Linear Algebra

Spring Semester 2014
Prof. Dr. Po-Ning Chen
http://shannon.cm.nctu.edu.tw/la.htm

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Problem 1 ***Diagonalizability of Rank one Matrices***

Consider the singular matrix A given by $\mathsf{A} = \mathbf{x}\mathbf{y}^{\mathsf{T}}$. Do the following questions.

a) (6%) Find two nonzero vectors $\mathbf{x}, \mathbf{y}$ such that the rank-1 matrix A is not diagonalizable.

Hint: For a 2×2 *matrix* $\mathsf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$, *the sum of two eigenvalues,* i.e., $\lambda_1 + \lambda_2$, *is equal to* $a_{1,1} + a_{1,2}$.

b) (6%) Determine three eigenvalues and their corresponding eigenvectors of A if

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

c) (4%) Is A in (b) diagonalizable? Explain why or why not.

Problem 2 ***Positive Definite, Negative Definite or Indefinite***

a) (4%) Determine the ranges of the unknown real numbers a, b such that

$$\mathsf{A} = \begin{bmatrix} 1 & 2 & 5 \\ 2 & a & 10 \\ 5 & 10 & b \end{bmatrix}.$$

is positive definite.

Hint: Leading principle minors.

b) (4%) Continue from (a). Fnd the ranges of real numbers of a, b such that A^2 is positive definite.

Hint: $\det(\mathsf{A}) = \prod_{i=1}^{3} \lambda_i$, *where* $\{\lambda_i\}_{i=1}^{3}$ *are eigenvalues of* A.

c) (6%) Prove that if B is positive definite and is similar to C, then C is also positive definite.

Hint: Positive definiteness $\equiv$ *positivity of all eigenvalues.*

d) (6%) Prove that if B and C are both positive definite, then the eigenvalues of BC are all positive.

Hint: Use $\mathbf{y}^{\mathsf{T}}\mathsf{B}\mathbf{y} > 0$ *and* $\mathbf{x}^{\mathsf{T}}\mathsf{C}\mathbf{x} > 0$ *to show that* $\mathsf{BC}\mathbf{x} = \lambda\mathbf{x}$ *can only be valid for positive* λ.

e) (6%) Suppose a 3×3 positive-definite matrix D has eigenvalues $0.8, 8, 80$. Determine the ranges of $p, q, r \in \mathbb{R}$ such that

- $\mathsf{D} - p\mathsf{I}$ is positive definite.
- $\mathsf{D} - q\mathsf{I}$ is indefinite.
- $\mathsf{D} - r\mathsf{I}$ is negative definite.

Here I denotes the 3×3 identity matrix.

Hint: Eigenvalues of $\mathsf{D} - c\mathsf{I}$.

Problem 3 ***Matrices with Repeated Eigenvalues***

a) (6%) Prove that if $\mathsf{A}_{n\times n}$ and $\lambda\mathsf{I}_{n\times n}$ are similar, then $\mathsf{A}_{n\times n} = \lambda\mathsf{I}_{n\times n}$.

   Hint: Similarity of A *and* B *implies* $\mathsf{A} = \mathsf{M}^{-1}\mathsf{B}\mathsf{M}$ *for some* M.

b) (6%) Prove that a diagonalizable matrix A whose eigenvalues are all equal to λ (*i.e.*, $\lambda_1 = \lambda_2 = \cdots = \lambda_n = \lambda$) must be equal to $\lambda\mathsf{I}$.

c) (4%) Use (b) to explain why

$$\mathsf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

   cannot be diagonalizable.

d) (6%) Use (b) to prove that the only symmetric matrix P satisfying both $i)$ $\mathsf{P}^2 = \mathsf{P}$ and $ii)$ all eigenvalues being positive is $\mathsf{P} = \mathsf{I}$.

Problem 4 ***Jordan Form, SVD and Pseudo-Inverse***

Let $\mathsf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

a) (6%) Find the Jordan form of A.

b) (6%) Determine the singular-value decomposition of A.

c) (4%) Find the pseudo-inverse of A.

d) (6%) Among all $\hat{\mathbf{x}}$'s that minimizes $\|A\hat{\mathbf{x}} - \mathbf{b}\|^2$, find the one with the smallest norm $\|\hat{\mathbf{x}}\|$, where

$$\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Problem 5 ***Invertible Property***

a) (6%) Prove that a matrix $\mathsf{A}_{n\times n} = \mathsf{U}_{n\times n}\Sigma_{n\times n}\mathsf{V}_{n\times n}$ is invertible if it has n singular values.

   Hint: A direct way to prove that a matrix has inverse is to provide its inverse.

b) (8%) Use (a) to prove that if the rank of $\mathsf{A}_{n\times m}$ is n and $n \leq m$, then $\mathsf{A}\mathsf{A}^\intercal$ is invertible.

   Hint: What is the form of $\Sigma_{n\times m}$ *for such* $\mathsf{A}_{n\times m}$*?*