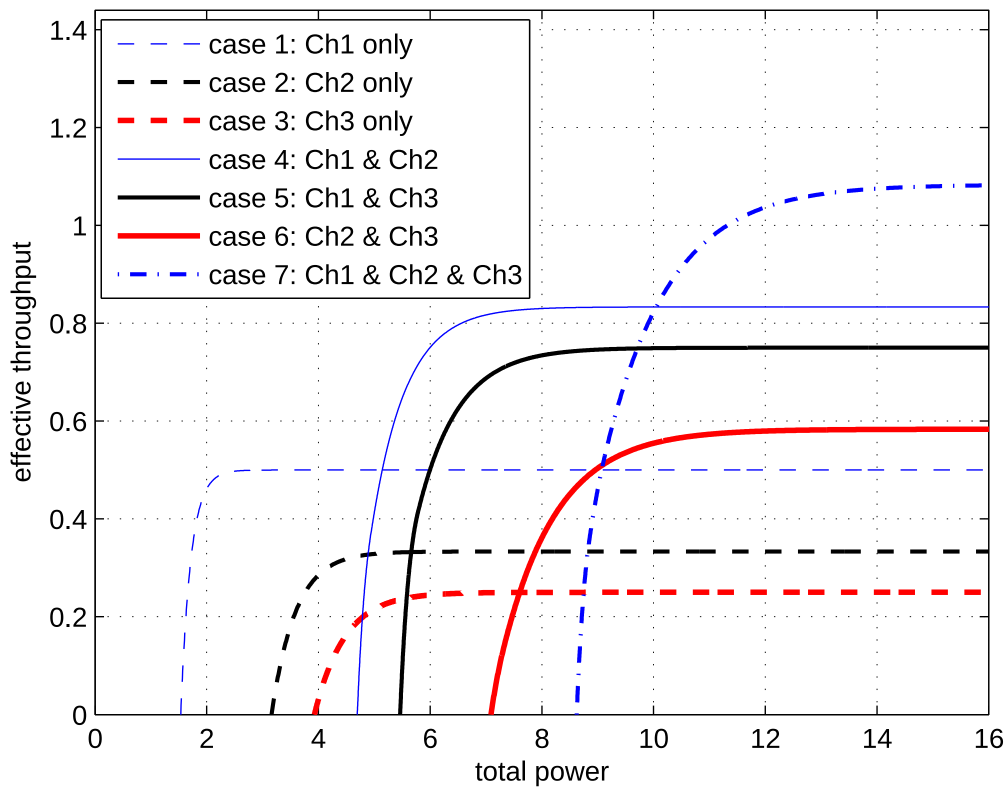

Figure 5: Case I: Effective throughputs for the seven choices of the active channel set $\mathcal{O}$.

- In Figure. 6, we compare the optimal effective throughput obtained from exhaustive search with the effective throughputs obtained from throughput-oriented water-filling I and the capacity-optimizing water-filling policy.

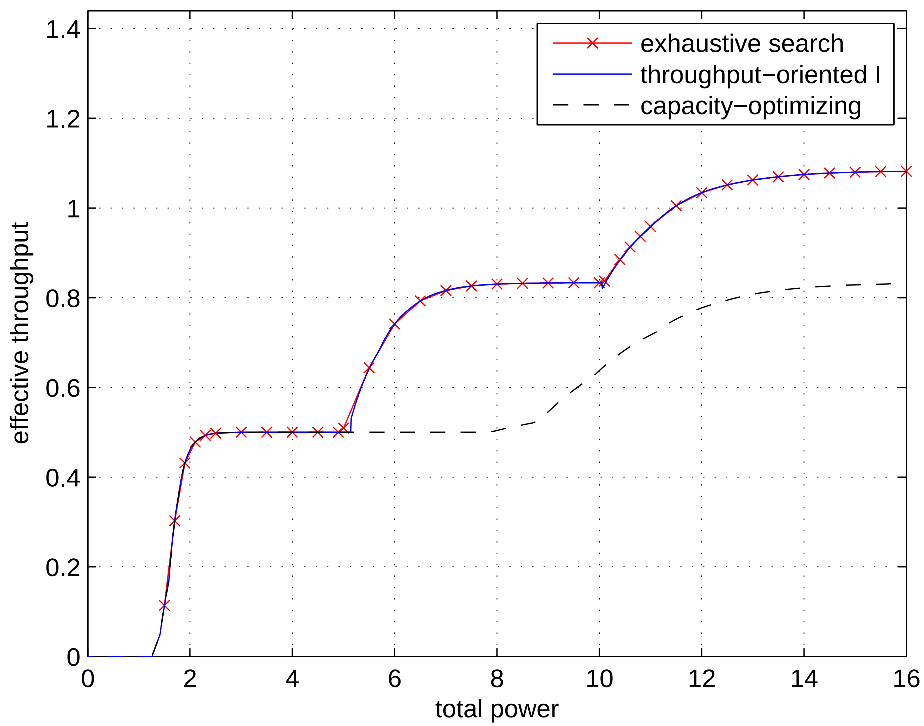

Figure 6: Case I: Optimal effective throughputs obtained from exhaustive search, the throughput-oriented I based on the FER approximation, and the capacity-optimizing water filling policy.

- In Figure. 7, we plot the optimal power ratio P_2^*/P_{total} with respect to different power allocation policies.

