Appendix A

Mathematical Background on Real Analysis

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The Concept of Sets

- Sets
  - A collection of objects, such as people, numbers, points, or any abstract objects
- A set with useful mechanism
  - There exists a *mechanism* (cf. the next slide) so that one can decide whether an object belongs to it or not.

The Concept of Field/Algebra

Definition (field/algebra) A set $\boldsymbol{\mathcal{F}}$ is said to be a *field* or *algebra* of a *sample space* Ω if it is a nonempty collection of subsets of Ω with the following properties:

1. $\emptyset \in \boldsymbol{\mathcal{F}}$ and $\Omega \in \boldsymbol{\mathcal{F}}$;
   - *Interpretation*: Mechanism to determine whether the *outcome* lies in an empty set (impossible) or the sample space (certain).
2. (closure under complement action) $A \in \boldsymbol{\mathcal{F}} \Rightarrow A^c \in \boldsymbol{\mathcal{F}}$;
   - *Interpretation*: "having a mechanism to determine whether the outcome lies in A" is equivalent to "having a mechanism to determine whether the outcome lines in A^c."
3. (closure under finite union) $A \in \boldsymbol{\mathcal{F}}$ and $B \in \boldsymbol{\mathcal{F}} \Rightarrow A \cup B \in \boldsymbol{\mathcal{F}}$.
   - *Interpretation*: If one has a mechanism to determine whether the outcome lies in A, and a mechanism to determine whether the outcome lies in B, then he can surely determine whether the outcome lies in the union of A and B.

- Elements of $\boldsymbol{\mathcal{F}}$ is referred to as *events*.

σ-field/algebra

- To work on a *field* may render some problems when a person is dealing with "*limit*".

  E.g., $\Omega = \Re$ (the real line) and $\boldsymbol{\mathcal{F}}$ is a collection of all *open*, *semi-open* and *closed* intervals whose two endpoints are rational numbers, including $\Re$ itself.

  Let

  $$A_i = [0, 1.\underbrace{111\ldots1}_{i \text{ of them}}).$$

  Then does the infinite union $\cup_{i=1}^{\infty} A_i$ of A_i belong to $\boldsymbol{\mathcal{F}}$? The answer is apparently no!

- We therefore need an extension definition of field, which is named σ-field.

Definition (σ-field/σ-algebra) A set $\boldsymbol{\mathcal{F}}$ is said to be a *σ-field* or *σ-algebra* of a sample space Ω if it is a nonempty collection of subsets of Ω with the following properties:

1. $\emptyset \in \boldsymbol{\mathcal{F}}$ and $\Omega \in \boldsymbol{\mathcal{F}}$;
2. (closure under complement action) $A \in \boldsymbol{\mathcal{F}} \Rightarrow A^c \in \boldsymbol{\mathcal{F}}$;
3. (closure under countable union) $A_i \in \boldsymbol{\mathcal{F}}$ for $i = 1, 2, 3, \ldots \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \boldsymbol{\mathcal{F}}$.

Set Operations

- Subset

$$A \subset B \quad (B \subset A)$$

- Equality

$$A = B$$

- Union

$$A \cup B$$

- Intersection

$$A \cap B$$

- Complement

$$A^c$$

Set Operations over Real Line $\Re$

- Set negation: Define the "negation" of a set $A \subset \Re$ to be

$$-A \stackrel{\triangle}{=} \{c \in \Re \;:\; c = -a \text{ for some } a \in A\}\,.$$

- Set addition: Define the "addition" of two sets $A \subset \Re$ and $B \subset \Re$ to be

$$A + B \stackrel{\triangle}{=} \{c \in \Re \;:\; c = a + b \text{ for some } a \in A \text{ and } b \in B\}\,.$$

- Set multiplication: Define the "multiplication" of two sets $A \subset \Re$ and $B \subset \Re$ to be

$$A \cdot B \stackrel{\triangle}{=} \{c \in \Re \;:\; c = a \cdot b \text{ for some } a \in A \text{ and } b \in B\}\,.$$

- Scalar multiplication for a set and a scalar: Define the "scalar multiplication" for a set $A \subset \Re$ and a scalar $k \in \Re$ to be

$$k \cdot A \stackrel{\triangle}{=} \{c \in \Re \;:\; c = k \cdot a \text{ for some } a \in A\}\,.$$

Supremum and Maximum

- Upper bound of a subset A of the real line $\Re$:
  - (Bounded above) A set $A \in \Re$ is said to be *bounded above* if

    $$(\exists\, u \in \Re) \text{ such that } (\forall\, a \in A)\ a \leq u.$$

    The number u is an *upper bound* of A.
  - Note: The "supremum" and "maximum" operations are always taken the set of all real numbers as a universal set.
- Least upper bound
  - Every non-empty bounded-above subset of the real line $\Re$ has a *least upper bound*, acronym as l.u.b.

