Information-Theoretic Approach to Unimodal Density Estimation

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Introduction:

1. It's useful to estimate the probability density function corresponding to unimodal .

2. Usually, a maximum entropy solution is attempted .

3. Disadvantages: the resulting density may not be unimodal. ex: Assume we use the following four percentile constraints on an unknown random variable Y:Pr$[3 \leq Y \leq 4]$=Pr$[6 \leq Y \leq 7]$=0.13, Pr$[0 \leq Y \leq 10]$=1.0, Pr$[0 \leq Y \leq 5]$=0.5, and estimate the density of Y via maximum entropy(as Fig. 1).

GOAL:

Present a new method for unimodal density estimation which extends the maximum entropy technique to guarantee unimodality of the resultant density.

Maximum Entropy and MDI Density Estimation:
An minimum discrimination information(MDI) estimation problem is to pick that density function f which is as close as possible to some other given function g, and for which f satisfies certain given moment constraints, e.g.

$$\min_f \int f(x) \ln \left(\frac{f(x)}{g(x)} \right) \lambda(dx)$$

subject to

$$\int a_i(x) f(x) \lambda(dx) = \theta_i. \qquad i = 0, 1, 2, \cdots, k.$$

OTE: If $g(x) = 1$, the MDI objective function is of the form

$$\int f(x) \ln f(x) \lambda(dx) = -\int f(x) \ln \left(\frac{1}{f(x)} \right) \lambda(dx),$$

and the MDI problem becomes an ME problem.

Introduction a Lagrange multiplier for each constraint, we wish to maximize

$$L = \int f(x) \ln\left(\frac{g(x)}{f(x)}\right) \lambda(dx) \quad - \quad \alpha_0 \left(\theta_0 - \int a_0(x) f(x) \lambda(dx)\right) + \cdots + \alpha_k \left(\theta_k - \int a_k(x) f(x) \lambda(dx)\right)$$

or equivalently

$$\begin{aligned} L - \sum_{i=1}^{k} \alpha_i \theta_i &= \int f(x) \left(\ln\left[\frac{g(x)}{f(x)}\right] - \alpha_0 - \alpha_1 a_1(x) - \cdots - \alpha_k a_k(x) \right) \lambda(dx) \\ &= \int f(x) \ln\left(\frac{g(x) \exp\left[-\sum_{i=0}^{k} \alpha_i a_i(x)\right]}{f(x)}\right) \lambda(dx) \\ &\leq \int f(x) \left(\frac{g(x) \exp\left[-\sum_{i=0}^{k} \alpha_i a_i(x)\right]}{f(x)} - 1\right) \lambda(dx) \end{aligned}$$

The equality holds when

$$f^*(x) = g(x) \exp\left(-\sum_{i=0}^{k} \alpha_i a_i(x)\right),$$

and $f^*(x)$ is the MDI density, where the constants α_i can be found using the moment constraints.

Method:

1. If Y is unimodal with mode m, then $Y - m$ is zero unimodal .We transform the moment constraints on Y into moment constraints on X where the new moment functions for X are
$$a_i^*(x) = \frac{1}{x}\int_0^x a_i(t+m)dt.$$

2. Then solve the estimation of f_x by the MDI problem.

3. if X is estimated by $\hat{X}$, then Y is estimated by $m + U \cdot \hat{X}$, where U is uniformly distributed over [0,1].

CONCLUSION:

1. The correspondence presents a method for transforming the estimation problem in the case of unimodal density estimation.

2. The transformed problem is then solved by information-theoretic methods and transformed back to obtain a unimodal density estimate.

3. Some qualitative characteristics of the desired density such as smoothness near the mode can also be incorporated into this unimodal information-theoretic density estimation technique.