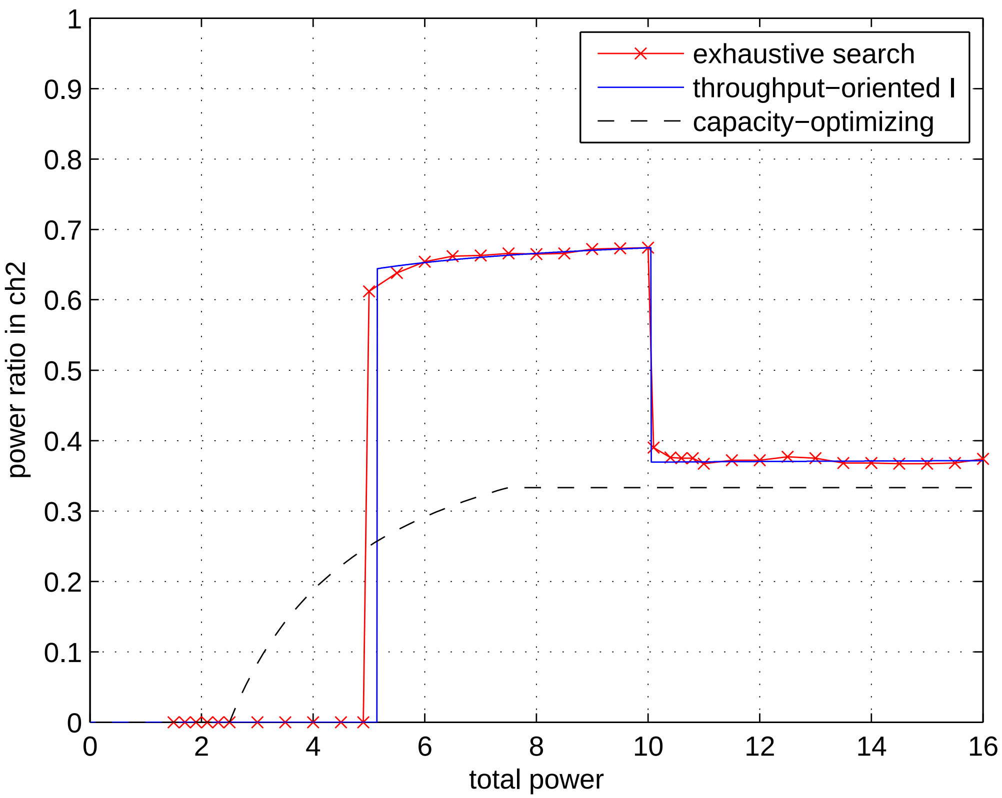

Figure 7: Case I: Optimal power ratio for channel 2.

Case I-Simulation Results

- The sudden increase for this ratio occurs in the simulation curve at $P_{\text{total}} = 4.98$ is exactly the instance the active channel set $\mathcal{O}$ changes from $\{1\}$ to $\{1, 2\}$ as shown in Figure. 8.

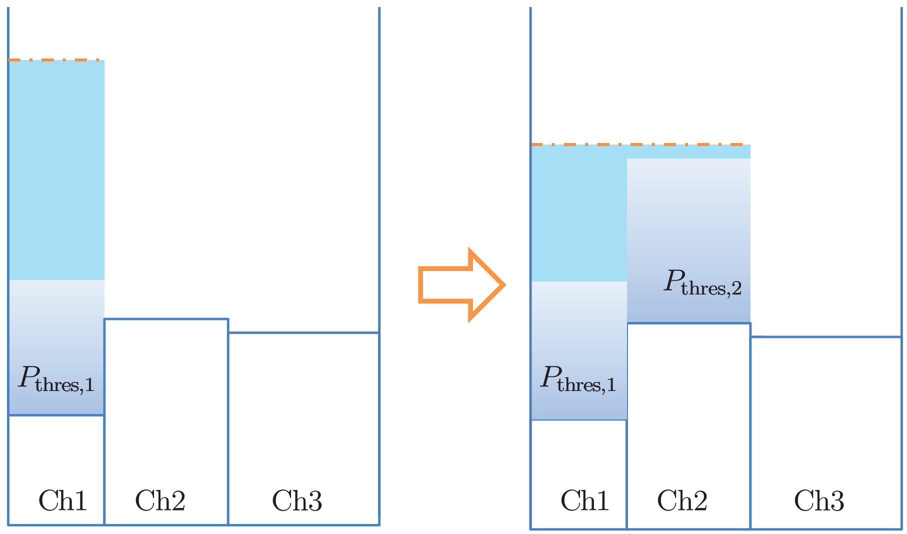

Figure 8: Case I: Illustration of the optimal active set $\mathcal{O}$ changing from $\{1\}$ to $\{1, 2\}$.

Case II-Settings

- $K = 3$
- $\sigma_1^2 = 1$, $\sigma_2^2 = 3.5$ and $\sigma_3^2 = 6$
- convolutional codes used in three channels
  - Ch1: convolutional (4,1,6) code
  - Ch2: convolutional (3,1,6) code
  - Ch3: convolutional (2,1,6) code
- $R_1 = 1/4$, $R_2 = 1/3$ and $R_3 = 1/2$
- $N_1 = 4(1000 + 6)$, $N_2 = 3(1000 + 6)$ and $N_3 = 2(1000 + 6)$

Case II-Simulation Results

- In Figure. 9, we depict the effective throughputs for the seven possible choices of the active channel set $\mathcal{O}$.