Supremum and Maximum

- Extension definition of l.u.b. for empty set and non-bounded-above set
  - l.u.b. of an empty set is $-\infty$.
  - l.u.b. of an non-bounded-above set is ∞.
- Under the extension definition, a common terminology for l.u.b. of A is $\sup A$, which termed the *supremum* of A.
- The *maximum* of a set

$$\text{if } \sup A \in A, \text{ then } \max A = \sup A; \text{ otherwise}, \max A \text{ does not exist.}$$

  - E.g., $A = (0, 1]$ and $\max A = \sup A = 1$.
  - E.g., if $A = (0, 1)$, then $\sup A = 1$ but $\max A$ does not exist!

Infimum and Minimum

- Lower bound of a subset A of the real line $\Re$:
  - (Bounded below) A set $A \in \Re$ is said to be *bounded below* if

    $$(\exists\ u \in \Re) \text{ such that } (\forall\ a \in A)\ a \geq u.$$

    The number u is a *lower bound* of A.
  - Note: The "infimum" and "minimum" operation are always taken the set of all real numbers as a universal set.
- Greatest lower bound
  - Every non-empty bounded-below subset of the real line $\Re$ has a *greatest lower bound*, acronym as g.l.b.

Infimum and Minimum

- Extension definition of g.l.b. for empty set and non-bounded-below set
  - g.l.b. of an empty set is ∞.
  - g.l.b. of an non-bounded-below set is $-\infty$.
- Under the extension definition, a common terminology for g.l.b. of A is $\inf A$, which termed the *infimum* of A.
- The *minimum* of a set

$$\text{if } \inf A \in A, \text{ then } \min A = \inf A; \text{otherwise } \min A \text{ does not exist.}$$

  - E.g., $A = [0, 1)$ and $\min A = \inf A = 0$.
  - E.g., if $A = (0, 1)$, then $\inf A = 0$ but $\min A$ does not exist!

Boundedness

- A subset A of the real line $\Re$ is said to be *bounded* if it is both bounded above and below.

Operations of Supremum and Infimum

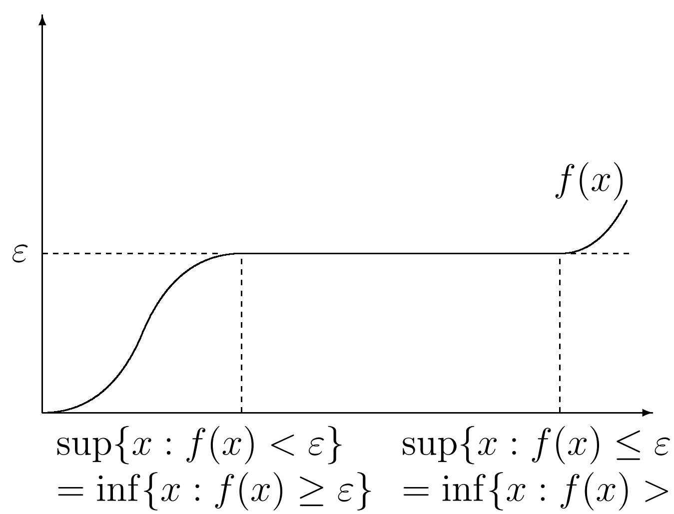
1. $A \subset B \Rightarrow \sup A \leq \sup B$;
2. $A \subset B \Rightarrow \inf A \geq \inf B$;
3. $\sup A = -\inf(-A)$;
4. $\inf A = -\sup(-A)$;
5. $\sup(A + B) = \sup A + \sup B$, if $|\sup A| < \infty$ and $|\sup B| < \infty$;
   (Although $\infty = \infty$ is acceptable in mathematics, the condition prevents from the occurrence of, e.g., $(-\infty) + (+\infty)$.)
6. $\sup(k \cdot A) = k \cdot \sup A$, if $|\sup A| < \infty$ and $0 < k < \infty$.
7. Both two cases of

