Linear Algebra

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Problem 1 ***Diagonalizability of Rank one Matrices***

Consider the singular matrix A given by $\mathsf{A} = \mathbf{x}\mathbf{y}^\intercal$. Do the following questions.

a) (6%) Find two nonzero vectors $\mathbf{x}, \mathbf{y}$ such that the rank-1 matrix A is not diagonalizable.

Hint: For a 2×2 *matrix* $\mathsf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$, *the sum of two eigenvalues,* i.e., $\lambda_1 + \lambda_2$, *is equal to* $a_{1,1} + a_{1,2}$.

b) (6%) Determine three eigenvalues and their corresponding eigenvectors of A if

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

c) (4%) Is A in (b) diagonalizable? Explain why or why not.

Solution ....

a) A 2×2 matrix is possibly not diagonalizable only when it has repeated eigenvalues. So, a rank-1 2×2 non-diagonalizable matrix must have eigenvalues $0, 0$. By this, an example satisfied the need of this subproblem is

$$\mathsf{A} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}.$$

Then $\text{rref}(\mathsf{A}) = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ implies A only has one eigenvector.

b) Since this matrix is of rank 1, it has two free columns; hence, two of its eigenvalues λ_1 and λ_2 are zeros and their corresponding eigenvectors are the basis of the nullspace of A. The non-zero eigenvalue can be determined via the trace of A, i.e.,

$$\text{tr}(\mathsf{A}) = \lambda_1 + \lambda_2 + \lambda_3 = 6;$$

it follows that $\lambda_3 = 6$.

The eigenvector corresponding to $\lambda_3 = 6$ is obviously $\mathbf{x}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, which can be confirmed via:

$$\mathsf{A}\mathbf{x}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \left(\begin{bmatrix} 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} 6 = 6\mathbf{x}_3.$$

The other two eigenvectors can be obtained using the following trick:

$$\lambda_1, \lambda_2 = 0, 0: \quad \mathrm{rref}(\mathsf{A}) = \begin{bmatrix} 1 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \Longrightarrow \quad \mathsf{N} = \begin{bmatrix} -\frac{1}{2} & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Longrightarrow \quad \mathbf{x}_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

c) A has three linearly independent eigenvectors; hence A is diagonalizable.

Additional Note: The following proposition provides a sufficient condition under which a rank-1 matrix is diagonalizable.

> **Proposition.** *Suppose* A *is given by*
>
> $$\mathsf{A} = \mathbf{x}\mathbf{y}^{\mathsf{T}} = \mathbf{x} \otimes \mathbf{y}$$
>
> *with some nonzero vectors* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. *Then if*
>
> $$\mathbf{y}^{\mathsf{T}}\mathbf{x} = \mathbf{x} \cdot \mathbf{y} = \mathrm{tr}(\mathbf{x} \otimes \mathbf{y}) = \mathrm{tr}(\mathsf{A}) \neq 0,$$
>
> A *is diagonalizable.*

Problem 2 — *Positive Definite, Negative Definite or Indefinite*

a) (4%) Determine the ranges of the unknown real numbers a, b such that

$$\mathsf{A} = \begin{bmatrix} 1 & 2 & 5 \\ 2 & a & 10 \\ 5 & 10 & b \end{bmatrix}.$$

is positive definite.

Hint: Leading principle minors.

b) (4%) Continue from (a). Fnd the ranges of real numbers of a, b such that A^2 is positive definite.

Hint: $\det(\mathsf{A}) = \prod_{i=1}^{3} \lambda_i$, *where* $\{\lambda_i\}_{i=1}^{3}$ *are eigenvalues of* A.

c) (6%) Prove that if B is positive definite and is similar to C, then C is also positive definite.

Hint: Positive definiteness $\equiv$ *positivity of all eigenvalues.*

d) (6%) Prove that if B and C are both positive definite, then the eigenvalues of BC are all positive.

Hint: Use $\mathbf{y}^{\mathsf{T}}\mathsf{B}\mathbf{y} > 0$ *and* $\mathbf{x}^{\mathsf{T}}\mathsf{C}\mathbf{x} > 0$ *to show that* $\mathsf{BC}\mathbf{x} = \lambda\mathbf{x}$ *can only be valid for positive* λ.

e) (6%) Suppose a 3×3 positive-definite matrix D has eigenvalues $0.8, 8, 80$. Determine the ranges of $p, q, r \in \mathbb{R}$ such that

- $\mathsf{D} - p\mathsf{I}$ is positive definite.

- $\mathsf{D} - q\mathsf{I}$ is indefinite.
- $\mathsf{D} - r\mathsf{I}$ is negative definite.