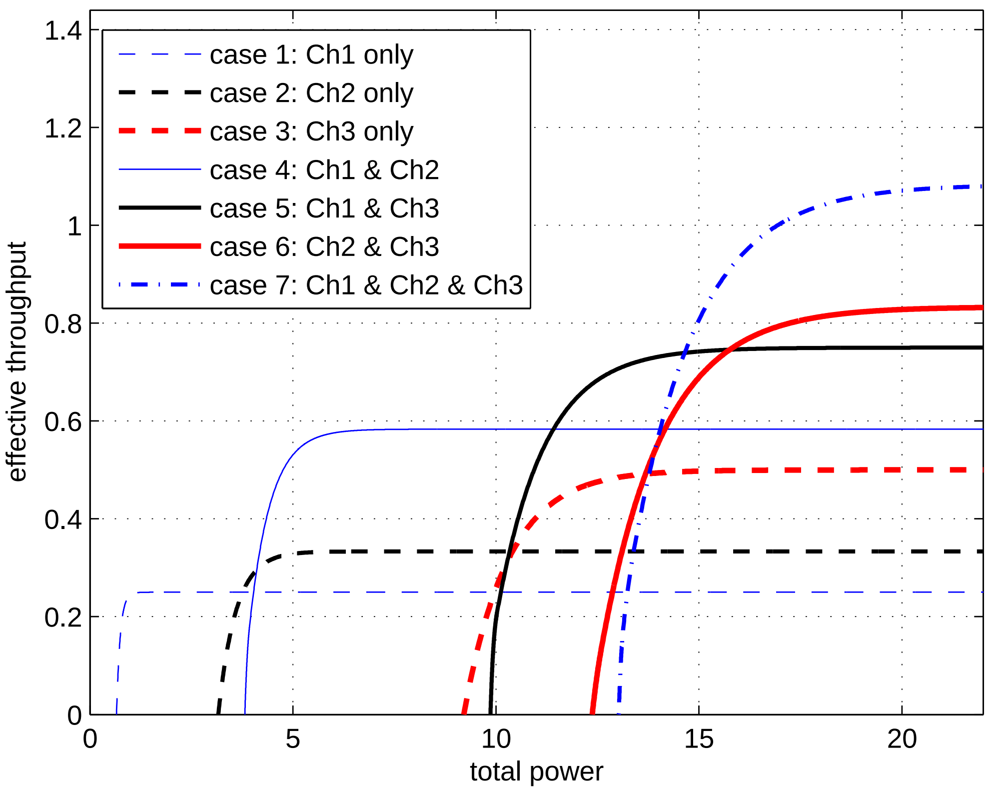

Figure 9: Case II: Effective throughputs for the seven choices of active channel set $\mathcal{O}$.

Case II-Simulation Results

- In Figure. 10, we compare the optimal effective throughput obtained from exhaustive search with the effective throughputs obtained from throughput-oriented water-filling I and the capacity-optimizing water-filling policy.

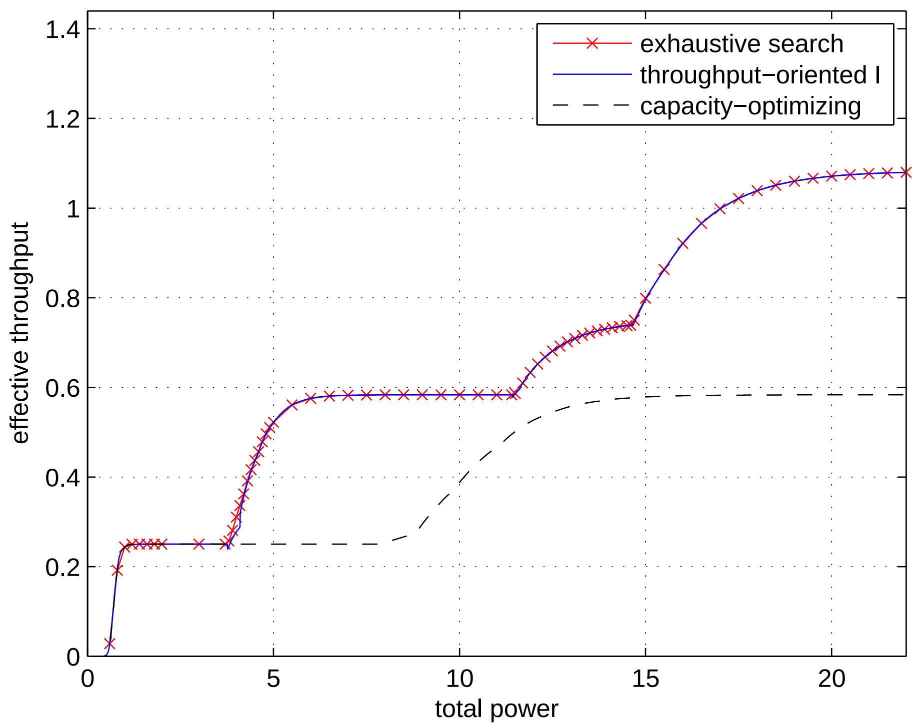

Figure 10: Case II: Optimal effective throughputs obtained from real simulation, the throughput-optimizing water-filling I on the FER approximation, and the capacity-optimizing water filling policy.

- In Figure. 11, we plot the optimal power ratio P_2^*/P_{total} with respect to different power allocation policies.