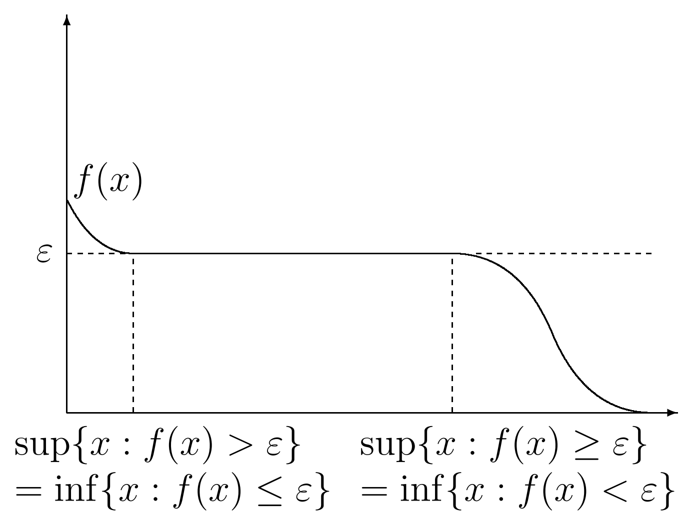
$$\sup(A \cdot B) > (\sup A) \cdot (\sup B)$$
$$\sup(A \cdot B) = (\sup A) \cdot (\sup B).$$

could happen. Can you give examples for both cases? (Exercise)

Operations of Supremum and Infimum

8. $\sup\{x \in \Re \ : \ f(x) < \varepsilon\} = \inf\{x \in \Re \ : \ f(x) \geq \varepsilon\}$ and
   $\sup\{x \in \Re \ : \ f(x) \leq \varepsilon\} = \inf\{x \in \Re \ : \ f(x) > \varepsilon\}$, if $f(x)$ non-decreasing;
9. $\sup\{x \in \Re \ : \ f(x) > \varepsilon\} = \inf\{x \in \Re \ : \ f(x) \leq \varepsilon\}$ and
   $\sup\{x \in \Re \ : \ f(x) \geq \varepsilon\} = \inf\{x \in \Re \ : \ f(x) < \varepsilon\}$, if $f(x)$ non-increasing.





Sequences and Their Limits

- Limit

$$\lim_{n\to\infty} a_n = L, \text{ if, and only if,}$$
$$(\forall\ \varepsilon > 0)(\exists\ N) \text{ such that } (\forall\ n > N)\ |a_n - L| < \varepsilon.$$

  - In terminology, we say that a_n converges to L with respect to n.
  - If there is no such L satisfying the above condition, we say that the limit of $\{a_n\}_{n\geq 1}$ does not exist.

- Convergence of a monotone sequence
  - If a_n non-decreasing, then either $\lim_{n\to\infty} a_n$ exists or $a_n \to \infty$.
  - If a_n non-increasing, then either $\lim_{n\to\infty} a_n$ exists or $a_n \to -\infty$.

  Note: Some researchers would not bother to separately consider the cases of $\lim_{n\to\infty} a_n$ exists in $\Re$ or $a_n \to \pm\infty$. Instead, they state that:

  - If a_n non-decreasing, then $\lim_{n\to\infty} a_n$ exists in $\Re \cup \{-\infty, \infty\}$.
  - If a_n non-increasing, then $\lim_{n\to\infty} a_n$ exists in $\Re \cup \{-\infty, \infty\}$.

  In the lecture notes, we will adopt the viewpoint on the set of *extended real numbers*, i.e., $\Re \cup \{-\infty, \infty\}$ due to its simplicity.

Limsup and Liminf

- Limit supremum

$$\limsup_{n\to\infty} a_n \triangleq \lim_{n\to\infty} \sup\{a_n, a_{n+1}, a_{n+2}, \ldots\} = \lim_{n\to\infty} \sup\{a_k : k \geq n\} = \lim_{n\to\infty} \sup_{k\geq n} a_k.$$

  – The sequence $b_n \triangleq \sup\{a_k : k \geq n\}$ is *non-increasing*; hence, its *limit* either exist or be $\pm\infty$.