Here I denotes the 3×3 identity matrix.

Hint: Eigenvalues of $\mathsf{D} - c\mathsf{I}$.

Solution . . . .

a) We can examine the leading principle minors and the determinant of A to determine the conditions under which A is positive definite:

$\det([a_{1,1}])$**:** $a_{1,1} = 1 > 0$

$\det\left(\begin{bmatrix} 1 & 2 \\ 2 & a \end{bmatrix}\right)$**:** $\det\left(\begin{bmatrix} 1 & 2 \\ 2 & a \end{bmatrix}\right) = a - 4 > 0 \Rightarrow a > 4.$

$\det(\mathsf{A})$**:** $\det(\mathsf{A}) = ab + 100 + 100 - 4b - 25 - 100 = (a-4)(b-25) > 0 \Rightarrow (a > 4 \text{ and } b > 25) \text{ or } (a < 4 \text{ and } b < 25).$

To sum up, we have $a > 4$ and $b > 25$.

b) The eigenvalues of A^2 are squares of eigenvalues of A. Hence, as long as A has no zero eigenvalues, eigenvalues of A^2 are all positive. Hence, in order to make A^2 positive definite, it requires $\det(\mathsf{A}) = (a-4)(b-25) = \prod_i \lambda_i \neq 0$, which implies $a \neq 4$ and $b \neq 25$.

c) Since B is positive definite, its eigenvalues are all positive. This conclusion, together with the fact that two similar matrices have the same eigenvalues, implies that C is positive definite.

d) It suffices to show that

$$\mathsf{BC}\mathbf{x} = \lambda \mathbf{x} \tag{1}$$

can only be valid for positive λ. Left-multiply (1) by $(\mathsf{C}\mathbf{x})^\intercal$ to obtain:

$$(\mathsf{C}\mathbf{x})^\intercal \mathsf{BC}\mathbf{x} = (\mathsf{C}\mathbf{x})^\intercal \lambda \mathbf{x} \Longrightarrow \underbrace{(\mathsf{C}\mathbf{x})^\intercal \mathsf{B}(\mathsf{C}\mathbf{x})}_{>0} = \lambda \mathbf{x}^\intercal \mathsf{C}^\intercal \mathbf{x} = \lambda \underbrace{\mathbf{x}^\intercal \mathsf{C}\mathbf{x}}_{>0}, \tag{2}$$

where (2) follows since B and C are both positive definite, and positive-definiteness is only defined over symmetric matrix. This proves that λ must be positive.

Additional Note: BC might not be symmetric; hence, it may be inappropriate to say BC is positive definite even if its eigenvalues are all positive.

e) The eigenvalues of $\mathsf{D} - c\mathsf{I}$ are equal to $\lambda_1 - c, \lambda_2 - c, \lambda_3 - c$ if $\lambda_1, \lambda_2, \lambda_3$ are eigenvalues of D.

- To make $\mathsf{D} - p\mathsf{I}$ positive definite, we must have $0.8 - p > 0, 8 - p > 0, 80 - p > 0$; hence, $p < 0.8$.
- To make $\mathsf{D} - q\mathsf{I}$ indefinite, then $0.8 - q < 0$ and $80 - q > 0$; hence, $0.8 < q < 80$.
- To make $\mathsf{D} - r\mathsf{I}$ negative definite, we must have $0.8 - r < 0, 8 - r < 0, 80 - r < 0$; hence $r > 80$.

Problem 3 ***Matrices with Repeated Eigenvalues***

a) (6%) Prove that if $\mathsf{A}_{n\times n}$ and $\lambda\mathsf{I}_{n\times n}$ are similar, then $\mathsf{A}_{n\times n} = \lambda\mathsf{I}_{n\times n}$.

Hint: Similarity of A *and* B *implies* $\mathsf{A} = \mathsf{M}^{-1}\mathsf{B}\mathsf{M}$ *for some* M.

b) (6%) Prove that a diagonalizable matrix A whose eigenvalues are all equal to λ (*i.e.*, $\lambda_1 = \lambda_2 = \cdots = \lambda_n = \lambda$) must be equal to $\lambda\mathsf{I}$.

c) (4%) Use (b) to explain why

$$\mathsf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

cannot be diagonalizable.

d) (6%) Use (b) to prove that the only symmetric matrix P satisfying both $i)$ $\mathsf{P}^2 = \mathsf{P}$ and $ii)$ all eigenvalues being positive is $\mathsf{P} = \mathsf{I}$.