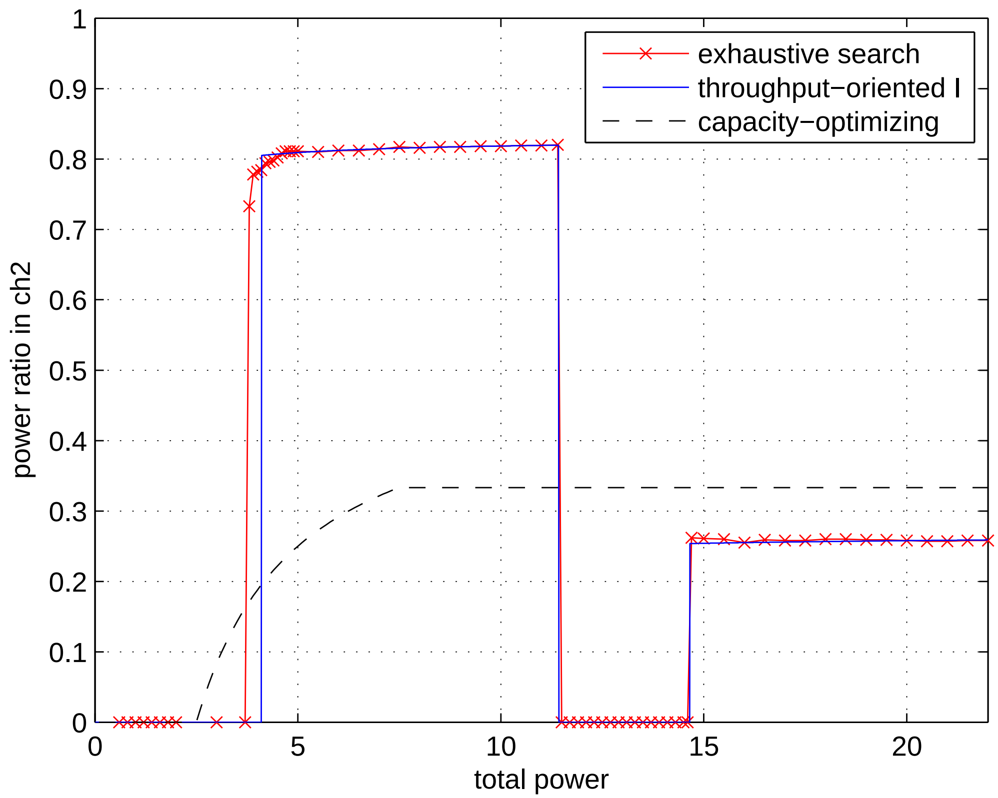

Figure 11: Case II: Optimal power ratio for channel 2.

Outline

Capacity-Optimizing: Only Total Noise Variance is Available

- σ^2_{total} is the total noise variance.
- The worst-case capacity of the system is

$$\min_{\sum_{i=1}^{K} \sigma_i^2 = \sigma^2_{\text{total}}} \sum_{i=1}^{K} \frac{1}{2} \log \left(1 + \frac{P_i}{\sigma_i^2} \right), \tag{8}$$

  where P_i is chosen such that (8) is maximized and the power constraint $\sum_{i=1}^{K} P_i = P_{\text{total}}$ is satisfied simultaneously.
- The resulting power allocation policy is to allocate equal power to each channel.

Outline

System Model and Problem Formulation II

- worst-case effective throughput: For any given power allocation, σ_i^2 is always chosen such that the effective throughput is minimized.

$$\mathrm{eff}_{\mathrm{worst}}\left(\underline{P}\right) \triangleq \min_{\sum_{i=1}^{K}\sigma_i^2=\sigma_{\mathrm{total}}^2} \sum_{i=1}^{K} R_i\left(1-\mathrm{FER}_i\right), \tag{9}$$

where P_i is chosen such that (9) is maximized under the power constraint $\sum_{i=1}^{K} P_i = P_{\mathrm{total}}$.

Throughput-Oriented Water-Filling II

- We focus on the situation that all of the channels are active.
  Power constrains I:

$$P_i \geq P_{\text{thres},i} \quad \forall\, 1 \leq i \leq K. \tag{10}$$

- Based on the adjusted approximation formula in (3), (9) becomes

$$\begin{aligned}
& \mathrm{eff}_{\text{worst}}\left(\underline{P}\right) & (11)\\
= & \min_{\sum_{i=1}^{K}\sigma_i^2=\sigma_{\text{total}}^2} \sum_{i=1}^{K} R_i \left(1 - \min\left\{1, A_i e^{-\frac{d_i P_i}{2\sigma_i^2}}\right\}\right) & (12)\\
= & \min_{\sum_{i=1}^{K}\sigma_i^2=\sigma_{\text{total}}^2} \sum_{i=1}^{K} R_i \left(1 - A_i e^{-\frac{d_i P_i}{2\sigma_i^2}}\right). & (13)
\end{aligned}$$

Throughput-Oriented Water-Filling II

- In general the worst case effective throughput is not a concave function over σ_i^2 from (14).

$$\frac{\partial^2 \text{eff}_{\text{worst}}\left(\underline{P}\right)}{\partial\left(\sigma_i^2\right)^2} = \frac{R_i A_i d_i P_i}{\left(\sigma_i^2\right)^3} e^{-\frac{d_i P_i}{2\sigma_i^2}} \left(1 - \frac{d_i P_i}{4\sigma_i^2}\right) \tag{14}$$

- **Power constrians II**

$$P_i \geq \frac{4\sigma_i^2}{d_i} \tag{15}$$

The problem is confined to be concave over σ_i^2 such that (14) becomes always negative.

- The σ_i^2 to achieve $\text{eff}_{\text{worst}}$ should be chosen either 0 or σ_{total}^2.

Throughput-Oriented Water-Filling II

- Consider one power allocation:

$$P_i^{\dagger} = \frac{2\sigma_{\text{total}}^2}{d_i}\left(\nu - \log \frac{1}{A_i R_i}\right), \tag{16}$$

where ν is chosen such that $\sum_{i=1}^{K} P_i^{\dagger} = P_{\text{total}}$.