- Limit infimum

$$\liminf_{n\to\infty} a_n \triangleq \lim_{n\to\infty} \inf\{a_n, a_{n+1}, a_{n+2}, \ldots\} = \lim_{n\to\infty} \inf\{a_k : k \geq n\} = \lim_{n\to\infty} \inf_{k\geq n} a_k.$$

  – The sequence $b_n \triangleq \inf\{a_k : k \geq n\}$ is *non-decreasing*; hence, its *limit* either exist or be $\pm\infty$.

Concept behind Limsup and Liminf

- Limsup = largest clustering point
- Liminf = smallest clustering point
- A clustering point is a point that the sequence a_n hits close for infinitely many times.

  E.g., $a_n = \sin(n\pi/2)$

  $\Rightarrow \{a_n\}_{n\geq 1} = \{1, 0, -1, 0, 1, 0, -1, 0, \ldots\}$

  There are three clustering points in such sequence, which are -1, 0 and 1.

  Consequently,

$$\begin{aligned} \limsup_{n\to\infty} a_n &= 1 = \text{the largest clustering point} \\ \liminf_{n\to\infty} a_n &= -1 = \text{the smallest clustering pint} \end{aligned}$$

  E.g., $a_n = -n$. Then $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = -\infty$.

  E.g., $a_n = n$. Then $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \infty$.

Limit: An Alternative Definition

- (Recall) The usual definition of limit

$$\lim_{n\to\infty} a_n = L, \text{ if, and only if,}$$
$$(\forall\ \varepsilon > 0)(\exists\ N) \text{ such that } (\forall\ n > N)\ |a_n - L| < \varepsilon.$$

- Limit: An alternative definition

$$\lim_{n\to\infty} a_n = L, \text{ if, and only if, } \limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n.$$

  - *Interpretation*: The fact that the largest clustering point coincides with the smallest clustering point surely implies that they are both equal to the-one-and-only clustering point.

- Can you prove that the two definitions are equivalent. Try it. (Exercise)

Properties regarding Limsup and Liminf

- The limit supremum either exists or equals $\pm\infty$.
  - Sometimes, mathematicians will consider the sample space of *extended real line* as $\Re \cup \{\infty, -\infty\}$.
  - In such case, they will say that the limsup always exists.
- The limit infimum either exists or equals $\pm$ infinity.
  - When considering the extended real line, the liminf always exists.

Properties regarding Limsup and Liminf

- $\liminf_{n\to\infty} a_n \le \limsup_{n\to\infty} a_n$.
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$$\begin{aligned}\liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n &\le \liminf_{n\to\infty}(a_n + b_n) \\ &\le \limsup_{n\to\infty} a_n + \liminf_{n\to\infty} b_n \\ &\le \limsup_{n\to\infty}(a_n + b_n) \\ &\le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.\end{aligned}$$

- If $\lim_{n\to\infty} a_n$ exists, then
  - $\liminf_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$
  - $\limsup_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$
  - $\lim_{n\to\infty}(\alpha \cdot a_n) = \alpha \cdot \lim_{n\to\infty} a_n$ (for a scalar α, either finite or $\pm\infty$).
- If both $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exists, then
  - $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$.
  - $\lim_{n\to\infty}(a_n \cdot b_n) = (\lim_{n\to\infty} a_n) \cdot (\lim_{n\to\infty} b_n)$.

Sufficiently Large and Infinitely Often

- "Something" is true for *all sufficiently large* $n \equiv$ It is true for every $n > N$ for some fixed N.
  - E.g., $a_n = \sqrt{n}$. Then $a_n > 20$ for all sufficiently large n.
- "Something" is true for *infinitely many* $n =$ For every N, it is true for some $n > N$.
  - E.g., $a_n = (-1)^n \sqrt{n}$. Then $a_n > 20$ for *infinitely many* n (or, $a_n > 20$ infinitely often in n).

Equivalence

1. $x < y + \varepsilon$ for all $\varepsilon > 0$ if, and only if, $x \leq y$;
2. $x < y - \varepsilon$ for some $\varepsilon > 0$ if, and only if, $x < y$;
3. $x > y - \varepsilon$ for all $\varepsilon > 0$ if, and only if, $x \geq y$;
4. $x > y + \varepsilon$ for some $\varepsilon > 0$ if, and only if, $x > y$;
5. $|a| < \varepsilon$ for all $\varepsilon > 0$ if, and only if, $a = 0$.

Key Notes

- Supremum and Infimum over a subset of real line
- Limsup and Liminf (and their properties)
- Sufficiently large and infinitely often
- Equivalence