Solution ....

a) Since A and $\lambda\mathsf{I}$ are similar, there exists some invertible M such that

$$\mathsf{A} = \mathsf{M}^{-1}(\lambda\mathsf{I})\mathsf{M} = \lambda\mathsf{M}^{-1}\mathsf{I}\mathsf{M} = \lambda\mathsf{I}.$$

b) A is diagonalizable; hence

$$\mathsf{A} = \mathsf{S}\Lambda\mathsf{S}^{-1} = \mathsf{S}(\lambda\mathsf{I})\mathsf{S}^{-1} = \lambda\mathsf{I}, \tag{3}$$

where (3) holds because all eigenvalues of A equal λ.

c) The eigenvalues of

$$\mathsf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

are all equal to 1. So according to (b), if it were diagonalizable, then $\mathsf{B} = \mathsf{I}$, which is apparently not true!

d) Note that a symmetric matrix is always diagonalizable (Spectral theorem in Slide 6-78). $\mathsf{P}^2 = \mathsf{P}$ implies that the eigenvalues satisfy $\lambda^2 = \lambda$, which indicates $\lambda = 0$ or 1. The second condition then implies $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 1$. By this result, subproblem (b) directly gives that $\mathsf{P} = \mathsf{I}$.

Problem 4 — *Jordan Form, SVD and Pseudo-Inverse*

Let $\mathsf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

a) (6%) Find the Jordan form of A.

b) (6%) Determine the singular-value decomposition of A.

c) (4%) Find the pseudo-inverse of A.

d) (6%) Among all $\hat{\mathbf{x}}$'s that minimizes $\|A\hat{\mathbf{x}} - \mathbf{b}\|^2$, find the one with the smallest norm $\|\hat{\mathbf{x}}\|$, where

$$\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Solution ....

a) See Slides 6-108~6-112.

b) See Slides 6-117~6-118.

c) Since all the singular values are equal to one, $\mathsf{A}^+ = \mathsf{A}^\mathsf{T}$.

d) $\mathbf{x}^+ = \mathsf{A}^+\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$

Problem 5 — *Invertible Property*

a) (6%) Prove that a matrix $\mathsf{A}_{n\times n} = \mathsf{U}_{n\times n}\Sigma_{n\times n}\mathsf{V}_{n\times n}$ is invertible if it has n singular values.

Hint: A direct way to prove that a matrix has inverse is to provide its inverse.

b) (8%) Use (a) to prove that if the rank of $\mathsf{A}_{n\times m}$ is n and $n \le m$, then $\mathsf{A}\mathsf{A}^\mathsf{T}$ is invertible.

Hint: What is the form of $\Sigma_{n\times m}$ for such $\mathsf{A}_{n\times m}$?

Solution ....

a) Since $\mathsf{A} = \mathsf{U}\Sigma\mathsf{V}^\mathsf{T}$ and A has n singular values, $\mathsf{B} = \mathsf{V}\Sigma^{-1}\mathsf{U}^\mathsf{T}$ must be its inverse, i.e., $\mathsf{AB} = \mathsf{U}\Sigma\mathsf{V}^\mathsf{T}\mathsf{V}\Sigma\mathsf{U}^\mathsf{T} = \mathsf{I}$ and similarly $\mathsf{BA} = \mathsf{I}$. Note that in case A has no n singular values, Σ has no inverse.

b) Since $\mathsf{A} = \mathsf{U}\begin{bmatrix}\Sigma_{n\times n} & 0\end{bmatrix}\mathsf{V}^\mathsf{T}$, where $\Sigma_{n\times n}$ is an n by n matrix with n (non-zero) singular values of A on its diagonal, we have

$$\begin{aligned}
\mathsf{A}\mathsf{A}^\mathsf{T} &= \mathsf{U}\begin{bmatrix}\Sigma_{n\times n} & 0\end{bmatrix}\mathsf{V}^\mathsf{T}\mathsf{V}\begin{bmatrix}\Sigma_{n\times n} \\ 0\end{bmatrix}\mathsf{U}^\mathsf{T} \\
&= \mathsf{U}\begin{bmatrix}\Sigma_{n\times n} & 0\end{bmatrix}\begin{bmatrix}\Sigma_{n\times n} \\ 0\end{bmatrix}\mathsf{U}^\mathsf{T} \\
&= \mathsf{U}\Sigma_{n\times n}^2\mathsf{U}^\mathsf{T}
\end{aligned}$$

Since $\Sigma_{n\times n}$ has no zero elements on its diagonals, neither does $\Sigma_{n\times n}^2$. Thus, Σ_n^2 is invertible, and so does $\mathsf{A}\mathsf{A}^\mathsf{T}$ by the statement in (a).