- From the power constraints I and II, $P_i^{\dagger}$ should satisfy

$$P_i^{\dagger} \geq \max\left\{\frac{2\sigma_i^2}{d_i}\log(A_i), \frac{4\sigma_i^2}{d_i}\right\} \quad \forall\, 1 \leq i \leq K. \tag{17}$$

- Since $\sigma_i^2 \leq \sigma_{\text{total}}^2$, we choose $P_i^{\dagger}$ sufficiently large such that the power constraints are also satisfied.

$$P_i^{\dagger} \geq \max\left\{\frac{2\sigma_{\text{total}}^2}{d_i}\log(A_i), \frac{4\sigma_{\text{total}}^2}{d_i}\right\} \quad \forall\, 1 \leq i \leq K. \tag{18}$$

Throughput-Oriented Water-Filling II

- The minimum power required in channel i as

$$P^{\dagger}_{\text{thres,i}} \triangleq \max\left\{\frac{2\sigma^2_{\text{total}}}{d_i}\log(A_i), \frac{4\sigma^2_{\text{total}}}{d_i}\right\} \quad \forall\, 1 \le i \le K.$$

Replacing $P_i^{\dagger}$ by (16), we further deduce (18) as a condition on ν,

$$\nu \ge \max\{-\log R_i, 2 - \log(A_i R_i)\} \quad \forall\, 1 \le i \le K. \tag{19}$$

- $\nu_{\min} \triangleq \max_i \{\max\{2 - \log A_i R_i, -\log R_i\}\}$.
- From the definition of $P_i^{\dagger}$ and (19), we have

$$P_i^{\dagger} \ge \frac{2\sigma_{\text{total}}}{d_i}\left(\nu_{\min} - \log\left(\frac{1}{A_i R_i}\right)\right) \quad \forall\, 1 \le i \le K. \tag{20}$$

(20) equivalently implies a constraint in system SNR.

$$\frac{\sum_{i=1}^{K} P_i^{\dagger}}{\sigma^2_{\text{total}}} \ge \sum_{i=1}^{K} \frac{2}{d_i}\left(\nu_{\min} - \log\left(\frac{1}{A_i R_i}\right)\right) = \gamma^{\dagger}_{\text{thres}}.$$

Throughput-Oriented Water-Filling II

- $\text{eff}^{\dagger}_{\text{worst}}$: worst-case effective throughput of $P_i^{\dagger}$
- $\sigma_i^{2\dagger}$: the noise variance that achieves $\text{eff}^{\dagger}_{\text{worst}}$

$$\sigma_i^{2\dagger} \triangleq \begin{cases} \sigma^2_{\text{total}}, & \text{if } i = m \\ 0, & \text{if } i \neq m \end{cases}, \tag{21}$$

where m can be chosen to be $1 \leq m \leq K$.

Throughput-Oriented Water-Filling II

$$
\begin{aligned}
& \mathrm{eff}^{\dagger}_{\mathrm{worst}}\left(\underline{P^{\dagger}}\right) \\
\triangleq\ & \min_{\sum \sigma_i^2=\sigma^2_{\mathrm{total}}} \sum_{i=1}^{K} R_i \left(1 - A_i e^{-\frac{d_i P_i^{\dagger}}{2\sigma_i^2}}\right) \\
=\ & \min_{\sum \sigma_i^2=\sigma^2_{\mathrm{total}}} \sum_{i=1}^{K} R_i \left(1 - A_i e^{-\frac{d_i \frac{2\sigma^2_{\mathrm{total}}}{d_i}\left(\nu - \log \frac{1}{R_i A_i}\right)}{2\sigma_i^2}}\right) && \left(\text{replacing } P_i^{\dagger}\right) \\
=\ & \sum_{i=1}^{K} R_i \left(1 - A_i e^{-\frac{d_i \frac{2\sigma^2_{\mathrm{total}}}{d_i}\left(\nu - \log \frac{1}{R_i A_i}\right)}{2(\sigma_i^{\dagger})^2}}\right) && (\text{removing minimization}) \\
=\ & \sum_{i\neq m} R_i + R_m - e^{-\left(\nu - \log \frac{1}{R_m A_m}\right) + \log R_m A_m} \\
=\ & \sum_{i=1}^{K} R_i - e^{-\nu}.
\end{aligned}
$$

Throughput-Oriented Water-Filling II

- The optimality of using $P_i^\dagger$ as power allocation is proved by the method of contradiction.
- $\underline{\hat{P}_i}$: any power allocation other than $\underline{P^\dagger}$
  - $\hat{P}_i = P_i^\dagger + \triangle P_i^\dagger$
  - $\triangle P_i^\dagger \neq 0$ for at least one channel

$$\sum_{i=1}^{K} \triangle P_i^\dagger = 0. \tag{22}$$

- $\hat{\sigma}_i$: a specific noise power allocation

$$\hat{\sigma_i}^2 = \begin{cases} \sigma_{\text{total}}^2 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases},$$

where

$$k \triangleq \arg \min_{1 \leq i \leq K} \left(\frac{d_i P_i}{2\sigma_{\text{total}}^2} - \log A_i R_i \right). \tag{23}$$

Throughput-Oriented Water-Filling II

- $\text{eff}_{\text{worst}}\left(\underline{\hat{P}}\right)$: the worst-case effective throughput of $\hat{P}_i$

$$\begin{aligned}
&\text{eff}_{\text{worst}}\left(\underline{\hat{P}}\right) \\
&= \min_{\sum \sigma_i^2 = \sigma_{\text{total}}^2} \sum_{i=1}^{K} R_i \left(1 - \min\left\{1, A_i e^{-\frac{d_i \hat{P}_i}{2\sigma_i^2}}\right\}\right) && (24) \\
&\leq \sum_{i=1}^{K} R_i \left(1 - \min\left\{1, A_i e^{-\frac{d_i \hat{P}_i}{2\hat{\sigma}_i{}^2}}\right\}\right) && (25) \\
&\leq \sum_{i \neq k} R_i + R_k \left(1 - \min\left\{1, A_k e^{-\frac{d_k \hat{P}_k}{2\sigma_{\text{total}}^2}}\right\}\right) && (26)
\end{aligned}$$

– If

$$\min\left\{1, A_k e^{-\frac{d_k \hat{P}_k}{2\sigma_{\text{total}}^2}}\right\} = 1,$$

(26) becomes $\sum_{i \neq k} R_i$ which is less than or equal to $\text{eff}_{\text{worst}}^{\dagger}$.

