Linear Algebra

Homework 1 of February 27, 2014

Spring Semester 2014
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Problem 1 (10%) ***Linear Combination of Polynomials***

The polynomials can also be regarded as "vectors." Define two vectors $\mathbf{u} = x^3 - 2x^2 - 5x - 3$ and $\mathbf{w} = 3x^3 - 5x^2 - 4x - 9$, show that the vector $\mathbf{w} = 2x^3 - 2x^2 + 12x - 6$ can be expressed as a linear combination of $\mathbf{u}$ and $\mathbf{v}$, while $\mathbf{s} = 3x^3 - 2x^2 + 7x + 8$ can not.

Problem 2 (30%) ***Parallelogram Law***

A parallelogram is a quadrilateral (四邊形) where the opposite sides are parallel and equal. Use vectors to show that:

a) The midpoints of the two diagonals (對角線) of a parallelogram are identical. In case the four sides of the parallelogram are equal in lengths, then the two diagonals are perpendicular.

b) Given a parallelogram, the sum of the squares of all four sides lengths is equal to the sum of the squares of two diagonals lengths.

Moreover, show that for any quadrilateral, connect the midpoints of its four sides in order, then the midpoints form a parallelogram.

Problem 3 (10%) ***Special Angles***

Pick any numbers x, y, z that add to $x + y + z = 0$ and define $\mathbf{v} = (x, y, z)$ and $\mathbf{w} = (z, x, y)$. Show that $\mathbf{v} \cdot \mathbf{w} / \|\mathbf{v}\|\|\mathbf{w}\|$ is always $-\frac{1}{2}$. *I.e.*, the angle between $\mathbf{v}$ and $\mathbf{w}$ is $120°$.

Problem 4 (10%) ***Cauchy–Schwarz Inequality***

For positive real numbers a, b, c. Use Schwarz Inequality to show that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Hint: Consider two vectors $\mathbf{v} = (\sqrt{a+b}, \sqrt{b+c}, \sqrt{c+a})$ *and* $\mathbf{w} = (\frac{1}{\sqrt{a+b}}, \frac{1}{\sqrt{b+c}}, \frac{1}{\sqrt{c+a}})$.

Problem 5 (10%) ***Perpendicular Unit Vectors***

Find four perpendicular unit vectors with all components equal to $\frac{1}{2}$ or $-\frac{1}{2}$.

Problem 6 (10%) ***Linear Combination of Matrices***

Claim that an arbitrary 2×2 matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad a_{ij} \in \mathbb{R}, \forall\, i, j,$$

can be expressed as a linear combination of the four given matrices as below:

$$\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}, \text{ and} \begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}.$$

I.e., find four scalars a, b, c, d in terms of a_{11}, a_{12}, a_{21}, and a_{22}, such that

$$\begin{bmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{bmatrix} = a\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix} + b\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix} + c\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix} + d\begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}.$$

Problem 7 (10%) *Elimination Matrices*

The matrix

$$\mathsf{A} = \begin{bmatrix}2 & -1 & 0 & 0\\-1 & 2 & -1 & 0\\0 & -1 & 2 & -1\\0 & 0 & -1 & 2\end{bmatrix}$$

will need elimination matrices $\mathsf{E}_{2,1}$, $\mathsf{E}_{3,2}$, and $\mathsf{E}_{4,3}$, to obtain

$$\mathsf{E}_{4,3}\mathsf{E}_{3,2}\mathsf{E}_{2,1}A = \begin{bmatrix}2 & -1 & 0 & 0\\0 & \frac{3}{2} & -1 & 0\\0 & 0 & \frac{4}{3} & -1\\0 & 0 & 0 & \frac{5}{4}\end{bmatrix} = \mathsf{U}.$$

a) What are those elimination matrices?

b) Multiply these three elimination matrices to get one matrix M that does the elimination in one below: $\mathsf{MA} = \mathsf{U}$.

c) What is the inverse of M? *Hint: Think about the inverses of* $\mathsf{E}_{2,1}$, $\mathsf{E}_{3,2}$, *and* $\mathsf{E}_{4,3}$ *first.*

Problem 8 (10%) *Proof of Associativity*

Suppose A is m by n, B is n by p, and C is p by q. To prove that $(\mathsf{AB})\mathsf{C} = \mathsf{A}(\mathsf{BC})$, first assume that $q = 1$.

a) Use the column vectors $\mathbf{b}_1, \ldots, \mathbf{b}_p$ of B and the components $c_1, \ldots, c_p$ of $\mathbf{c}$ to express $(\mathsf{AB})\mathbf{c}$.

b) Use the column vectors $\mathbf{b}_1, \ldots, \mathbf{b}_p$ of B and the components $c_1, \ldots, c_p$ of $\mathbf{c}$ to express $\mathsf{A}(\mathsf{B}\mathbf{c})$.

c) Explain why these two expressions are identical.

d) Now drop the simplification of $q = 1$ and explain why $(\mathsf{AB})\mathsf{C} = \mathsf{A}(\mathsf{BC})$ in general.