– Else if

$$\min\left\{1, A_k e^{-\frac{d_k \hat{P}_k}{2\sigma_{\text{total}}^2}}\right\} = A_k e^{-\frac{d_k \hat{P}_k}{2\sigma_{\text{total}}^2}},$$

(26) becomes

$$\sum_{i=1}^{K} R_i - e^{-\left(\frac{d_k \hat{P}_k}{2\sigma_{\text{total}}^2} - \log\left(R_k A_k\right)\right)}. \tag{27}$$

From the definition of k in (23), we have

$$\frac{d_k \hat{P}_k}{2\hat{\sigma}_k^2} - \log\left(R_k A_k\right) = \min_{1 \le i \le K}\left(\frac{d_i \hat{P}_i}{2\sigma_{\text{total}}^2} - \log A_i R_i\right) \tag{28}$$

Note that

$$\frac{d_i \hat{P}_i}{2\sigma_{\text{total}}^2} - \log R_i A_i = \frac{d_i\left(P_i^{\dagger} + \triangle P_i\right)}{2\hat{\sigma_i}^2} - \log R_i A_i \tag{29}$$

$$= \nu + \frac{d_i \triangle P_i}{2\sigma_{\text{total}}^2} \quad \forall\, 1 \le i \le K. \tag{30}$$

Throughput-Oriented Water-Filling II

There always exists at least a channel has its $\frac{d_i \triangle P_i}{2\sigma^2_{\text{total}}}$ being negative since $\hat{P}_i \neq P_i^{\dagger}$ and $\triangle P_i$ should satisfy the constraint in (22). Taking minimization over (31), we have

$$\frac{d_k \hat{P}_k}{2\hat{\sigma}^2_k} - \log\left(R_k A_k\right) = \min_{1 \le i \le K} \left(\frac{d_i \hat{P}_i}{2\sigma^2_{\text{total}}} - \log A_i R_i \right) < \nu \quad (31)$$

Thus (27) becomes

$$\begin{aligned} & \sum_{i=1}^{K} R_i - e^{-\left(\frac{d_k \hat{P}_k}{2\sigma^2_{\text{total}}} - \log\left(R_k A_k\right)\right)} \\ \le & \sum_{i=1}^{K} R_i + e^{-\nu} \\ = & \text{eff}^{\dagger}_{\text{worst}}\left(\underline{P^{\dagger}}\right) \end{aligned}$$

- From above, we have proved that

$$\text{eff}_{\text{worst}}\left(\underline{\hat{P}}\right) \le \text{eff}^{\dagger}_{\text{worst}}\left(\underline{P^{\dagger}}\right).$$

Throughput-Oriented Water-Filling II

- When the total power goes without bound, the optimal allotted power in channel i should be inversely proportional to its d_i

$$\lim_{P_{\text{total}} \to \infty} \frac{P_i^\dagger}{P_j^\dagger} = \frac{\frac{1}{d_i}}{\frac{1}{d_j}}$$

- When each channel using the same code, the proposed power allocation policy can be simplified to be the equal power allocation For $i \neq j$

$$P_i^\dagger = \frac{2\sigma_{\text{total}}^2}{d_i} \left(\nu - \log \frac{1}{A_i R_i} \right) = P_j^\dagger. \tag{32}$$

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$(2, 1, 6)$	10	11	10.63	1478.07	2(500+6)	[133 171]
$(3, 1, 6)$	14	1	15.79	593.83	3(500+6)	[133 171 145]
$(4, 1, 6)$	20	2	22.42	962.51	4(500+6)	[117 127 155 171]
$(2, 1, 2)$	5	1	5.31	111.56	2(500+2)	[5 7]
$(3, 1, 11)$	24	13	29.04	41373.67	3(500+11)	[5475 6471 7553]
$(4, 1, 10)$	29	3	35.54	11266.62	4(500+10)	[2565 2747 3311 3723]

- The information of the used convolutional codes in the simulation in listed in Table 2.

Case I-Settings

- $K = 3$
- convolutional codes used in three channels
  - Ch1: convolutional (2,1,6) code
  - Ch2: convolutional (3,1,6) code
  - Ch3: convolutional (4,1,6) code
- $\sigma^2_{\text{total}} = 10$
- maximum rate: $\sum_{i=1}^{K} R_i$

Case I-Simulations

- Compare the worst-case effective throughputs of exhaustive search, throughput-oriented water-filling II and equal power allocation.

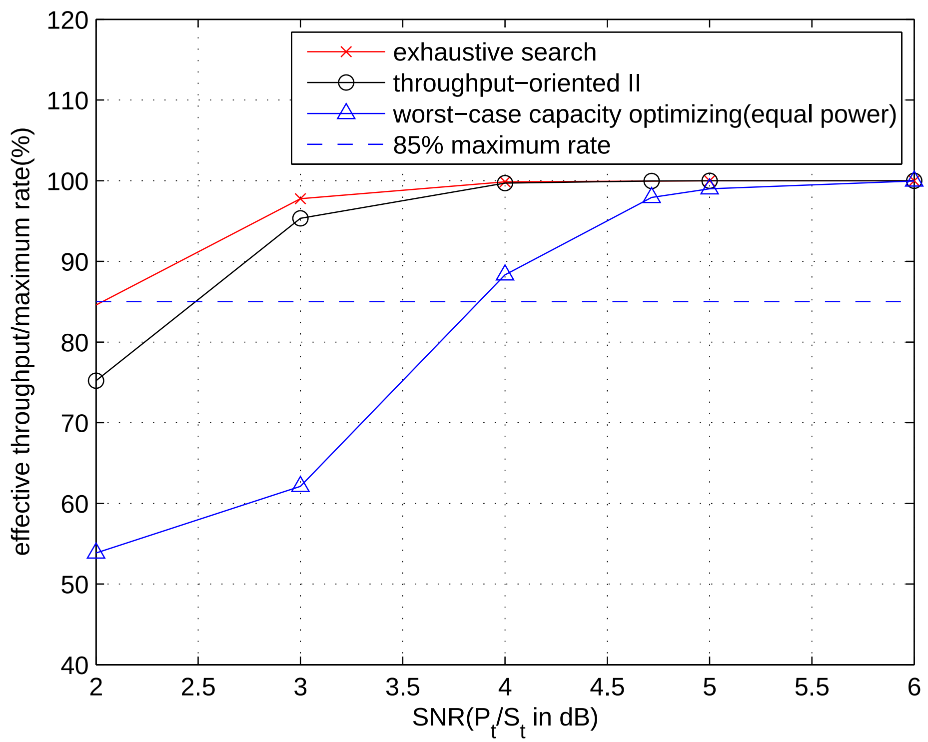

Figure 12: Case I: The worst-case effective throughputs obtained from exhaustive search, throughput-oriented water-filling II based on the FER approximation, and the worst-case-capacity-optimizing equal power allocation.

Case I-Simulation Results

Table 3: The $\sigma_i^{2\dagger}$ of each channel($K = 3$) for system SNR ranging from 2 dB to 6 dB.

	2 dB	3 dB	4 dB	5 dB	6 dB
$\sigma_1^{2\dagger}$	0	0	0	0	0
$\sigma_2^{2\dagger}$	10	10	10	10	10
$\sigma_3^{2\dagger}$	0	0	0	0	0

- The $\gamma_{\text{thres}}^{\dagger}$ where our proposed power allocation becomes optimal is 4.72 dB.
- The throughput-oriented II has around 1.4 dB gain over the equal power allocation when achieving 85% of the maximum rate.
- In Table 3 we provide the $\sigma_i^{2\dagger}$ in three channels for system SNR varying from 2 dB to 6 dB when using the throughput-oriented II.

Case II-Settings

- $K = 3$
- convolutional codes used in three channels
  - Ch1: convolutional (2,1,2) code
  - Ch2: convolutional (3,1,11) code
  - Ch3: convolutional (4,1,10) code
- $\sigma^2_{\text{total}} = 10$

Case II-Simulation Results

- Compare the worst-case effective throughputs of exhaustive search, throughput-oriented II and equal power allocation.

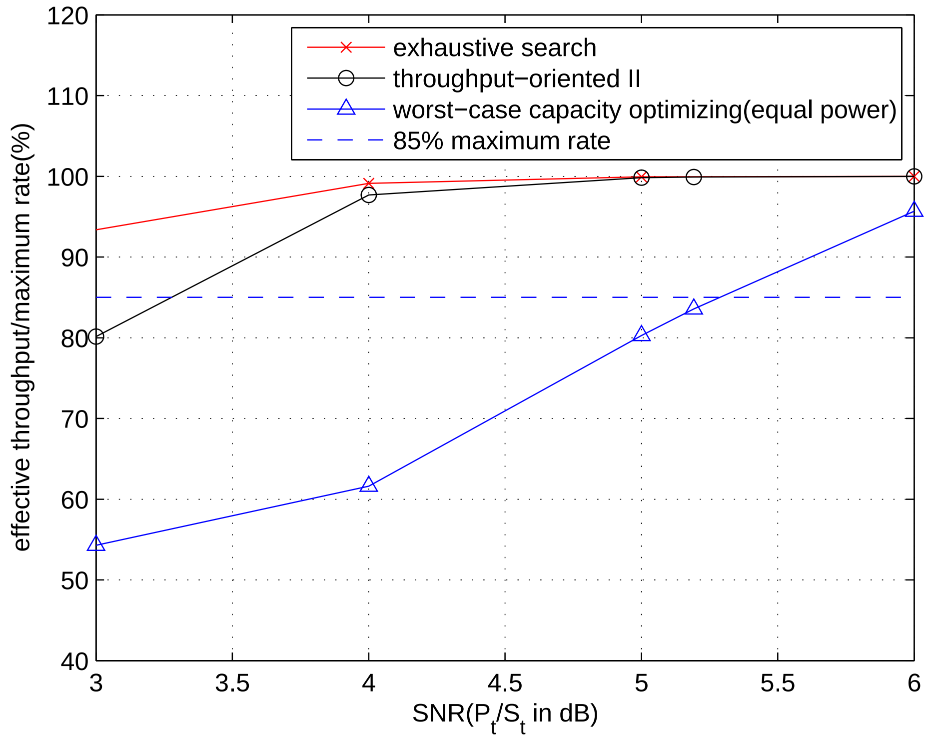

Figure 13: Case II: The worst-case effective throughputs obtained from exhaustive search, throughput-oriented water-filling II based on the FER approximation, and the worst-case-capacity-optimizing equal power allocation.

Case II-Simulation Results

- The $\gamma^{\dagger}_{\text{thres}}$ where our proposed power allocation becomes optimal is 5.19 dB.
- The throughput-oriented water-filling II has around 2 dB gain over the equal power allocation when achieving 85% of the maximum rate.

Outline

Conclusions

- Two power allocation policies that aiming to maximize the **effective throughput** and so-defined worst-case effective throughput over K **coded** parallel AWGN channels subject to practical finite-length and fixed-rate **coding constraints** are proposed.
- These policies preserves the notion of the water-filling principle by additionally taking into consideration the **code characteristics**.
- Part I: The proposed power allocation policy achieves a near-optimal throughput for all value of the total power.
- Part II: The proposed power allocation policy achieves a near-optimal throughput for system SNR greater than certain threshold system SNR. And from the simulation, it still performs better than tradition equal power allocation for moderate SNR.

Future Works

- Part I: The future work is to provide a quick determination of the optimal active channel set $\mathcal{O}$ (instead of examining all $(2^K - 1)$ cases).
- Part II: The future work is to find the optimal power allocation policy for system SNR before the threshold system SNR.

Thank You for Your Listening

Appendix I: Linear Least Square Estimators

- Linear least square estimators
  The approximated FER in log scale is a linear combination of d_i and $\log A$ in the operating SNR region. Thus linear least square estimator can be applied to retrieve d and $\log A$.
  - $\boldsymbol{x} = [x[0]\, x[1] \ldots x[M-1]]^T$: M true FER values which are in log scale
  - $\boldsymbol{g} = [g[0]\, g[1] \ldots g[M-1]]^T$: its corresponding $\frac{E_b}{N_0}$.
  - $\boldsymbol{s} = [s[0]\, s[1] \ldots s[M-1]]^T$: the approximated FER values in log scale. From (3), the $s[j]$ can be modelled by

    $$s[j] = -Rg[j]\, d + \log A \quad \forall\, 1 \leq j \leq M$$

    or in matrix form

    $$\boldsymbol{s} = \boldsymbol{H}\boldsymbol{\theta},$$

where

$$\boldsymbol{H} = \begin{bmatrix} -Rg[0] & 1 \\ -Rg[1] & 1 \\ \cdot & \\ \cdot & \\ \cdot & \\ -Rg[M-1] & 1 \end{bmatrix}, \boldsymbol{\theta} = \begin{bmatrix} d \\ \log A \end{bmatrix}.$$

– Least square error

$$J(\boldsymbol{\theta}) = (\boldsymbol{x} - \boldsymbol{H\theta})^T (\boldsymbol{x} - \boldsymbol{H\theta}). \tag{33}$$

Setting the gradient of $J(\boldsymbol{\theta})$ to be zero yields the least square estimator

$$\hat{\boldsymbol{\theta}} = \left(\boldsymbol{H}^T \boldsymbol{H}\right)^{-1} \boldsymbol{H}^T \boldsymbol{x}$$

The refined parameters A and d for the code is then found.

- Capacity is convex over σ_i^2. Thus the Kuhn-Tucker conditions can be applied to find σ_i^2 such that the worst-case capacity is achieved for given P_i.

$$C_1(\underline{P}) = \sum_{i=1}^{n} \frac{1}{2} \log \left(1 + \frac{P_i}{\sigma_i^2}\right) + \lambda_1 \left(\sum_{i=1}^{n} \sigma_i^2 - \sigma_{\text{total}}^2\right), \quad (34)$$

where λ_1 is chosen such that $\sum_{i=1}^{K} \sigma_i^2 = \sigma_{\text{total}}^2$. Taking derivative of (34) w.r.t. σ_i^2, we have

$$\begin{cases} \left(\sigma_i^2\right)^2 + P_i \sigma_i^2 + \frac{P_i}{2\lambda_1} = 0, & \text{if } \sigma_i^2 > 0; \\ \left(\sigma_i^2\right)^2 + P_i \sigma_i^2 + \frac{P_i}{2\lambda_1} < 0, & \text{if } \sigma_i^2 = 0. \end{cases} \quad (35)$$

Hence,

$$\sigma_i^2 = \left(\frac{-P_i + \sqrt{(P_i)^2 - \frac{2P_i}{\lambda_1}}}{2}\right)^+ . \quad (36)$$

We then replace the σ_i^2 in (8) by (36) and thus take away the minimization. (8) becomes

$$\sum_{i=1}^{n} \frac{1}{2} \log \left(1 + \frac{P_i}{\frac{-P_i + \sqrt{(\gamma P_i)^2 - \frac{2P_i}{\lambda_1}}}{2}} \right) \tag{37}$$

Since (37) is concave over P_i, Kuhn-Tucker conditions can be applied again as the following. Let

$$C_2(\underline{P}) = \sum_{i=1}^{n} \frac{1}{2} \log \left(1 + \frac{P_i}{\frac{-P_i + \sqrt{(P_i)^2 - \frac{2P_i}{\lambda_1}}}{2}} \right) \tag{38}$$

$$+\lambda_2 \left(\sum_{i=1}^{n} P_i - 1 \right), \tag{39}$$

where λ_2 is chosen such that $\sum_{i=1}^{K} P_i = P_{\text{total}}$.

The derivative of (38) w.r.t. P_i becomes

$$\frac{\partial C_2(\underline{P})}{\partial P_i} = \left(\frac{1}{\left(-1+\sqrt{1-\frac{2}{\lambda_1 P_i}}\right)^2 + 2(-1+\sqrt{1-2\frac{2}{\lambda_1 P_i}})} \right) \left(\frac{1}{\sqrt{(1-\frac{2}{\lambda_1 P_i}}\lambda_1 P_i^2} \right) + \lambda_2. \quad (40)$$

And (40) should also satisfy

$$\begin{cases} \frac{\partial C_2(\underline{P})}{\partial P_i} = 0, \text{ if } P_i > 0; \\ \frac{\partial C_2(\underline{P})}{\partial P_i} < 0, \text{ if } P_i = 0. \end{cases}$$

For any $j \neq i$, we have

$$\frac{\partial C_2(\underline{P})}{\partial P_i} = \frac{\partial C_2(\underline{P})}{\partial P_j} \quad (41)$$

if $P_i > 0$ and $P_j > 0$. One of possible choice of P_j for (41) to hold and the power constraint to be satisfied is choosing $P_i = P_j$. Thus, the resulting power allocation policy is the equal power allocation.